Interpolated Derivatives

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In a previous paper [Spain, *Proc. Roy. Soc. Edinburgh*, Vol. LX (1940), 134], I have shown that the application of the cardinal function to the problem of interpolating the derivatives yields the result

$$D^n f(x) = \frac{1}{\Gamma(-n)} \int_a^x \frac{f(u)\,du}{(x-u)^{n+1}} + \frac{1}{\Gamma(-n)} \int_{-\infty}^a du\, f(u) \int_1^\infty \frac{t^n e^{-t(x-u)}\,dt}{\Gamma(n+1)}.$$

This formula is valid for $x > a$ (the constant of integration), and $R(n) < 0$. The analytical continuation for $R(n) \geqslant 0$ is indicated in the paper just quoted. The first term is the familiar expression for a fractional derivative, but the second term is not Riemann's complementary function. Furthermore, this result is unsatisfactory because it is impossible to perform the repeated operation of a fractional derivative of a fractional derivative.

The cardinal function interpolation requires the derivatives to be given for both positive and negative values of n. Instead let us try and interpolate by the Gregory-Newton formula for negative values of n. The sth repeated integral of $f(x)$ can be written

$$F(s) = \frac{1}{(s-1)!} \int_a^x (x-u)^{s-1} f(u)\,du.$$

The Gregory-Newton formula is

$$F(n) = \sum_{r=0}^{\infty} \frac{n(n-1)\dots(n-r+1)}{r!} \Delta^r F$$

where

$$\Delta^r F = \sum_{s=0}^{r} (-1)^{r+s} \binom{r}{s} F(b+ws)$$

and $F(n)$ the function to be interpolated is given at the points $b,\ b+w,\ \dots,\ b+ws,\ \dots$. In our problem $w = b = 1$ and so $b+ws = 1+s$ and by substitution we have

$$\begin{aligned} F(n+1) &= \sum_{r=0}^{\infty} \frac{n(n-1)\dots(n-r+1)}{r!} \sum_{s=0}^{r} (-1)^{r+s} \binom{r}{s} \frac{1}{s!} \int_a^x (x-u)^s f(u)\,du \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r n(n-1)\dots(n-r+1)}{r!} \int_a^x du\, f(u) \sum_{s=0}^{r} (-1)^s \binom{r}{s} \frac{(x-u)^s}{s!} \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r n(n-1)\dots(n-r+1)}{r!} \int_a^x f(u) L_r(x-u)\,du \\ &= \int_a^x du\, f(u) \sum_{r=0}^{\infty} \frac{(-1)^r n(n-1)\dots(n-r+1)}{r!} L_r(x-u), \end{aligned}$$

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where $L_r(x)$ is the normalised Laguerre polynomial of degree r (Kaczmarz and Steinhaus, *Theorie der Orthogonalreihen*, 140) defined by

$$L_r(x)=\frac{1}{r!}\,e^x\,\frac{d^r}{dx^r}\,(e^{-x}x^r)$$

with the orthogonal properties

$$\int_0^\infty e^{-x}\,L_r(x)\,L_t(x)\,dx=\begin{cases}0 \text{ if } r\neq t,\\ 1 \text{ if } r=t.\end{cases}$$

The summation we require is readily obtained if we expand x^n for non-integral $n>0$ in a series of Laguerre polynomials. That is, write

$$x^n=\sum_{r=0}^{\infty}\lambda_r\,L_r(x)$$

and by the orthogonal properties of the Laguerre polynomials we have

$$\lambda_r=\int_0^\infty e^{-x}\,x^n\,L_r(x)\,dx=\frac{1}{r!}\int_0^\infty x^n\,\frac{d^r}{dx^r}\,(e^{-x}x^r)\,dx.$$

Since $n>0$, repeated application of integration by parts yields

$$\lambda_r=\frac{(-1)^r n(n-1)\dots(n-r+1)}{r!}\int_0^\infty x^{n-r}(e^{-x}x^r)\,dx$$

$$=\frac{(-1)^r n(n-1)\dots(n-r+1)\,\Gamma(n+1)}{r!}.$$

Thus $$\frac{x^n}{\Gamma(n+1)}=\sum_{r=0}^{\infty}\frac{(-1)^r n(n-1)\dots(n-r+1)}{r!}\,L_r(x)$$

and so $$F(n+1)=\frac{1}{\Gamma(n+1)}\int_a^x (x-u)^n f(u)\,du.$$

That is, $$D^{-n}f(x)=\frac{1}{\Gamma(n)}\int_a^x (x-u)^{n-1}f(u)\,du.$$

We see that the Gregory-Newton interpolation formula does give the familiar generalisation of the fractional derivative.

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