CHARACTERIZATIONS OF SPHERICAL NEIGHBOURHOODS

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Introduction. If Σ is a specified class of metric spaces and $M \in \Sigma$, then the characterization problem is to find necessary and sufficient conditions which distinguish the spherical neighbourhoods (open spheres) of M among a specified class of subsets of M.

In a metric space M the notation pqr means $p \neq q \neq r$ and $pq + qr = pr$. M is said to be uniformly locally externally convex if there exists $\delta > 0$ such that if $p, q \in M$, $p \neq q$, and $pq < \delta$, then there exists $r \in M$ such that the relation pqr subsists. We will prove the following result.

THEOREM 1. *Let M be a metric space which is complete, metrically convex, and uniformly locally externally convex. A non-empty, bounded, open subset S of M with diameter $D < \delta$ is a spherical neighbourhood if and only if for each two distinct points $p, q \in S$ there exists a spherical neighbourhood U, $U \subset S$, such that p and q are boundary points of U.*

A similar condition was used by Hsiang [**6**] to characterize circles among Jordan curves in E_2.

Equichordal points of convex sets in normed linear spaces are studied and a question of Blaschke, Rothe, and Weitzenböck [**1**] concerning the possible existence of a convex set in E_2 with more than one equichordal point is shown (Theorem 3) to have an affirmative answer in this more general setting. Finally, the property of possessing an equichordal point is adjoined to the property of constant width to obtain characterizing conditions for spherical neighbourhoods where Σ is the class of real finite-dimensional Banach spaces.

2. Proof of Theorem 1. The following lemma is a corollary to the proof of a result of Blumenthal [**2**, p. 55].

LEMMA. *Let M be a metric space which is complete and uniformly locally externally convex. If $p, q \in M$ and $0 < pq < \delta$, then for $\epsilon > 0$ there exists $r \in M$ such that pqr subsists and $pr > \delta - \epsilon$.*

Suppose that S is a spherical neighbourhood $U(p; \rho)$ of p with radius ρ. Let $a, b \in S$, $a \neq b$, where $pa \geqq pb$. Since M is complete and metrically

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convex, there exists a metric segment $S_{p,a}$ joining p to a (see [**2**, p. 41]). For $x \in Sp, a$ the function $f(x) = xa - xb$ is continuous on $S_{p,a}$ and $f(p) \geqq 0$, $f(a) < 0$. Therefore, there exists a point $z \in S_{p,a}$, $z \neq a$, for which $f(z) = 0$. Let $q \in U(z; za)$. Then $pq \leqq pz + zq < pz + za = pa < \rho$. Therefore $U(z; za) \subset U(a; \rho) = S$ and a and b are boundary points of $U(z; za)$ since metric segments $S_{z,a}$ and $S_{z,b}$ exist.

For the sufficiency part of the proof, we may assume that M has at least two points and since $D < \delta$, it follows from the lemma that S is a proper subset of M. Consequently, $R = \text{lub}\, \rho$, $U(p; \rho) \subset S$, is finite, and it follows from the two-point property stated in the theorem and the triangle inequality that $2R \geqq D$.

Let $\{U(p_i; \rho_i)\}$ be a sequence of spherical neighbourhoods, each contained in S, such that $\rho_i \to R$. For $\epsilon > 0$, $\epsilon < \delta - D$, let $R - \rho_i < \epsilon/3$ for $i > N_\epsilon$. We will show that $\{p_i\}$ is a Cauchy sequence. For $p_i \neq p_j$, since $p_i p_j \leqq D < \delta$ there exists $s_j \in M$ such that $p_i p_j s_j$ subsists and $p_i s_j > \delta - \epsilon > D$. Therefore $s_j \notin S$ and $p_j s_j \geqq \rho_j$. Since $p_i p_j s_j$ holds, there exists, by a known result [**2**, p. 44], a metric segment with endpoints p_i, s_j and which contains p_j. On this segment there is a point $t_j \in U(p_j; \rho_j) \subset S$ such that $p_i p_j t_j$ subsists and $p_j t_j > \rho_j - \epsilon/6$. Similarly, using t_j, p_i in place of p_i, p_j, there exists $t_i \in U(p_i; \rho_i)$ such that $t_j p_i t_i$ subsists and $p_i t_i > \rho_i - \epsilon/6$. Consequently, $2R \geqq D \geqq t_i t_j = t_i p_i + p_i t_j = t_i p_i + p_i p_j + p_j t_j > p_i p_j + \rho_i + \rho_j - \epsilon/3$. Therefore $\epsilon > p_i p_j$ for $i, j > N_\epsilon$ and by completeness of M there exists $p \in M$, $p = \lim p_n$.

Let $q \in U(p; R)$. For $\epsilon > 0$, $2\epsilon < R - pq$, let $pp_i < \epsilon$, $R - \rho_i < \epsilon$, for some i. Then $p_i q \leqq p_i p + pq < \rho_i$ and $q \in U(p_i; \rho_i)$, and hence $U(p; R) \subset S$.

Now suppose that $q \in S$ and $pq - R = \epsilon > 0$. Since $qp \leqq D < \delta$, we may proceed as before and prolong a segment $S_{q,p}$ through p and finally obtain a point $t \in S$ such that qpt subsists and $pt > R - \epsilon$. Consequently, $qt = qp + pt > 2R \geqq D$, which yields a contradiction. Also $pq \neq R$, $q \in S$, since there exists in M a segment with endpoint p and having q as an interior point. Hence $S = U(p; R)$, completing the proof.

The necessity part of the proof did not require external convexity. However, it may be observed that if M is the unit disk in the Euclidean plane and the complement of S is a closed proper subset of the perimeter containing at least two points, then S satisfies the condition in the theorem but S is not a spherical neighbourhood. Also, the constraint $D < \delta$ cannot in general be relaxed. For if M is the spherical space $S_{2,r}$ with $\delta = \pi r$ and S consists of all of M except for a pair of antipodal points, then S satisfies the condition in the theorem, but S is not a spherical neighbourhood.

3. Equichordal points. In a real normed linear space M, a point $p \in M$ is called an equichordal point of a bounded convex subset S if every algebraic line through p intersects S in a chord of constant positive length, the length being determined from the metric $xy = ||x - y||$. If M has dimension 1, then every point $p \in M$ is an equichordal point of a bounded convex subset S which contains more than one point. Convex sets in E_2, E_3 with equichordal points have been studied by Süss [**9**], Dirac [**4**], Dulmage [**5**], and Wirsing [**10**]. Also, see [**7**] for a critique and other references.

THEOREM 2. *If M is a real normed linear space of dimension greater than* 1, *then a bounded convex subset S of M has at most two equichordal points.*

Proof. We assume that S has three equichordal points. Let F be a 2-flat containing these three points. We translate F to the origin and thereby reduce the theorem to showing that a bounded convex set S^* in a Banach space B_2 cannot have three equichordal points.

Let e_1, e_2, e_3 denote three equichordal points of S^*. Clearly each e_i is an interior point of S^* and any two chords of S^* which contain an equichordal point have the same length. Let a_{ij}, a_{ji} be the boundary points of S^* on the chord through e_i, e_j $(i \neq j)$ such that the four points appear in the order a_{ij}, e_i, e_j, a_{ji}. Let x be any boundary point of S^* distinct from the a_{ij}. The line through x and one of the e_i, say e_2, must intersect the segment with endpoints e_1, e_3 in an interior point. The chords of S^* through e_1 and e_3 parallel to the line $L(x, e_2)$ form opposite sides of a parallelogram P, and it follows that x is an interior point of an adjacent side of P. Thus the only extreme boundary points of S^* are among the points a_{ij}. Therefore, the points e_i are non-collinear and the boundary of S^* consists of the six segments $\overline{a_{12}a_{13}}$, $\overline{a_{13}a_{23}}$, $\overline{a_{23}a_{21}}$, $\overline{a_{21}a_{31}}$, $\overline{a_{31}a_{32}}$, $\overline{a_{32}a_{12}}$.

Since the chord of S^* through e_2, e_3 is equal in length to the parallel chord through e_1, it follows that the boundary segments $\overline{a_{32}a_{12}}$ and $\overline{a_{13}a_{23}}$ lie on parallel lines. Similarly, the segments $\overline{a_{13}a_{23}}$ and $\overline{a_{21}a_{31}}$ lie on parallel lines. Consequently, at least four of the points a_{ij} are collinear. But this is impossible since the points e_i are non-collinear interior points of S^*.

THEOREM 3. *There exist real normed linear spaces of arbitrary dimension greater than one in which there are bounded convex sets with exactly two equichordal points.*

Proof. Let R be a real linear space with inner product (x, y). An example of an inner product space with a Hamel (or vector) basis of given cardinality k is readily constructed from the real-valued functions, over a set of cardinality k, which are zero except at a finite number of points (compare [**8**, p. 95]).

Let R have dimension greater than one and let $p \in R$, $(p, p) = \delta$, $0 < \delta \leqq 1/2$, and set $\alpha = 1 - \delta$. Define the function $||x||_p$ by

$$||x||_p = \frac{(x, x)}{[\alpha(x, x) + (x, p)^2]^{\frac{1}{2}}}, \qquad x \neq \phi,$$

$$||\phi||_p = 0.$$

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We will show that $||x||_p$ is a norm. The properties $||x||_p > 0$, $x \neq \phi$, and $||tx|| = |t|\,||x||_p$ are immediate and we will establish $||x + y||_p \leqq ||x||_p + ||y||_p$ by showing that the set U of $x \in R$ for which $||x||_p \leqq 1$ is convex.

Let $\bar{p} = \delta^{-\frac{1}{2}}p$ and let $z \in R$ be any point such that $(z, z) = 1$, $(z, \bar{p}) = 0$. The condition on the real number pair (λ, μ) such that $\lambda\bar{p} + \mu z \in U$ is either $\lambda = \mu = 0$ or $(\lambda^2 + \mu^2)(\lambda^2 + \alpha\mu^2)^{-\frac{1}{2}} \leqq 1$. Let C be the set of points (λ, μ) in the Cartesian plane satisfying this condition. The boundary curve of C is given in polar coordinates by $r = [1 - \delta \sin^2\theta]^{\frac{1}{2}}$, and by calculating its curvature we see that C is convex for $\delta \leqq 1/2$.

For any $x, y \in R$ it is easily shown that there exist $z_1, z_2 \in R$ such that $(z_i, z_i) = 1$, $(z_i, \bar{p}) = 0$, and $x = \lambda_1\bar{p} + \mu_1 z_1$, $y = \lambda_2\bar{p} + \mu_2 z_2$, where $\mu_i \geqq 0$. If $x, y \in U$, then $(\lambda_i, \mu_i) \in C$ and we will show that $w = (1 - t)x + ty \in U$ for $0 \leqq t \leqq 1$. If $(1 - t)\mu_1 z_1 + t\mu_2 z_2 = \phi$, then $w \in U$ since C is convex and symmetric with respect to the λ-axis. Otherwise, we may write $w = \alpha\bar{p} + \beta z$, where $(z, z) = 1$, $(z, \bar{p}) = 0$, $\alpha = (1 - t)\lambda_1 + t\lambda_2$ and $0 < \beta \leqq (1 - t)\mu_1 + t\mu_2$ by the Schwarz inequality. Therefore, $w \in U$ and $||x||_p$ is a norm.

Let S be the set of $x \in R$ for which $(x, x) \leqq 1$. The set S is convex and if x and y are endpoints of a chord of S through p, then $y - x = t(p - x)$, where $t = 2(1 - (x, p))(1 + \delta - 2(x, p))^{-1}$. It follows that $||y - x||_p = 2$ and p is an equichordal point of S. But $(-p)$ is also an equichordal point of S since $||x||_p = ||x||_{(-p)}$, and by Theorem 2 the proof is complete.

THEOREM 4. *Let M be a real finite-dimensional Banach space. A bounded, open, convex subset S of M is a spherical neighbourhood if and only if S has constant width and possesses an equichordal point.*

Proof. Let $n + 1$ be the dimension of M, d the equichordal length, and w the width of S. We may assume that the origin ϕ is an equichordal point of S and therefore $\phi \in S$. We first observe that there is some chord of S through ϕ such that there exist parallel supporting planes to S at the endpoints of the chord. To prove this statement we carry out our argument in Euclidean space E_{n+1} with unit sphere S_n. First suppose that the boundary of S is smooth. A ray from ϕ in the direction u cuts the boundary of S at a point with outer unit normal vector $f(u)$. The function f gives a continuous mapping of S_n into S_n such that $(u, f(u)) > 0$. This mapping is homotopic to the identity mapping [**11**, p. 809], and therefore has degree 1. Consequently, some pair of antipodal points is mapped into a pair of antipodal points [**11**, p. 810]. The general case is then obtained by approximating S by convex sets with smooth boundaries and applying standard arguments.

Now let x_1 and x_2 be endpoints of a chord of S through ϕ with parallel supporting planes π_1, π_2 at x_1, x_2, respectively. Let $z_i \in \pi_i$, $i = 1, 2$, and let s_1 and s_2 be endpoints of a chord of S through ϕ parallel to the line $L(z_1, z_2)$. Since $w = \text{glb}||z_2 - z_1||$, $z_i \in \pi_i$, and $||z_2 - z_1|| \geqq ||s_2 - s_1|| = ||x_2 - x_1|| = d$, it follows that $w = d$.

Let y_1 and y_2 be the endpoints of any chord of S through ϕ. The parallel chord of $U(\phi; w/2)$ through ϕ has parallel supporting planes at its endpoints. Consequently, the pair of supporting planes to S parallel to these must contain y_1 and y_2. This property implies by a known result [**3**, p. 89] that ϕ is the midpoint of every chord of S through ϕ and it follows that S is a spherical neighbourhood.

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