ONE PARAMETRIC MOTION IN KLEIN SPACES

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On the euclidean plane one-parametric motion is in general a roulett motion, exceptions being a translation at each instant and a rotation with a fixed center; here we mean by a roulett motion a motion in which a certain curve rolls on another fixed curve without slipping. In this paper we extend this fact to the case of Klein spaces and investigate in detail especially the cases of the euclidean space and the projective space. We repeatedly refer to

1. E. Cartan: La théorie des groupes finis et continus et la géométrie différentielle.

1. Let the fundamental Lie-group and its element be respectively G and S_a, and let the fundamental frame be R. Take the frames S_aTR and S_bTR, which are relatively fixed to S_aR and S_bR respectively. Then the transformation which shifts S_aTR to S_bTR is $(S_aT)^{-1}S_bT = T^{-1}S_a^{-1}S_bT$. If $T^{-1}S_a^{-1}S_bT$ is an element of a certain subgroup H of G, the set of frames HR is invariant by the motion $T^{-1}S_a^{-1}S_bT$. This HR can be considered as an element of the homogeneous space that is generated by the cosets of H. This process is a generalization of the fact that the motion on the euclidean plane is either a translation or a rotation.

Now we consider a one-parametric motion S_a in a Klein space, which we assume to be differentiable to the order we need. Put $\delta S = S_a^{-1}S_{a+da}$ and let T_a be a transformation such that $T_a^{-1}\delta ST_a$ belongs to a fixed subgroup H of G, which is independent of a. We call S_aT_aR the *instantaneous center* of this motion at an instant a. Then for the relative displacement $\delta(ST)$ of the instantaneous center we have

$$\delta(ST) = (S_aT_a)^{-1}(S_{a+da}T_{a+da}) = T_a^{-1}\cdot\delta S\cdot T_a\cdot\delta T, \tag{1}$$

where we put $\delta T = T_a^{-1}T_{a+da}$. T_aR is the position of the instantaneous center relative to the moving frame S_aR, and δT is the relative displacement of the instantaneous center on the moving space. Then we consider the relative components $\omega_i(a, da)$ of $\delta S = S_a^{-1}S_{a+da}$. As the relative components of the product of two infinitesimal motions are the sum of those of each motion, we get from (1) the result that the relative components of $\delta(ST)$ are equal to the sum of those of $T_a^{-1}\cdot\delta S\cdot T_a$ and δT. If we take the homogeneous space with cosets of

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H as its elements and consider the principal relative components of this homogeneous space, then the principal relative components of $\delta(ST)$ are equal to those of δT. $S_a T_a R$ generate in the fixed space a one-parametric figure C_1 (basic figure) with cosets of H as its elements, and similarly $T_a R$ generate a one-parametric figure C_0 (rolling figure) in the moving space. From the above consideration we get:

THEOREM. Let S_a be a one-parametric motion, for which there exists such T_a that $T_a^{-1}\cdot\delta S\cdot T_a$ belongs to a fixed subgroup H of the fundamental group G. Then the motion S_a can be interpreted as the motion which is realized by the rolling of a certain figure C_0 along another figure C_1 without slipping. Here C_0 and C_1 are figures generated by the one-parametric set of cosets of H, and the rolling without slipping means that the principal components of C_0 and C_1 coinside at every instant with certain choice of frames attached to each elements of C_0 and C_1.

2. Spherical Motion

Let the orthogonal axes be $I_1, I_2, \ldots, I_n$, and their relative displacement be $dI_i = \sum_j \omega_{ij} I_j$ $(\omega_{ij} + \omega_{ji} = 0)$ which we write $dI = \Omega I$. By the coordinate transformation $\bar{I} = PI$ (P, orthogonal), Ω is transformed to $\bar{\Omega} = P\Omega P'$. If we take P suitably, we have

$$\bar{\Omega} = 0 \dotplus 0 \dotplus \ldots \dotplus 0 \dotplus \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} \dotplus \ldots \dotplus \begin{pmatrix} 0 & \lambda_k \\ -\lambda_k & 0 \end{pmatrix}. \tag{2}$$

So we may take as the subgroup H the one which consists of the transformations

$$1 \dotplus 1 \dotplus \ldots \dotplus 1 \dotplus \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \dotplus \ldots \dotplus \begin{pmatrix} \cos\theta_k & \sin\theta_k \\ -\sin\theta_k & \cos\theta_k \end{pmatrix},$$

and the instantaneous center is the set of points

$$(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1, 0, \ldots, 0),$$

and several two dimensional planes. The one-parametric spherical motion is the one by which a one-parametric figure consisting of the above elements rolls on another figure of the same kind.

As an example we take the case $n = 4$. Then an instantaneous center is a two-dimensional Grassmann manifold and if we take along the one-parametric set of these elements the frame I_1, I_2, I_3, I_4 suitably and put $dI_i = \sum_j \alpha_{ij} I_j$, then the principal components are $\alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}$, their invariants of 1st order being $\alpha_{12}^2 + \alpha_{14}^2 + \alpha_{23}^2 + \alpha_{24}^2$, and $\alpha_{13}\alpha_{24} - \alpha_{23}\alpha_{14}$. For the basic figure and the rolling figure these two invariants must be the same.

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3. Euclidean Motion

Let the vertex of the frame be A, the axes $I_1, \ldots, I_n$, and let their relative instantaneous motion be $dA = \sum \omega_i I_i$, $dI = \Omega I$. If, by the frame transformation $\bar{A} = A + \sum x_i I_i$ and $\bar{I} = PI$, ω_i and $\Omega = (\omega_{ij})$ are transformed respectively to $\bar{\omega}_i$ and $\bar{\Omega} = (\bar{\omega}_{ij})$, we have

$$\omega + x\Omega = \bar{\omega}P, \quad \bar{\Omega}P = P\Omega,$$

where $\omega = (\omega_1, \ldots, \omega_n)$, $\bar{\omega} = (\bar{\omega}_1, \ldots, \bar{\omega}_n)$ and $x = (x_1, \ldots, x_n)$. Now we take an orthogonal matrix P such that $\bar{\Omega}$ is of the form (2). When $|\Omega| \neq 0$, we can take $x_1, \ldots, x_n$ such that $\bar{\omega}_1 = 0, \ldots, \bar{\omega}_n = 0$. When $|\Omega| = 0$, we take $\bar{\omega}$ such that (3) can be solved for the unknown x_i. We perform the above consideration for the case $n = 3$. In this case (2) is of the form $\bar{\Omega} = 0 \dotplus \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$. When $\Omega = 0$, our motion is a translation at every instant. Hereafter we treat the case $\Omega \neq 0$, namely $\lambda \neq 0$. Owing to the speciality of dimensions 3, we have by the second equation of (3) $(\Omega = (\omega_{ij}),\ P = (p_{ij}))$,

$$\omega_{23} = \lambda p_{11}, \quad \omega_{31} = \lambda p_{12}, \quad \omega_{12} = \lambda p_{13}, \tag{4}$$

$$\lambda^2 = \omega_{23}^2 + \omega_{31}^2 + \omega_{12}^2. \tag{5}$$

The first equation of (3) is in this case

$$\omega_i + \sum_{j=1}^{3} x_j \omega_{ji} = \sum_{j=1}^{3} p_{ji} \bar{\omega}_j \quad (i = 1, 2, 3), \tag{6}$$

which by (4) gives rise to

$$\omega_1 \omega_{23} + \omega_2 \omega_{31} + \omega_3 \omega_{12}^2 = \lambda \bar{\omega}_1. \tag{7}$$

This is the only condition that (6) should be consistent. So if we take x_1, x_2, x_3 suitably we have $\bar{\omega}_2 = 0$, $\bar{\omega}_3 = 0$ except (7).

The line through (x_1, x_2, x_3) with the direction (p_{11}, p_{12}, p_{13}) is the instantaneous center, while the basic figure C_1 and the rolling figure C_0 are ruled surfaces. If we take along the basic ruled surface the cartesian frame with I_1 on the generic line, we have

$$dA = \sum \alpha_i I_i, \quad dI_i = \sum \alpha_{ij} I_j, \tag{8}$$

where $\alpha_2, \alpha_3, \alpha_{12}, \alpha_{13}$ are principal components. When we transform this to the Frenet's frame, we have ([1], p. 51)

$$dA = d\sigma(\alpha I_1 + kI_3), \quad dI_1 = d\sigma \cdot I_2, \quad dI_2 = d\sigma(-I_1 + \beta I_3), \quad dI_3 = d\sigma(-\beta I_2). \tag{9}$$

The relative components of $T_a^{-1} \cdot \delta S \cdot T_a$ is $\bar{\omega}_1$ and λ, and we have

$$dA = \bar{\omega}_1 I_1, \quad dI_1 = 0, \quad dI_2 = \lambda I_3, \quad dI_3 = -\lambda I_2. \tag{10}$$

So for the basic ruled surface and the rolling one, $d\sigma$ and $kd\sigma$ are the same, while α and β are in general different.

If $\omega_1 \omega_{23} + \omega_2 \omega_{31} + \omega_3 \omega_{13} = 0$, then by (7) $\bar{\omega}_1 = 0$, and by (10) there is no

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slipping on the generic lines of the two surfaces, α being the same, which shows that the lines of striction of the two surfaces coinside. When the two surfaces are developable, namely $k=0$, we have, from (9), $dA = \alpha d\sigma \cdot I_1$, $dI_1 = d\sigma \cdot I_2$ and α is the radius of curvature of the edge of regression of the surface. So in the case of developable surfaces the condition $\bar{\omega}_1 = 0$ reduces to the coincidence of the radius of curvature of the edge of regression. A spherical motion can be characterised by the relation $k=0$ and $\alpha=0$.

Now we represent the invariants of our basic ruled surface by the relative component of our motion. We write (p_1, p_2, p_3) in place of (p_{11}, p_{12}, p_{13}), which are by (4)

(11) $$p_1 = \omega_{23}/\lambda, \quad p_2 = \omega_{31}/\lambda, \quad p_3 = \omega_{12}/\lambda \quad (\lambda^2 = \omega_{23}^2 + \omega_{31}^2 + \omega_{12}^2).$$

By (7) we have

(12) $$\bar{\omega}_1 = (1/\lambda)(\omega_1\omega_{23} + \omega_2\omega_{31} + \omega_3\omega_{12}), \quad \bar{\omega}_2 = 0, \quad \bar{\omega}_3 = 0.$$

Hence (6) can be written as

(13) $$r_1 - p_3x_2 + p_2x_3 = 0, \quad r_2 + p_3x_1 - p_1x_3 = 0, \quad r_3 - p_2x_1 + p_1x_2 = 0,$$

where we put

(14) $$r_i = (1/\lambda)(\omega_i - p_i\bar{\omega}_1) \quad (i = 1, 2, 3).$$

Then by the consideration of (8), (9), (11), (13) and the process [1] p. 49, 50 we get the following: (calculation being omitted,)

$$d\sigma^2 = \alpha_{12}^2 + \alpha_{13}^2 = dp_1^2 + dp_2^2 + dp_3^2,$$

$$k d\sigma^2 = \alpha_2\alpha_{13} - \alpha_3\alpha_{12} = \begin{vmatrix} p_1 & dp_1 & dx_1 \\ p_2 & dp_2 & dx_2 \\ p_3 & dp_3 & dx_3 \end{vmatrix} = dp_1dr_1 + dp_2dr_2 + dp_3dr_3,$$

$$\alpha d\sigma = \alpha_1 - d\left(\frac{\alpha_2\alpha_{12} + \alpha_3\alpha_{13}}{\alpha_{12}^2 + \alpha_{13}^2}\right) = p_1dx_1 + p_2dx_2 + p_3dx_3 - d\left(\frac{dx_1dp_1 + dx_2dp_2 + dx_3dp_3}{dp_1^2 + dp_2^2 + dp_3^2}\right)$$

$$= -\begin{vmatrix} dp_1 & p_1 & r_1 \\ dp_2 & p_2 & r_2 \\ dp_3 & p_3 & r_3 \end{vmatrix} - d\left(\begin{vmatrix} dp_1 & p_1 & dr_1 \\ dp_2 & p_2 & dr_2 \\ dp_3 & p_3 & dr_3 \end{vmatrix} \Big/ (dp_1^2 + dp_2^2 + dp_3^2)\right),$$

$$\beta d\sigma = \alpha_{23} + d\tan^{-1}\frac{\alpha_{13}}{\alpha_{12}} = \begin{vmatrix} p_1 & dp_1 & d^2p_1 \\ p_2 & dp_2 & d^2p_2 \\ p_3 & dp_3 & d^2p_3 \end{vmatrix} \Big/ (dp_1^2 + dp_2^2 + dp_3^2).$$

4. Projective Motion (complex number field)

Let the frame be $A_0, A_1, \ldots, A_n$ and the instantaneous relative displacement be $dA_i = \sum_{j=0}^{n} \omega_{ij}A_j$ $(i = 0, \ldots, n)$, which we write briefly $dA = \Omega A$. By the frame transformation $\bar{A} = TA$ Ω is transformed to $\bar{\Omega}$, which is given by $\bar{\Omega} = T\Omega T^{-1}$. Here we assume that Ω has only simple elementary divisors. Then for suitably chosen T, we have $\bar{\Omega} = \alpha_0 \dot{+} \alpha_1 \dot{+} \ldots \dot{+} \alpha_n$, and our instantaneous center is the set of $n+1$ points $A_0, \ldots, A_n$. For the one-parametric figure of these sets of $n+1$ points, we attach the frame $A_0 A_1 \ldots A_n$, for which $dA_i = \sum \alpha_{ij} A_j$. When $\alpha_{i0} \neq 0$, $\alpha_{0i} \neq 0$ $(i \neq 0)$, we see that the complete system of invariants is the following,

$$\alpha_{0i}\alpha_{i0} \ (i \neq 0), \quad \alpha_{0i}\alpha_{j0}\alpha_{ij} \ (i \neq j \neq 0). \tag{15}$$

We omit here the details. Our motion is the one, by which two figures of one-parametric set of $n+1$ points, for which (15) are the same, are basic and rolling figures.

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