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A THEOREM ON COMPATIBLE N-GROUPS

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A near-ring N is a set N with binary operations $+$ and $\cdot$ satisfying the conditions (1) $(N, +)$ is a group, (2) $(N, \cdot)$ is a semigroup, and (3) $\cdot$ satisfies one of the distributive laws over $+$. $(N, +)$ need not be an abelian group and if the left distributive law holds, i.e. $a \cdot (b+c) = a \cdot b + a \cdot c$ for all $a, b, c \in N$, then N is called a left near-ring. Similarly, the notion of a right near-ring may be defined.

N is said to be zero-symmetric if $0 \cdot n = n \cdot 0 = 0$ for all $n \in N$. The prototype for zero-symmetric near rings is the set $M_0(V)$ of zero preserving maps from the group V to itself with the operations being pointwise addition and composition of mappings. The near-ring $M_0(V)$ is left or right depending upon which side of a mapping one places the argument. Throughout this article, the term near-ring means left near-ring.

For a near-ring N, an N-group is defined to be a group V with a mapping $u: V \times N \to V$ via $(v, n) \to vn$ such that for every $v \in V$ and for all $n, n' \in N$, $v(n+n') = vn + vn'$ and $v(nn') = vn(n')$. N-groups need not be abelian. If V is an N-group for the near-ring N with identity 1, then V is called a unitary N-group provided $v1 = v$ for all $v \in V$. The existence of certain kinds of N-groups for the near-ring N has much to say about the structure of N, and conversely. The main result of this paper forces a structural condition on a particular kind of N-group, called a compatible N-group, under the conditions that N is zero-symmetric with identity and satisfies the minimal condition on right ideals.

Compatible N-groups have been defined in [**3**] and [**4**]. These are just those unitary N-groups V for which, given $v \in V$ and $\alpha \in N$, there exist $\beta \in N$ such that $(v+w)\alpha - v\alpha = w\beta$ for all $w \in V$. There are many classes of near-rings that have compatible N-groups. For example, if V is a group and N is the near-ring generated additively by a semigroup S of endomorphisms of V where S contains the set of inner automorphisms of V, then V is compatible (see Section 6 of [**3**]). Included in this collection are the near-rings $I(V)$, $A(V)$, and $E(V)$, respectively, the near-rings generated additively by the inner automorphisms, the automorphisms, and the endomorphisms of V.

Further definitions and notation are standard and follow Pilz's book [**2**] except for his right near-ring convention. For the remainder of this paper all near-rings will be zero-symmetric and have an identity. Groups will be written additively, but this does not imply commutivity. The following theorem will be proved:

Theorem. *Let V be a compatible N-group. If N has minimal condition on right ideals then there exists a nilpotent normal subgroup P of V such that V/P is finite.*

For the proof of this theorem, some preliminary results and definitions are required. Much of what follows makes use of results in [3].

If V is a group and S a subset of V then the set of all $v \in V$ such that $-v+\sigma+v=\sigma$ for all $\sigma \in S$ is just the group centralizer of S in V and will be denoted by $\mathscr{C}_V(S)$. The (near-ring) centralizer $C_V(U)$ of a submodule U of a compatible N-group V is defined in [3] as all $v \in V$ for which $[vN, U]=\{0\}$. Alternatively, $C_V(U)$ is the sum of all submodules of V contained in $\mathscr{C}_V(U)$. Clearly, $C_V(U)$ is a submodule of V. H/W is a factor of V provided H and W are submodules of V and $H \supseteq W$. H/W is a minimal factor of V if there are no submodules of V between H and W. If H/W is a factor of V then $C_V(H/W)$ is just the submodule $H_1 \supseteq W$ of V for which $H_1/W = C_{V/W}(H/W)$ and $\mathscr{C}_V(H/W)$ may be defined similarly.

The first proposition is a straightforward consequence of the fact that in the compatible situation the near-ring induces the inner automorphisms.

Proposition 1. *If H_1/H_2 and W_1/W_2 are two N-isomorphic factors of a compatible N-group V, then $\mathscr{C}_V(H_1/H_2)=\mathscr{C}_V(W_1/W_2)$ and $C_V(H_1/H_2)=C_V(W_1/W_2)$.*

In (3) the abelian factors of an N-group V are defined as those factors that, as N-groups, are simply ring modules. That is to say that $N/(0\colon V)=A$ is a ring and V is an A-module in the ring sense.

Proposition 2. *If V is a compatible N-group and N has minimal condition on right ideals, then a non-abelian minimal factor H of V is finite.*

Proof. Now H is a 2-tame N-group of type 2 (see Section 6 of [3] for a definition of 2-tame). By Theorem 7.4 of [3], $N/(0\colon H)$ $(=N')$ is either a ring or is isomorphic to $M_0(H)$. If N' is a ring then $H=hN'$ for a non-zero element $h \in H$ and H is a ring module, hence abelian. Therefore, $N' \cong M_0(H)$ and the finiteness of H follows [2, Th. 7.19].

Corollary. *With V, H, and N as in Proposition 2, the index $|V\colon \mathscr{C}_V(H)|$ of $\mathscr{C}_V(H)$ in V is finite.*

Proof. The group action of V on H under conjugation is simply that of $V/\mathscr{C}_V(H)$. As H is finite, $V/\mathscr{C}_V(H)$ is finite.

For the sake of completeness we state the next result due to S. D. Scott as it is crucial to the development.

Theorem 3 (Scott, [3, Theorem 8.9]). *If the following conditions hold:* (a) *N is a near-ring with minimal condition on right ideals,* (b) *V is a faithful compatible N-group,* (c) *U is an abelian minimal N-subgroup of V, and* (d) *$(U\colon V) \nsubseteq (0\colon U)$, then the index of $C_V(U)$ in V is finite.*

Lemma 4. *If V is a compatible N-group, N has minimal condition on right ideals, and U is an abelian minimal submodule of V, then $|V\colon C_V(U)|$ is finite provided V/U has no factors N-isomorphic to U.*

Proof. Clearly we may regard V as being faithful. We shall now show that conditions (a)–(d) of Scott's result hold. In fact we need only check condition (d). If $(U: V) \subseteq (0: U)$ then $V(U: V) \subseteq U$ and so $V(U: V)^2 \subseteq U(0: U) = \{0\}$, and $(U: V)^2 = \{0\}$. Thus $(U: V)$ is nilpotent and $(U: V) \subseteq J(N)$. Now V/U is a faithful compatible $N/(U: V)$ $(=N')$ group and $N/J(N) \cong N'/J(N')$ as $J(N') = J(N)/(U: V)$. However, the minimal factors of V are precisely those of $N/J(N)$ (see [**4**]) the same being true for V/U and $N'/J(N')$. Consequently, V/U must have a factor isomorphic to U. This contradiction yields $(U: V) \nsubseteq (0: U)$. The lemma now follows from Theorem 3.

We now use Lemma 4 to extend the finite index condition to abelian minimal factors of V.

Lemma 5. *If V is a compatible N-group, N has minimal condition on right ideals, and H is an abelian minimal factor of V, then $|V: C_V(H)|$ is finite.*

Proof. Since V is compatible, hence tame, V has a finite socle series [**1**] given by $\{0\} \subset U_1 \subset U_2 \subset \ldots \subset U_k = V$. Let i be the largest integer $1 \leq i \leq k-1$ such that U_{i+1}/U_i has direct summands N-isomorphic to H. Then $U_{i+1}/U_i = W_1 \oplus W_2$ where W_1 and W_2 are submodules of V/U_i and W_1 is N-isomorphic to H. Now $(V/U_i)/W_2$ has $(W_1 \oplus W_2)/W_2 (\cong_N H)$ as a minimal N-subgroup and, by the isomorphism theorem, no factor of $[(V/U_i)/W_2]/[(W_1 \oplus W_2)/W_2]$ is N-isomorphic to H. By Lemma 4 and Proposition 1, $|V: C_V(H)|$ is finite.

Proof of the theorem

The number of N-isomorphism types of minimal factors of V is finite. Thus, the intersection $\bigcap \mathscr{C}_V(H)$ $(=P)$ over all minimal factors H is by Proposition 1 a finite intersection of normal subgroups. As each $\mathscr{C}_V(H) \supseteq C_V(H)$, it follows from the Corollary of Proposition 2, and Lemma 5, that for each H, $|V: \mathscr{C}_V(H)|$ is finite. Thus $|V: P|$ is finite. Let $\{0\} \subset U_1 \subset U_2 \subset \ldots \subset U_k = V$ be a finite sequence of submodules of V the factors of which are direct sums of N-isomorphic minimal submodules. Consider $\{0\} \subseteq U_1 \cap P \subseteq U_2 \cap P \subseteq \ldots \subseteq U_k \cap P = P$. As P centralizes U_1 it centralizes $U_1 \cap P$, i.e., $U_1 \cap P \subseteq Z(P)$. But P centralizes U_2/U_1 and thus $(U_2 \cap P)/(U_1 \cap P) \subseteq Z(P/U_1 \cap P)$ etc. It follows that each factor $(U_i \cap P)/(U_{i-1} \cap P)$ is central and P is nilpotent. The proof is complete.

Corollary. *If all the factors of V are abelian, then by Lemma* 4, *we may take P as a submodule.*

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