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ON WITTEN MULTIPLE ZETA-FUNCTIONS ASSOCIATED WITH SEMI-SIMPLE LIE ALGEBRAS V

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Abstract. We study the values of the zeta-function of the root system of type G_2 at positive integer points. In our previous work we considered the case when all integers are even, but in the present paper we prove several theorems which include the situation when some of the integers are odd. The underlying reason why we may treat such cases, including odd integers, is also discussed.

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1. Introduction. Let $\mathbb{N}$ be the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the rational number field, $\mathbb{R}$ the real number field and $\mathbb{C}$ the complex number field.

The present paper is the continuation of our series of papers [7, 10, 11, 15] (and also [5,6,8,9]), in which we have developed the theory of zeta-functions of root systems. Motivated by the work of Witten [23] in quantum gauge theory, Zagier [24] defined the Witten zeta-function as

$$\zeta_W(s, \mathfrak{g}) = \sum_{\varphi} (\dim \varphi)^{-s}$$

associated with any complex semi-simple Lie algebra $\mathfrak{g}$, where the sum runs over all finite-dimensional irreducible representations φ of $\mathfrak{g}$. The notion of zeta-functions of root systems was introduced as a multi-variable generalisation of the Witten zeta-functions. We will give the rigorous definition of the zeta-function of the root system Δ in the next section, which we will denote by $\zeta_r(\mathbf{s}, \mathbf{y}; \Delta)$, where r is the rank of Δ. By Weyl's dimension formula, it is possible to obtain the explicit form of $\zeta_r(\mathbf{s}, \mathbf{y}; \Delta)$. For example, when the root system is of type G_2 and $\mathbf{y} = \mathbf{0}$, then $r = 2$, $\mathbf{s} = (s_1, s_2, s_3, s_4, s_5, s_6) \in \mathbb{C}^6$

and

$$\begin{aligned}\zeta_2(\mathbf{s}; G_2) &= \zeta_2(\mathbf{s}, \mathbf{0}; G_2)\\ &= \sum_{m,n\geq 1} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3} (m+2n)^{s_4} (m+3n)^{s_5} (2m+3n)^{s_6}}. \end{aligned} \tag{1.1}$$

In our former papers, besides the general theory, we studied some individual cases of low rank. Zeta-functions of root systems of type A_r ($r = 2, 3$) were studied in [5, 10, 15], and those of type B_r, C_r ($r = 2, 3$) were studied in [7, 9, 10]. Then in [11], the zeta-function of the root system of type G_2, the simplest exceptional algebra, was discussed.

The main topic in [11] is the situation when $s_1, \ldots, s_6$ are positive even integers. From our general result given in [10, Theorem 8], it is possible to show that

$$\zeta_2(2a, 2b, 2b, 2b, 2a, 2a; G_2) \in \mathbb{Q} \cdot \pi^{6(a+b)} \qquad (a, b \in \mathbb{N}). \tag{1.2}$$

Moreover, the rational coefficients can be determined explicitly. In [11], using the idea developed in [9], we proved certain functional relations [11, Theorem 5.1] which include (1.2) as special cases.

However, it is possible to treat the case when some of $s_1, \ldots, s_6$ are odd integers. Zhao [26] expressed the values $\zeta_2(\mathbf{k}; G_2)$ for $\mathbf{k} \in \mathbb{N}_0^6$ (under certain conditions) in terms of double polylogarithms. Using his formula, Zhao calculated numerically some of those special values, for example

$$\zeta_2(2, 1, 1, 1, 1, 1; G_2) = 0.0099527234\cdots. \tag{1.3}$$

The parity result for $\zeta_2(\mathbf{k}; G_2)$ in some extended sense has been shown by Okamoto [19]. We will discuss his result more closely in the last section of the present paper.

In the present paper, we also study the situation when some of $s_1, \ldots, s_6$ are odd integers. In Section 2, after preparing the basic notations and definitions, we will prove a general theorem (Theorem 2.1), which gives the underlying reason why sometimes it is possible to evaluate the values of multiple zeta-functions at odd integer points. In Section 3, we will apply Theorem 2.1 to $\zeta_2(\mathbf{s}; G_2)$. Sections 4 to 6 are devoted to the proof of functional relations among $\zeta_2(\mathbf{s}; G_2)$, the Riemann zeta-function $\zeta(s)$ and a certain Dirichlet L-function. Those functional relations especially imply explicit evaluations of special values of $\zeta_2(\mathbf{s}; G_2)$, such as

$$\zeta_2(2, 1, 1, 1, 1, 1; G_2) = \frac{1}{18}\zeta(2)\zeta(5) - \frac{109}{1296}\zeta(7) \tag{1.4}$$

(see Example 6.4), which we announced in [11]. Our result (1.4) agrees with Zhao's [**26**] numerical value (1.3).

2. A general formula.

We use the same notation as in [5, 7, 10, 11]. We first recall several basic definitions and facts about root systems and Weyl groups (for details, see [2–4]).

Let V be an r-dimensional real vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. The norm $\|\cdot\|$ is defined by $\|v\| = \langle v, v \rangle^{1/2}$. The dual space V^* is identified with V via the inner product of V. Let Δ be a finite reduced root system in V, which may not be

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irreducible, and $\Psi = \{\alpha_1, \ldots, \alpha_r\}$ its fundamental system. We fix Δ_+ and Δ_- as the set of all positive roots and negative roots respectively. Then we have a decomposition of the root system $\Delta = \Delta_+ \coprod \Delta_-$. Let $Q = Q(\Delta)$ be the root lattice, $Q^\vee$ the coroot lattice, $P = P(\Delta)$ the weight lattice, $P^\vee$ the coweight lattice, P_+ the set of integral dominant weights and P_{++} be the set of integral strongly dominant weights respectively defined by

$$Q = \bigoplus_{i=1}^{r} \mathbb{Z}\,\alpha_i, \qquad Q^\vee = \bigoplus_{i=1}^{r} \mathbb{Z}\,\alpha_i^\vee, \tag{2.1}$$

$$P = \bigoplus_{i=1}^{r} \mathbb{Z}\,\lambda_i, \qquad P^\vee = \bigoplus_{i=1}^{r} \mathbb{Z}\,\lambda_i^\vee, \tag{2.2}$$

$$P_+ = \bigoplus_{i=1}^{r} \mathbb{N}_0\,\lambda_i, \qquad P_{++} = \bigoplus_{i=1}^{r} \mathbb{N}\,\lambda_i, \tag{2.3}$$

where the fundamental weights $\{\lambda_j\}_{j=1}^r$ and the fundamental coweights $\{\lambda_j^\vee\}_{j=1}^r$ are the dual bases of $\Psi^\vee$ and Ψ satisfying $\langle\alpha_i^\vee, \lambda_j\rangle = \delta_{ij}$ and $\langle\lambda_i^\vee, \alpha_j\rangle = \delta_{ij}$ respectively. Let

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+} \alpha = \sum_{j=1}^{r} \lambda_j \tag{2.4}$$

be the lowest strongly dominant weight. Then $P_{++} = P_+ + \rho$.

Let σ_α be the reflection with respect to a root $\alpha \in \Delta$ defined as

$$\sigma_\alpha : V \to V, \qquad \sigma_\alpha : v \mapsto v - \langle\alpha^\vee, v\rangle\alpha. \tag{2.5}$$

For a subset $A \subset \Delta$, let $W(A)$ be the group generated by reflections σ_α for all $\alpha \in A$. In particular, $W = W(\Delta)$ is the Weyl group, and $\{\sigma_j (= \sigma_{\alpha_j}) \mid 1 \le j \le r\}$ generates W. For $w \in W$, denote $\Delta_w = \Delta_+ \cap w^{-1}\Delta_-$.

Let $\mathrm{Aut}(\Delta)$ be the subgroup of all the automorphisms $\mathrm{GL}(V)$ which stabilises Δ. Then the Weyl group W is a normal subgroup of $\mathrm{Aut}(\Delta)$ and there exists a subgroup $\Omega \subset \mathrm{Aut}(\Delta)$ such that $\mathrm{Aut}(\Delta) = \Omega \ltimes W$. The subgroup Ω is isomorphic to the group $\mathrm{Aut}(\Gamma)$ of automorphisms of the Dynkin diagram Γ (see [3, Section 12.2]).

Now we can define the zeta-function of the root system Δ. For $\mathbf{s} = (s_\alpha)_{\alpha\in\Delta_+} \in \mathbb{C}^{|\Delta_+|}$ and $\mathbf{y} \in V$, it is defined by

$$\zeta_r(\mathbf{s}, \mathbf{y}; \Delta) = \sum_{\lambda\in P_{++}} e^{2\pi\sqrt{-1}\langle\mathbf{y},\lambda\rangle} \prod_{\alpha\in\Delta_+} \frac{1}{\langle\alpha^\vee, \lambda\rangle^{s_\alpha}}. \tag{2.6}$$

Let

$$\mathcal{S} = \{\mathbf{s} = (s_\alpha) \in \mathbb{C}^{|\Delta_+|} \mid \Re s_\alpha > 1 \text{ for } \alpha \in \Delta_+\}.$$

Then $\zeta_r(\mathbf{s}, \mathbf{y}; \Delta)$ is absolutely convergent in the region $\mathcal{S}$ and is holomorphic there [10, Lemma 9].

Next, let $I \subset \{1, \ldots, r\}$ with $I \neq \emptyset$, and define a certain linear combination $S(\mathbf{s}, \mathbf{y}; I; \Delta)$ of the zeta-function associated with I. Let $\Psi_I = \{\alpha_i \mid i \in I\} \subset \Psi$, and let V_I be the linear subspace of V spanned by Ψ_I. Then $\Delta_I = \Delta \cap V_I$ is a root system in V_I

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whose fundamental system is Ψ_I. For the root system Δ_I, we denote the corresponding coroot lattice by $Q_I^\vee = \bigoplus_{i\in I}\mathbb{Z}\,\alpha_i^\vee$. Let

$$P_I = \bigoplus_{i\in I}\mathbb{Z}\,\lambda_i, \tag{2.7}$$

$$P_{I+} = \bigoplus_{i\in I}\mathbb{N}_0\,\lambda_i. \tag{2.8}$$

The natural embedding $\iota : Q_I^\vee \to Q^\vee$ induces the projection $\iota^* : P \to P_I$. Namely for $\lambda \in P$, $\iota^*(\lambda)$ is defined as a unique element of P_I satisfying $\langle \iota(q), \lambda\rangle = \langle q, \iota^*(\lambda)\rangle$ for all $q \in Q_I^\vee$. Let W_I be the subgroup of W generated by all the reflections associated with the elements in Ψ_I, and

$$W^I = \{w \in W \mid \Delta_{I+}^\vee \subset w\Delta_+^\vee\}. \tag{2.9}$$

For $\mathbf{s} = (s_\alpha)_{\alpha\in\Delta_+} \in \mathbb{C}^{|\Delta_+|}$, we define an action of $\mathrm{Aut}(\Delta)$ by

$$(w\mathbf{s})_\alpha = s_{w^{-1}\alpha}, \tag{2.10}$$

where we have set $s_{-\alpha} = s_\alpha$. Now define

$$S(\mathbf{s}, \mathbf{y}; I; \Delta) = \sum_{\lambda\in\iota^{*-1}(P_{I+})\setminus H_{\Delta^\vee}} e^{2\pi\sqrt{-1}\langle \mathbf{y},\lambda\rangle} \prod_{\alpha\in\Delta_+} \frac{1}{\langle\alpha^\vee, \lambda\rangle^{s_\alpha}}, \tag{2.11}$$

where $H_{\Delta^\vee} = \{v \in V \mid \langle\alpha^\vee, v\rangle = 0 \text{ for some } \alpha \in \Delta\}$ is the set of all walls of the Weyl chambers. By [10, Theorem 5], for $\mathbf{s} \in \mathcal{S}$ and $\mathbf{y} \in V$, we have

$$S(\mathbf{s}, \mathbf{y}; I; \Delta) = \sum_{v\in W^I} \Big(\prod_{\alpha\in\Delta_{v^{-1}}} (-1)^{-s_\alpha}\Big) \zeta_r(v^{-1}\mathbf{s}, v^{-1}\mathbf{y}; \Delta). \tag{2.12}$$

The following theorem implies that under certain conditions, the number of terms on the right-hand side of (2.12) can be reduced.

THEOREM 2.1. *Assume that there exist* $w_1 \in \mathrm{Aut}(\Delta)$*,* $\mathbf{s} \in \mathcal{S}$ *and* $\mathbf{y} \in V$ *which satisfy the conditions that* $s_\alpha \in \mathbb{Z}$ *for* $\alpha \in \Delta_{w_1^{-1}}$*,*

$$\Big(\prod_{\alpha\in\Delta_{w_1^{-1}}} (-1)^{s_\alpha}\Big) = -1 \tag{2.13}$$

and

$$w_1^{-1}\mathbf{s} = \mathbf{s}, \qquad w_1^{-1}\mathbf{y} = \mathbf{y}. \tag{2.14}$$

Then we have

$$\begin{aligned} S(\mathbf{s}, \mathbf{y}; I; \Delta) = \frac{1}{2}\Big(&\sum_{v\in W^I\setminus w_1 W^I} \Big(\prod_{\alpha\in\Delta_{v^{-1}}} (-1)^{-s_\alpha}\Big)\zeta_r(v^{-1}\mathbf{s}, v^{-1}\mathbf{y}; \Delta) \\ &+ \sum_{v\in W^I\setminus w_1^{-1} W^I} \Big(\prod_{\alpha\in\Delta_{v^{-1}}} (-1)^{-s_\alpha}\Big)\zeta_r(v^{-1}\mathbf{s}, v^{-1}\mathbf{y}; \Delta)\Big). \end{aligned} \tag{2.15}$$

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Furthermore, if $w_1^{-1}W^I = w_1 W^I$*, then*

$$S(\mathbf{s}, \mathbf{y}; I; \Delta) = \sum_{v \in W^I \setminus w_1 W^I} \Bigl(\prod_{\alpha \in \Delta_{v^{-1}}} (-1)^{-s_\alpha}\Bigr) \zeta_r(v^{-1}\mathbf{s}, v^{-1}\mathbf{y}; \Delta). \tag{2.16}$$

Proof. We first prove that

$$\begin{aligned} &\Bigl(\prod_{\alpha \in \Delta_{w^{-1}}} (-1)^{s_\alpha}\Bigr) S(w^{-1}\mathbf{s}, w^{-1}\mathbf{y}; I; \Delta) \\ &\quad = \sum_{v \in wW^I \cap W^I} \Bigl(\prod_{\alpha \in \Delta_{v^{-1}}} (-1)^{-s_\alpha}\Bigr) \zeta_r(v^{-1}\mathbf{s}, v^{-1}\mathbf{y}; \Delta) \\ &\qquad + \Bigl(\prod_{\alpha \in \Delta_{w^{-1}}} (-1)^{s_\alpha}\Bigr) \sum_{v \in wW^I \setminus W^I} \Bigl(\prod_{\alpha \in \Delta_{v^{-1}w}} (-1)^{-s_{w\alpha}}\Bigr) \zeta_r(v^{-1}\mathbf{s}, v^{-1}\mathbf{y}; \Delta) \end{aligned} \tag{2.17}$$

for any $w \in \mathrm{Aut}(\Delta)$. Since $\iota^{*-1}(P_{I+}) = \bigcup_{v \in W^I} vP_+$ by [10, Lemma 2], we can write the left-hand side of (2.17) as

$$\Bigl(\prod_{\alpha \in \Delta_{w^{-1}}} (-1)^{s_\alpha}\Bigr) \sum_{u \in W^I} \sum_{\lambda \in uP_{++}} e^{2\pi\sqrt{-1}\langle w^{-1}\mathbf{y}, \lambda\rangle} \prod_{\alpha \in \Delta_+} \frac{1}{\langle \alpha^\vee, \lambda\rangle^{s_{w\alpha}}}. \tag{2.18}$$

The inner sum of (2.18) is equal to

$$\sum_{\lambda \in uP_{++}} e^{2\pi\sqrt{-1}\langle \mathbf{y}, w\lambda\rangle} \prod_{\alpha \in \Delta_+} \frac{1}{\langle (w\alpha)^\vee, w\lambda\rangle^{s_{w\alpha}}} = \sum_{\lambda \in wuP_{++}} e^{2\pi\sqrt{-1}\langle \mathbf{y}, \lambda\rangle} \prod_{\alpha \in w\Delta_+} \frac{1}{\langle \alpha^\vee, \lambda\rangle^{s_\alpha}},$$

and concerning the last product, since $w\Delta_+ \cap \Delta_- = w\Delta_w = -\Delta_{w^{-1}}$, we have

$$\prod_{\alpha \in w\Delta_+} \frac{1}{\langle \alpha^\vee, \lambda\rangle^{s_\alpha}} = \Bigl(\prod_{\alpha \in \Delta_{w^{-1}}} (-1)^{-s_\alpha}\Bigr) \prod_{\alpha \in \Delta_+} \frac{1}{\langle \alpha^\vee, \lambda\rangle^{s_\alpha}}. \tag{2.19}$$

Using this expression when $wu \in W^I$, we see that (2.18) is equal to

$$\begin{aligned} &\sum_{\substack{u \in W^I \\ wu \in W^I}} \sum_{\lambda \in wuP_{++}} e^{2\pi\sqrt{-1}\langle \mathbf{y}, \lambda\rangle} \prod_{\alpha \in \Delta_+} \frac{1}{\langle \alpha^\vee, \lambda\rangle^{s_\alpha}} \\ &\quad + \Bigl(\prod_{\alpha \in \Delta_{w^{-1}}} (-1)^{s_\alpha}\Bigr) \sum_{\substack{u \in W^I \\ wu \notin W^I}} \sum_{\lambda \in uP_{++}} e^{2\pi\sqrt{-1}\langle w^{-1}\mathbf{y}, \lambda\rangle} \prod_{\alpha \in \Delta_+} \frac{1}{\langle \alpha^\vee, \lambda\rangle^{s_{w\alpha}}} \\ &= \Sigma_1 + \Bigl(\prod_{\alpha \in \Delta_{w^{-1}}} (-1)^{s_\alpha}\Bigr) \Sigma_2, \end{aligned} \tag{2.20}$$

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say. Putting $wu = v$, we have

$$\begin{aligned}\Sigma_1 &= \sum_{v\in wW^I\cap W^I}\sum_{\lambda\in vP_{++}} e^{2\pi\sqrt{-1}\langle \mathbf{y},\lambda\rangle}\prod_{\alpha\in\Delta_+}\frac{1}{\langle\alpha^\vee,\lambda\rangle^{s_\alpha}}\\ &= \sum_{v\in wW^I\cap W^I}\sum_{\lambda\in P_{++}} e^{2\pi\sqrt{-1}\langle \mathbf{y},v\lambda\rangle}\prod_{\alpha\in\Delta_+}\frac{1}{\langle\alpha^\vee,v\lambda\rangle^{s_\alpha}}\\ &= \sum_{v\in wW^I\cap W^I}\sum_{\lambda\in P_{++}} e^{2\pi\sqrt{-1}\langle v^{-1}\mathbf{y},\lambda\rangle}\prod_{\alpha\in v^{-1}\Delta_+}\frac{1}{\langle\alpha^\vee,\lambda\rangle^{s_{v\alpha}}},\end{aligned}$$

and, as in (2.19), the signature appears from the last product when $\alpha\in v^{-1}\Delta_+\cap\Delta_- = -\Delta_v$, which is

$$\prod_{\alpha\in\Delta_v}(-1)^{-s_{v\alpha}} = \prod_{\alpha\in\Delta_{v^{-1}}}(-1)^{-s_\alpha}$$

because $v\Delta_v = -\Delta_{v^{-1}}$. Therefore, we obtain

$$\begin{aligned}\Sigma_1 &= \sum_{v\in wW^I\cap W^I}\Bigl(\prod_{\alpha\in\Delta_{v^{-1}}}(-1)^{-s_\alpha}\Bigr)\sum_{\lambda\in P_{++}} e^{2\pi\sqrt{-1}\langle v^{-1}\mathbf{y},\lambda\rangle}\prod_{\alpha\in\Delta_+}\frac{1}{\langle\alpha^\vee,\lambda\rangle^{s_{v\alpha}}}\\ &= \sum_{v\in wW^I\cap W^I}\Bigl(\prod_{\alpha\in\Delta_{v^{-1}}}(-1)^{-s_\alpha}\Bigr)\zeta_r(v^{-1}\mathbf{s},v^{-1}\mathbf{y};\Delta).\end{aligned}\tag{2.21}$$

Similarly, we can show

$$\Sigma_2 = \sum_{v\in wW^I\setminus W^I}\Bigl(\prod_{\alpha\in\Delta_{v^{-1}w}}(-1)^{-s_{w\alpha}}\Bigr)\zeta_r(v^{-1}\mathbf{s},v^{-1}\mathbf{y};\Delta).\tag{2.22}$$

Substituting (2.21) and (2.22) into (2.20), we obtain (2.17).

Now consider the equation

$$2S(\mathbf{s},\mathbf{y};I;\Delta) = S(\mathbf{s},\mathbf{y};I;\Delta) - \Bigl(\prod_{\alpha\in\Delta_{w_1^{-1}}}(-1)^{s_\alpha}\Bigr)S(w_1^{-1}\mathbf{s},w_1^{-1}\mathbf{y};I;\Delta),\tag{2.23}$$

which trivially follows from the assumptions (2.13) and (2.14). Substitute the expansions (2.12) and (2.17) (with $w = w_1$) to the right-hand side of (2.23). The first sum on the right-hand side of (2.17) is cancelled with the part $v\in w_1W^I\cap W^I$ of (2.12), and hence

$$\begin{aligned}2S(\mathbf{s},\mathbf{y};I;\Delta) &= \sum_{v\in W^I\setminus wW^I}\Bigl(\prod_{\alpha\in\Delta_{v^{-1}}}(-1)^{-s_\alpha}\Bigr)\zeta_r(v^{-1}\mathbf{s},v^{-1}\mathbf{y};\Delta)\\ &\quad-\Bigl(\prod_{\alpha\in\Delta_{w^{-1}}}(-1)^{s_\alpha}\Bigr)\sum_{v\in wW^I\setminus W^I}\Bigl(\prod_{\alpha\in\Delta_{v^{-1}w}}(-1)^{-s_{w\alpha}}\Bigr)\zeta_r(v^{-1}\mathbf{s},v^{-1}\mathbf{y};\Delta).\end{aligned}\tag{2.24}$$

We see that the second term on the right-hand side of (2.24) is, after renaming $w^{-1}v$ by v and using (2.13) and (2.14), equal to

$$\begin{aligned}&\Big(\prod_{\alpha\in\Delta_{w^{-1}}}(-1)^{s_\alpha}\Big)\sum_{v\in W^I\backslash w^{-1}W^I}\Big(\prod_{\alpha\in\Delta_{v^{-1}}}(-1)^{-s_{w\alpha}}\Big)\zeta_r(v^{-1}w^{-1}\mathbf{s},v^{-1}w^{-1}\mathbf{y};\Delta)\\&\quad=-\sum_{v\in W^I\backslash w^{-1}W^I}\Big(\prod_{\alpha\in\Delta_{v^{-1}}}(-1)^{-s_\alpha}\Big)\zeta_r(v^{-1}\mathbf{s},v^{-1}\mathbf{y};\Delta).\end{aligned}\tag{2.25}$$

The desired results follow from (2.24) and (2.25). □

Remark 2.2. The above Theorem 2.1 is stated under the condition $\mathbf{s}\in\mathcal{S}$. Treating more carefully, however, we can generalise this theorem to the case when $s_\alpha=1$ for some of the α's (cf. [10, Remark 2]).

Remark 2.3. Since the right-hand sides of (2.15) and (2.16) include signature factors, sometimes the right-hand side might be zero. If so, then Theorem 2.1 gives no useful information. In the next section we will give examples when the right-hand side does not vanish. This is the key point why we can sometimes treat the situation when some of the variables are odd integers.

3. Application of Theorem 2.1 to the case G_2. Hereafter in the present paper we concentrate on the study of the zeta-function of the root system G_2. The fundamental system of G_2 is $\Psi=\{\alpha_1,\alpha_2\}$, where $|\alpha_2|=\sqrt{3}|\alpha_1|$ and the angle between α_1 and α_2 is $5\pi/6$. Denote the positive roots by $\alpha_1,\ldots,\alpha_6$, where

$$\begin{aligned}\alpha_3&=3\alpha_1+\alpha_2, &\quad \alpha_3^\vee&=\alpha_1^\vee+\alpha_2^\vee,\\ \alpha_4&=3\alpha_1+2\alpha_2, & \alpha_4^\vee&=\alpha_1^\vee+2\alpha_2^\vee,\\ \alpha_5&=\alpha_1+\alpha_2, & \alpha_5^\vee&=\alpha_1^\vee+3\alpha_2^\vee,\\ \alpha_6&=2\alpha_1+\alpha_2, & \alpha_6^\vee&=2\alpha_1^\vee+3\alpha_2^\vee,\end{aligned}\tag{3.1}$$

and we abbreviate $\sigma_j=\sigma_{\alpha_j}$. Applying Weyl's dimension formula with the above data to (2.6), we find that the form of the zeta-function of G_2 is given by (1.1), with $s_j=s_{\alpha_j}$ $(1\le j\le 6)$.

Now we show an application of Theorem 2.1 to the G_2 case. Assume that s_i are all positive integers (≥ 2). Let $I=\{2\}$, $\mathbf{y}=\mathbf{0}$ and $w_1=w_0\sigma_1$ with the longest element

$$w_0=\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2=\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1=-1.$$

Then we have $w_1=w_1^{-1}=\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2=-\sigma_1$, so $w_1\alpha_1=\alpha_1$, $w_1\alpha_2=-\alpha_3$, $w_1\alpha_3=-\alpha_2$, $w_1\alpha_4=-\alpha_4$, $w_1\alpha_5=-\alpha_6$, $w_1\alpha_6=-\alpha_5$, and hence $w_1^{-1}\mathbf{s}=(s_1,s_3,s_2,s_4,s_6,s_5)$. Therefore, when $s_2=s_3$ and $s_5=s_6$, we have $w_1^{-1}\mathbf{s}=\mathbf{s}$. Since

$$\Delta_{w_1^{-1}}=\Delta_+\cap w_1\Delta_-=\Delta_+\cap\sigma_1\Delta_+=\Delta_+\setminus\{\alpha_1\},$$

we have

$$\Big(\prod_{\alpha\in\Delta_{w_1^{-1}}}(-1)^{s_\alpha}\Big)=(-1)^{s_2+s_3+s_4+s_5+s_6}.$$

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Therefore, we can apply Theorem 2.1 when $s_2 = s_3$, $s_5 = s_6$, and $s_2 + s_3 + s_4 + s_5 + s_6$ is odd. It is easy to see that

$$W_I = \{1, \sigma_2\},$$
$$W^I = \{1, \sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\sigma_2, \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\},$$
$$w_1 W^I = \{\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2, \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2, \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\sigma_2, \sigma_1\sigma_2\sigma_1, \sigma_1\sigma_2\},$$

and hence $W^I \setminus w_1 W^I = \{1, \sigma_1\}$. Thus, by (2.16) we have

$$S(\mathbf{s}, \mathbf{0}; \{2\}; G_2) = \zeta_2(\mathbf{s}, \mathbf{0}; G_2) + (-1)^{s_1}\zeta_2(\sigma_1^{-1}\mathbf{s}, \mathbf{0}; G_2).$$

Since $s_2 = s_3$ and $s_5 = s_6$, we have $\sigma_1^{-1}\mathbf{s} = \mathbf{s}$. Therefore, if s_1 is even, then the right-hand side of the above is $2\zeta_2(\mathbf{s}, \mathbf{0}; G_2)$. The conclusion is as follows.

PROPOSITION 3.1. *Let $p, q, r, u \in \mathbb{N}_{\geq 2}$ with even p and odd r. Then*

$$S((p, q, q, r, u, u), \mathbf{0}; \{2\}; G_2) = 2\zeta_2(p, q, q, r, u, u; G_2). \tag{3.2}$$

Similarly, we can treat the case $I = \{1\}$, $\mathbf{y} = \mathbf{0}$. Then $W_I = \{1, \sigma_1\}$ and

$$W^I = \{1, \sigma_2, \sigma_2\sigma_1, \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_1\sigma_2\sigma_1, \sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\}. \tag{3.3}$$

In this case we choose $w_1 = w_0\sigma_2$. Then $W^I \setminus w_1 W^I = \{1, \sigma_2\}$. We can apply Theorem 2.1 when $s_1 = s_5$, $s_3 = s_4$, and $s_1 + s_3 + s_4 + s_5 + s_6$ is odd. We obtain the following.

PROPOSITION 3.2. *Let $p, q, r, u \in \mathbb{N}_{\geq 2}$ with even q and odd u. Then*

$$S((p, q, r, r, p, u), \mathbf{0}; \{1\}; G_2) = 2\zeta_2(p, q, r, r, p, u; G_2). \tag{3.4}$$

4. A functional relation corresponding to $I = \{1\}$.

The results in the previous sections are valid only in the case $\mathbf{s} \in \mathcal{S}$. Hereafter we study the situation which includes the case when some of the variables take the value 1.

In this section we will show a functional relation which corresponds, in some sense, to the case $I = \{1\}$ in the preceding section. The discussion on the general situation would require more pages, so here we restrict ourselves to the following one special example.

EXAMPLE 4.1. The functional relation

$$\begin{aligned}
&\zeta_2(s, 2, 1, 1, 1, 1; G_2) + \zeta_2(s, 1, 2, 1, 1, 1; G_2) + \zeta_2(1, 2, 1, 1, s, 1; G_2) \\
&\quad - \zeta_2(1, 1, 2, 1, 1, s; G_2) + \zeta_2(1, 1, 1, 2, 1, s; G_2) - \zeta_2(1, 1, 1, 2, s, 1; G_2) \\
&= \zeta(2)\zeta(s+4) - \left(\frac{651}{8} - 2^{-s-1} - \frac{5 \cdot 3^{-s-2}}{2}\right)\zeta(s+6) \\
&\quad + \frac{9\pi}{2}\sum_{m\geq 1}\frac{\sin(2\pi m/3)}{m^{s+5}} - 135\sum_{m\geq 1}\frac{\cos(2\pi m/3)}{m^{s+6}} \\
&= \zeta(2)\zeta(s+4) - \left(\frac{111}{8} - 2^{-s-1}\right)\zeta(s+6) + \frac{81}{4}L(1, \chi_3)L(s+5, \chi_3)
\end{aligned} \tag{4.1}$$

holds for $s \in \mathbb{C}$ except for singularities on both sides, where $L(\cdot, \chi_3)$ denotes the Dirichlet L-function attached to the primitive Dirichlet character χ_3 of conductor 3.

Proof. We calculate

$$S((s, 2, 1, 1, 1, 1), \mathbf{0}, \{1\}; G_2) \tag{4.2}$$

in two ways. Using (2.12) and (3.3), we find that (4.2) is equal to the left-hand side of (4.1). On the other hand, since $P_{\{1\}+} = \mathbb{N}_0\lambda_1$, from (2.11) it follows that (4.2) is equal to

$$\sum_{m\geq 1} \sum_{\substack{n\neq 0 \\ m+n\neq 0 \\ m+2n\neq 0 \\ m+3n\neq 0 \\ 2m+3n\neq 0}} \frac{1}{m^s n^2 (m+n)(m+2n)(m+3n)(2m+3n)}. \tag{4.3}$$

Therefore, the remaining task is to show that (4.3) is equal to the right-hand side of (4.1). A direct way of the proof is to rewrite (4.3) as

$$\sum_{m\geq 1} \sum_{\substack{n\neq 0 \\ l_1\neq 0 \\ l_2\neq 0 \\ l_3\neq 0 \\ l_4\neq 0}} \frac{1}{m^s n^2 l_1 l_2 l_3 l_4} \int_0^1 \int_0^1 \int_0^1 \int_0^1 e^{2\pi i x_1(m+n-l_1)} e^{2\pi i x_2(m+2n-l_2)}$$
$$\times\, e^{2\pi i x_3(m+3n-l_3)} e^{2\pi i x_4(2m+3n-l_4)} dx_1 dx_2 dx_3 dx_4 \tag{4.4}$$

and compute this by using

$$\lim_{M\to\infty} \sum_{m=-M}^{M} \frac{e^{2\pi i m\theta}}{m^k} = -\frac{(2\pi i)^k}{k!} B_k(\theta - [\theta]) \quad (k \in \mathbb{N};\ \theta \in \mathbb{R}) \tag{4.5}$$

[1, Theorem 12.19], where $[\theta]$ is the integer part of θ, and $\{B_n(x)\}$ are the Bernoulli polynomials defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{4.6}$$

This method is based on an idea initiated by Zagier [25] and systematically used by Nakamura [16–18] (and also in [13, 14]). However, if we follow this way, the necessary computations are really enormous. Therefore, in order to reduce the total amount of computations, we first modify (4.3) using partial fraction decompositions, before applying the idea of Zagier–Nakamura.

First, using the partial fraction decomposition

$$\frac{1}{(m+3n)(2m+3n)} = \frac{1}{m(m+3n)} - \frac{1}{m(2m+3n)},$$

we divide (4.3) into two sums

$$\sum\sum{}^{*} \frac{1}{m^{s+1} n^2 (m+n)(m+2n)(m+3n)} - \sum\sum{}^{*} \frac{1}{m^{s+1} n^2 (m+n)(m+2n)(2m+3n)},$$

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where $\sum\sum^*$ denotes the same double sum as in (4.3). We then apply the same type of partial fraction decompositions some more times to find that (4.3) is equal to

$$\begin{aligned} &2\sum\sum^* \frac{1}{m^{s+1}n^3(m+n)(m+2n)} - \frac{5}{2}\sum\sum^* \frac{1}{m^{s+1}n^4(m+n)} \\ &\quad + \frac{1}{2}\sum\sum^* \frac{1}{m^{s+1}n^4(m+3n)} + 4\sum\sum^* \frac{1}{m^{s+1}n^4(2m+3n)} \\ &= 2\Sigma_1 - \frac{5}{2}\Sigma_2 + \frac{1}{2}\Sigma_3 + 4\Sigma_4, \end{aligned} \tag{4.7}$$

say. We divide Σ_1 as

$$\Sigma_1 = \sum\sum^{**} - \sum_{m+3n=0}\sum^{**} - \sum_{2m+3n=0}\sum^{**},$$

where $\sum\sum^{**}$ denotes the sum over $m \geq 1$, $n \neq 0$, $m+n \neq 0$ and $m+2n \neq 0$. Denote the first term by Σ_{11}. Putting $m = 3l$ and $n = -l$ in the second sum, we see that the second term is

$$-\sum_{l=1}^{\infty} \frac{1}{(3l)^{s+1}(-l)^3(3l-l)(3l-2l)} = \frac{1}{2\cdot 3^{s+1}}\zeta(s+6).$$

Similarly, the third term is $-3^{-s-1}2^{-3}\zeta(s+6)$. Therefore, we have

$$\Sigma_1 = \Sigma_{11} + \left(\frac{1}{2\cdot 3^{s+1}} - \frac{1}{2^3 3^{s+1}}\right)\zeta(s+6) = \Sigma_{11} + \frac{1}{2^3 3^s}\zeta(s+6). \tag{4.8}$$

As for Σ_2, we divide it as

$$\Sigma_2 = \sum\sum^{***} - \sum_{m+2n=0}\sum^{***} - \sum_{m+3n=0}\sum^{***} - \sum_{2m+3n=0}\sum^{***}, \tag{4.9}$$

where $\sum\sum^{***}$ denotes the sum over $m \geq 1$, $n \neq 0$, $m+n \neq 0$. Denote the first term by Σ_{21} and evaluate the remaining three sums as above to obtain

$$\Sigma_2 = \Sigma_{21} - \left(\frac{1}{2^{s+1}} - \frac{1}{2^4 3^{s-1}}\right)\zeta(s+6). \tag{4.10}$$

Similarly,

$$\Sigma_3 = \Sigma_{31} + \left(\frac{1}{2} + \frac{1}{2^{s+1}} + \frac{1}{2^4 3^{s+2}}\right)\zeta(s+6) \tag{4.11}$$

and

$$\Sigma_4 = \Sigma_{41} + \left(1 - \frac{1}{2^{s+1}} - \frac{1}{3^{s+2}}\right)\zeta(s+6), \tag{4.12}$$

where

$$\Sigma_{31} = \sum_{m\geq 1}\sum_{\substack{n\neq 0\\ m+3n\neq 0}} \frac{1}{m^{s+1}n^4(m+3n)}, \quad \Sigma_{41} = \sum_{m\geq 1}\sum_{\substack{n\neq 0\\ 2m+3n\neq 0}} \frac{1}{m^{s+1}n^4(2m+3n)}.$$

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Applying a partial fraction decomposition once more, we obtain

$$\Sigma_{11} = \sum\sum\nolimits^{**} \frac{1}{m^{s+1}n^4(m+n)} - \sum\sum\nolimits^{**} \frac{1}{m^{s+1}n^4(m+2n)}.$$

On the right-hand side, we separate the part $m+2n=0$ from the first double sum and separate the part $m+n=0$ from the second double sum. We obtain

$$\Sigma_{11} = \Sigma_{21} - \Sigma^\sharp - \left(1+\frac{1}{2^{s+1}}\right)\zeta(s+6), \tag{4.13}$$

where

$$\Sigma^\sharp = \sum_{m\geq 1}\sum_{\substack{n\neq 0\\ m+2n\neq 0}} \frac{1}{m^{s+1}n^4(m+2n)}.$$

We evaluate Σ_{21}. Recall the definition of the zeta-function of the root system A_2 (or the Mordell–Tornheim double sum)

$$\zeta_2(s_1,s_2,s_3;A_2) = \sum_{m,n\geq 1} \frac{1}{m^{s_1}n^{s_2}(m+n)^{s_3}}.$$

The part corresponding to positive n of the sum Σ_{21} is exactly $\zeta_2(s+1,4,1;A_2)$. The part corresponding to negative n is, putting $m-n=l$ when $m>n$ and $n-m=k$ when $m<n$, equal to

$$\sum_{n,l\geq 1}\frac{1}{(n+l)^{s+1}n^4 l} - \sum_{m,k\geq 1}\frac{1}{m^{s+1}(m+k)^4 k}$$
$$= \zeta_2(4,s+1,1;A_2) - \zeta_2(1,s+1,4;A_2).$$

Therefore,

$$\begin{aligned}\Sigma_{21} &= \zeta_2(s+1,4,1;A_2) + \zeta_2(4,s+1,1;A_2) - \zeta_2(1,s+1,4;A_2)\\ &= -5\zeta(s+6) + 2\zeta(2)\zeta(s+4) + 2\zeta(4)\zeta(s+2),\end{aligned} \tag{4.14}$$

where the second equality can be seen by [9, Theorem 3.1].

As for $\Sigma^\sharp$, we apply the method of Zagier–Nakamura. Write $\Sigma^\sharp$ as

$$\begin{aligned}\Sigma^\sharp &= \sum_{\substack{m\geq 1\\ n\neq 0\\ l\neq 0}} \frac{1}{m^{s+1}n^4 l}\int_0^1 e^{2\pi i(m+2n-l)\theta}\,d\theta\\ &= \sum_{m\geq 1}\frac{1}{m^{s+1}}\int_0^1 e^{2\pi i m\theta}\sum_{n\neq 0}\frac{e^{2\pi i n\cdot 2\theta}}{n^4}\sum_{l\neq 0}\frac{e^{2\pi i l(-\theta)}}{l}d\theta,\end{aligned} \tag{4.15}$$

and apply (4.5). We obtain

$$\Sigma^\sharp = \frac{(2\pi i)^5}{24}\sum_{m\geq 1}\frac{1}{m^{s+1}}(J_1+J_2), \tag{4.16}$$

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where

$$J_1 = \int_0^{1/2} e^{2\pi i m\theta} B_4(2\theta) B_1(1-\theta) d\theta, \quad J_2 = \int_{1/2}^{1} e^{2\pi i m\theta} B_4(2\theta - 1) B_1(1-\theta) d\theta.$$

Since $B_1(x) = x - 1/2$ and $B_4(x) = x^4 - 2x^3 + x^2 - 1/30$, the factors $B_4(2\theta)B_1(1-\theta)$ and $B_4(2\theta-1)B_1(1-\theta)$ are polynomials in θ of degree 5. It is easy to see recursively that

$$\int_0^{1/2} e^{2\pi i m\theta} \theta^k d\theta = \sum_{j=1}^{k+1} \frac{(-1)^{j-1+m} k!}{(2\pi i m)^j 2^{k+1-j} (k+1-j)!} - \frac{(-1)^k k!}{(2\pi i m)^{k+1}} \tag{4.17}$$

and

$$\int_{1/2}^{1} e^{2\pi i m\theta} \theta^k d\theta = \sum_{j=1}^{k+1} \frac{(-1)^{j-1} k!}{(2\pi i m)^j (k+1-j)!} \left(1 - \frac{(-1)^m}{2^{k+1-j}}\right). \tag{4.18}$$

Using these formulas, we can evaluate J_1 and J_2. Substituting the results into (4.16), we find that $\Sigma^\sharp$ can be written in terms of $\zeta(s)$ and $\phi(s) = \sum_{m=1}^{\infty} (-1)^m m^{-s} = (2^{1-s} - 1)\zeta(s)$, more explicitly,

$$\Sigma^\sharp = \frac{\pi^4}{45}\zeta(s+2) + \frac{4\pi^2}{3}\zeta(s+4) - \left(16 + \frac{1}{2^s}\right)\zeta(s+6). \tag{4.19}$$

Substituting (4.14) and (4.19) into (4.13) we obtain

$$\Sigma_{11} = -\pi^2\zeta(s+4) + \left(10 + \frac{1}{2^{s+1}}\right)\zeta(s+6), \tag{4.20}$$

and so

$$2\Sigma_{11} - \frac{5}{2}\Sigma_{21} = -\frac{\pi^4}{18}\zeta(s+2) - \frac{17\pi^2}{6}\zeta(s+4) + \left(\frac{65}{2} + \frac{1}{2^s}\right)\zeta(s+6). \tag{4.21}$$

The evaluation of Σ_{31} and Σ_{41} is similar to that of $\Sigma^\sharp$. In these cases, instead of (4.17) and (4.18), the integrals over the intervals $[0, 1/3]$, $[1/3, 2/3]$ and $[2/3, 1]$ appear, and hence the 3rd root of unity appears. We obtain

$$\begin{aligned} \frac{1}{2}\Sigma_{31} + 4\Sigma_{41} = {} & \frac{\pi^4}{18}\zeta(s+2) + 3\pi^2\zeta(s+4) + \left(-\frac{911}{8} - \frac{1}{2^{s+1}} + \frac{5}{2\cdot 3^{s+2}}\right)\zeta(s+6) \\ & + \frac{9\pi}{2}\sum_{m\geq 1}\frac{\sin(2\pi m/3)}{m^{s+5}} - 135\sum_{m\geq 1}\frac{\cos(2\pi m/3)}{m^{s+6}}. \end{aligned} \tag{4.22}$$

Moreover, it is easy to see that

$$\sum_{m\geq 1}\frac{\cos(2\pi m/3)}{m^{s+6}} = \frac{3^{-s-5}-1}{2}\zeta(s+6), \tag{4.23}$$

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$$\sum_{m\geq 1}\frac{\sin(2\pi m/3)}{m^{s+5}}=\frac{\sqrt{3}}{2}\sum_{m\geq 1}\frac{\chi_3(m)}{m^{s+5}}=\frac{\sqrt{3}}{2}L(s+5,\chi_3). \tag{4.24}$$

Consequently, we can conclude that (4.3) coincides with the right-hand side of (4.1). □

In particular, setting $s=1$ in (4.1), we obtain that the left-hand side is equal to $2\zeta_2(1,2,1,1,1,1;G_2)$ (see (3.2)). Hence, we have

$$\zeta_2(1,2,1,1,1,1;G_2)=\frac{1}{2}\zeta(2)\zeta(5)-\frac{109}{16}\zeta(7)+\frac{81}{8}L(1,\chi_3)L(6,\chi_3). \tag{4.25}$$

REMARK 4.2. A little digression. Recall that the zeta-function of the root system C_2 is defined by

$$\zeta_2(s_1,s_2,s_3,s_4;C_2)=\sum_{m,n\geq 1}\frac{1}{m^{s_1}n^{s_2}(m+n)^{s_3}(m+2n)^{s_4}}.$$

Divide Σ_{11} into two subsums accordingly as $n\geq 1$ and $n\leq -1$. Then the former part is exactly $\zeta_2(s+1,3,1,1;C_2)$. The latter is further divided accordingly as $m-n>0$ and $m-n<0$. The part corresponding to $m-n<0$ is $-\zeta_2(s+1,1,3,1;C_2)$, while the remaining part is again divided into two subsums. The conclusion is that

$$\begin{aligned}\Sigma_{11}=&\zeta_2(s+1,3,1,1;C_2)-\zeta_2(s+1,1,3,1;C_2)\\&+\zeta_2(1,1,3,s+1;C_2)-\zeta_2(1,3,1,s+1;C_2).\end{aligned} \tag{4.26}$$

On the other hand, we have shown that Σ_{11} can be written in terms of $\zeta(s)$ (see (4.20)). Combining these two formulas (4.26) and (4.20), we obtain a functional relation between the zeta-function of C_2 and the Riemann zeta-function, which is different from the previously known relations ([9, Section 8], [18, Section 5]).

5. Some lemmas. In the next section, we will deduce a functional relation corresponding to the case $I=\{2\}$ by a method different from that described in the preceding section. In this section, we prepare several lemmas which are necessary in the next section. First, the following lemma is a slight modification of [11, Lemma 4.2], which can be proved similarly.

LEMMA 5.1. *Let $\{P_m\}$, $\{Q_m\}$, $\{R_m\}$ be sequences such that*

$$P_m=\sum_{j=0}^{[m/2]}R_{m-2j}\frac{(i\pi)^{2j}}{(2j)!},\quad Q_m=\sum_{j=0}^{[m/2]}R_{m-2j}\frac{(i\pi)^{2j}}{(2j+1)!}$$

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for any $m \in \mathbb{N}_0$. *Then*

$$P_m = -2 \sum_{\tau=0}^{[m/2]} \zeta(2\tau) Q_{m-2\tau}, \tag{5.1}$$

$$Q_{2h} = \frac{2}{\pi^2} \sum_{\tau=0}^{h} \left(2^{2h-2\tau+2} - 1\right) \zeta(2h - 2\tau + 2) P_{2\tau} \tag{5.2}$$

for any $h \in \mathbb{N}_0$.

Following is an important key to the argument in the next section.

LEMMA 5.2 *[9, Lemma 6.3]. Let* $h \in \mathbb{N}$, $\lambda_j = (1 + (-1)^j)/2$ *for* $j \in \mathbb{Z}$ *and*

$$\begin{aligned}
\mathfrak{C} &:= \left\{C(l) \in \mathbb{C} \mid l \in \mathbb{Z},\ l \neq 0\right\}, \\
\mathfrak{D} &:= \{D(N; m; \eta) \in \mathbb{R} \mid N, m, \eta \in \mathbb{Z},\ N \neq 0,\ m \geq 0,\ 1 \leq \eta \leq h\}, \\
\mathfrak{A} &:= \{a_\eta \in \mathbb{N} \mid 1 \leq \eta \leq h\}
\end{aligned}$$

be sets of numbers indexed by integers. Assume that the infinite series appearing in

$$\begin{aligned}
&\sum_{\substack{N \in \mathbb{Z} \\ N \neq 0}} (-1)^N C(N) e^{iN\theta} - 2 \sum_{\eta=1}^{h} \sum_{k=0}^{a_\eta} \phi(a_\eta - k) \lambda_{a_\eta - k} \\
&\qquad \times \sum_{\xi=0}^{k} \left\{ \sum_{\substack{N \in \mathbb{Z} \\ N \neq 0}} (-1)^N D(N; k - \xi; \eta) e^{iN\theta} \right\} \frac{(i\theta)^\xi}{\xi!}
\end{aligned} \tag{5.3}$$

are absolutely convergent for $\theta \in [-\pi, \pi]$*, and that (5.3) is a constant function for* $\theta \in [-\pi, \pi]$*. Then, for* $d \in \mathbb{N}_0$*,*

$$\begin{aligned}
&\sum_{\substack{N \in \mathbb{Z} \\ N \neq 0}} \frac{(-1)^N C(N) e^{iN\theta}}{N^d} - 2 \sum_{\eta=1}^{h} \sum_{k=0}^{a_\eta} \phi(a_\eta - k) \lambda_{a_\eta - k} \\
&\quad \times \sum_{\xi=0}^{k} \left\{ \sum_{\omega=0}^{k-\xi} \binom{\omega + d - 1}{\omega} (-1)^\omega \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \frac{(-1)^m D(m; k - \xi - \omega; \eta) e^{im\theta}}{m^{d+\omega}} \right\} \frac{(i\theta)^\xi}{\xi!} \\
&\quad + 2 \sum_{k=0}^{d} \phi(d - k) \lambda_{d-k} \sum_{\xi=0}^{k} \left\{ \sum_{\eta=1}^{h} \sum_{\omega=0}^{a_\eta - 1} \binom{\omega + k - \xi}{\omega} (-1)^\omega \right. \\
&\quad \times \left. \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \frac{D(m; a_\eta - 1 - \omega; \eta)}{m^{k-\xi+\omega+1}} \right\} \frac{(i\theta)^\xi}{\xi!} = 0
\end{aligned} \tag{5.4}$$

holds for $\theta \in [-\pi, \pi]$*, where the infinite series appearing on the left-hand side of (5.4) are absolutely convergent for* $\theta \in [-\pi, \pi]$.

We prepare another lemma with the same feature, which is a slight generalisation of [11, Lemma 4.4].

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LEMMA 5.3. *Let* $h \in \mathbb{N}$,

$$\begin{aligned}&\mathcal{A} := \left\{\alpha(l) \in \mathbb{C} \mid l \in \mathbb{Z},\ l \neq 0\right\},\\ &\mathcal{B} := \{\beta(N; m; \eta) \in \mathbb{R} \mid N, m, \eta \in \mathbb{Z},\ N \neq 0,\ m \geq 0,\ 1 \leq \eta \leq h\},\\ &\mathcal{C} := \{c_\eta \in \mathbb{N} \mid 1 \leq \eta \leq h\}\end{aligned}$$

be sets of numbers indexed by integers, and

$$\begin{aligned}R_\pm(\theta) = &\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} (\pm i)^m \alpha(m) e^{im\theta/2} - 2\sum_{\eta=1}^{h}\sum_{k=0}^{c_\eta} \phi(c_\eta - k)\lambda_{c_\eta - k}\\ &\times \sum_{\xi=0}^{k} \left\{\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} (\pm i)^m \beta(m; k-\xi; \eta) e^{im\theta/2}\right\} \frac{(i\theta)^\xi}{\xi!}.\end{aligned} \tag{5.5}$$

Assume that both of the right-hand sides of $R_\pm(\theta)$ *in (5.5) are absolutely convergent for* $\theta \in [-\pi, \pi]$*, and that both* $R_+(\theta)$ *and* $R_-(\theta)$ *are constant functions on* $[-\pi, \pi]$*. Then, for* $d \in \mathbb{N}$,

$$\begin{aligned}&\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \frac{\alpha(m)}{m^d} - 2\sum_{\eta=1}^{h}\sum_{k=0}^{[c_\eta/2]} \zeta(2k) \sum_{\omega=0}^{c_\eta - 2k} \binom{\omega + d - 1}{\omega} (-2)^\omega \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \frac{\beta(m; c_\eta - 2k - \omega; \eta)}{m^{d+\omega}}\\ &+ 2\sum_{\eta=1}^{h}\sum_{k=0}^{[d/2]} \zeta(2k) 2^{-2k} \sum_{\omega=0}^{c_\eta - 1} \binom{\omega + d - 2k}{\omega} (-2)^\omega\\ &\times \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \frac{((-1)^m + 1)\beta(m; c_\eta - 1 - \omega; \eta)}{m^{d-2k+\omega+1}}\\ &- 2\sum_{\eta=1}^{h}\sum_{k=0}^{[(d+1)/2]} \zeta(2k)\left(1 - 2^{-2k}\right) \sum_{\omega=0}^{c_\eta - 1} \binom{\omega + d - 2k}{\omega} (-2)^\omega\\ &\times \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \frac{((-1)^m - 1)\beta(m; c_\eta - 1 - \omega; \eta)}{m^{d-2k+\omega+1}} = 0\end{aligned} \tag{5.6}$$

for $\theta \in [-\pi, \pi]$*, where the infinite series appearing on the left-hand side of (5.6) are absolutely convergent for* $\theta \in [-\pi, \pi]$.

Proof. We just indicate how to modify the proof of [11, Lemma 4.4] to obtain the above lemma. Let $\mathcal{G}_N^\pm(\theta)$ and $\mathfrak{C}_n^\pm$ be as in the proof of [11, Lemma 4.4]. Putting $N = d + 1$ for $d \in \mathbb{N}$ and $\theta = \pi$ in [11, (4.11)], we obtain

$$\frac{i^d}{2\pi}\left\{\mathcal{G}_{d+1}^+(\pi) - \mathcal{G}_{d+1}^+(-\pi)\right\} = \sum_{\nu=0}^{[d/2]} \mathfrak{C}_{d-2\nu}^+ 2^{d-2\nu} \frac{(i\pi)^{2\nu}}{(2\nu+1)!}. \tag{5.7}$$

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Similarly, we have

$$\frac{i^{d+1}}{2}\left\{\mathcal{G}^{+}_{d+1}(\pi)+\mathcal{G}^{+}_{d+1}(-\pi)\right\}=\sum_{\nu=0}^{[(d+1)/2]}\mathfrak{C}^{+}_{d+1-2\nu}2^{d+1-2\nu}\frac{(i\pi)^{2\nu}}{(2\nu)!}. \tag{5.8}$$

These are analogues of [11, (4.12)] and [11, (4.13)], with $2d$ replaced by d and $d+1$ respectively. By Lemma 5.1, we have

$$\begin{aligned}&\frac{i^{d}}{2}\left\{\mathcal{G}^{+}_{d}(\pi)+\mathcal{G}^{+}_{d}(-\pi)\right\}\\&\quad=-\frac{i^{d}}{\pi}\sum_{\tau=0}^{[d/2]}\zeta(2\tau)(-1)^{\tau}\left\{\mathcal{G}^{+}_{d+1-2\tau}(\pi)-\mathcal{G}^{+}_{d+1-2\tau}(-\pi)\right\},\end{aligned} \tag{5.9}$$

and similarly,

$$\begin{aligned}&\frac{1}{2}\left\{\mathcal{G}^{-}_{d}(\pi)-\mathcal{G}^{-}_{d}(-\pi)\right\}\\&=\frac{1}{\pi}\sum_{\rho=0}^{[(d-1)/2]}\left(2^{2\rho+2}-1\right)\zeta(2\rho+2)(-1)^{\rho}\left\{\mathcal{G}^{-}_{d-1-2\rho}(\pi)+\mathcal{G}^{-}_{d-1-2\rho}(-\pi)\right\}\\&=-\frac{1}{\pi}\sum_{\tau=1}^{[(d+1)/2]}\left(2^{2\tau}-1\right)\zeta(2\tau)(-1)^{\tau}\left\{\mathcal{G}^{-}_{d+1-2\tau}(\pi)+\mathcal{G}^{-}_{d+1-2\tau}(-\pi)\right\}.\end{aligned} \tag{5.10}$$

We evaluate each side of (5.9) in the same way as in the proof of [11, Lemma 4.4] with obvious modifications. The result is almost the same as [11, (4.17)], just replacing $2d$ by d, and the summation $\sum_{\xi=0}^{d}$ by $\sum_{\xi=0}^{[d/2]}$. Similarly, from (5.10) we obtain a formula which is almost the same as [11, (4.19)], just replacing $2d$ by d, and the summation $\sum_{\xi=0}^{d}$ by $\sum_{\xi=0}^{[(d+1)/2]}$. Combining those two formulas we obtain (5.6). □

6. A functional relation corresponding to $I=\{2\}$. Using the lemmas in the previous section, we now construct a functional relation among $\zeta_2(s;G_2)$, $\zeta(s)$ and $\phi(s)=(2^{1-s}-1)\zeta(s)$, which corresponds to the case $I=\{2\}$ in Section 2.

THEOREM 6.1. *For $p,q,r,u,v\in\mathbb{N}$,*

$$\begin{aligned}&\zeta_2(p,s,q,r,u,v;G_2)+(-1)^{p}\zeta_2(p,q,s,r,v,u;G_2)+(-1)^{p+q}\zeta_2(v,q,r,s,p,u;G_2)\\&\quad+(-1)^{p+q+v}\zeta_2(v,r,q,s,u,p;G_2)+(-1)^{p+q+r+v}\zeta_2(u,r,s,q,v,p;G_2)\\&\quad+(-1)^{p+q+r+u+v}\zeta_2(u,s,r,q,p,v;G_2)\\&\quad+I_1+I_2+\cdots+I_8=0\end{aligned} \tag{6.1}$$

holds for all $s\in\mathbb{C}$ except for singularities of functions on the left-hand side, where I_j $(1\leq j\leq 8)$, defined below, are linear combinations of $\zeta(s)$ and $\phi(s)$.

The definition of I_j is given by

$$I_j=A_j+B_{1j}+B_{2j}\qquad(1\leq j\leq 8),$$

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where A_j, B_{1j}, B_{2j} $(1 \leq j \leq 8)$ are defined as follows:

$$A_j = 2(-1)^{p+a_1} \sum_{k=0}^{[a_2/2]} \zeta(2k) \sum_{\sigma=0}^{a_2-2k} \binom{\sigma+v-1}{\sigma} \sum_{\rho=0}^{a_3-a_4} \binom{\rho+u-a_5}{\rho}$$
$$\times \sum_{\omega=0}^{a_6-a_7} \binom{\omega+r-a_8}{\omega} \binom{p+q-1-\omega-a_7}{a_9-1}$$
$$\times (-1)^{a_{10}} 2^{a_{11}} 3^{a_{12}} \zeta(s+p+q+r+u+v-2k),$$

$$B_{1j} = 2(-1)^{p+b_1} \sum_{k=0}^{[v/2]} 2^{-2k} \zeta(2k) \sum_{\sigma=0}^{a_2-1} \binom{\sigma+v-2k}{\sigma} \sum_{\rho=0}^{a_3-b_4} \binom{\rho+u-b_5}{\rho}$$
$$\times \sum_{\omega=0}^{a_6-b_7} \binom{\omega+r-b_8}{\omega} \binom{p+q-1-\omega-b_7}{a_9-1} (-1)^{a_{10}} 2^{b_{11}} 3^{b_{12}}$$
$$\times \{\zeta(s+p+q+r+u+v-2k) + \phi(s+p+q+r+u+v-2k)\}$$

and

$$B_{2j} = 2(-1)^{p+b_1} \sum_{k=0}^{[(v+1)/2]} (1-2^{-2k}) \zeta(2k) \sum_{\sigma=0}^{a_2-1} \binom{\sigma+v-2k}{\sigma} \sum_{\rho=0}^{a_3-b_4} \binom{\rho+u-b_5}{\rho}$$
$$\times \sum_{\omega=0}^{a_6-b_7} \binom{\omega+r-b_8}{\omega} \binom{p+q-1-\omega-b_7}{a_9-1} (-1)^{a_{10}} 2^{b_{11}} 3^{b_{12}}$$
$$\times \{\zeta(s+p+q+r+u+v-2k) - \phi(s+p+q+r+u+v-2k)\},$$

where $a_l = a_l(j)$, $b_l = b_l(j)$ are as follows: According to $j = 1, \ldots, 8$, a_l $(1 \leq l \leq 12)$ take the values

$$\begin{aligned}
a_1 &= 1, v+1, 1, v, v+1, v, v, v+1,\\
a_2 &= p, u, q, u, r, u, r, u,\\
a_3 &= p, p, q, q, r, r, r, r,\\
a_4 &= 2k+\sigma, 1, 2k+\sigma, 1, 2k+\sigma, 1, 2k+\sigma, 1,\\
a_5 &= 1, 2k+\sigma, 1, 2k+\sigma, 1, 2k+\sigma, 1, 2k+\sigma,\\
a_6 &= p, p, q, q, p, p, q, q,\\
a_7 &= 2k+\sigma+\rho, 1+\rho, 2k+\sigma+\rho, 1+\rho, 1, 1, 1, 1,\\
a_8 &= 1, 1, 1, 1, 2k+\sigma+\rho, 1+\rho, 2k+\sigma+\rho, 1+\rho,\\
a_9 &= q, q, p, p, q, q, p, p,\\
a_{10} &= 0, 0, \sigma+\rho+\omega, \rho+\omega, \rho, \rho, \rho+\omega, \rho+\omega,
\end{aligned}$$

$$\begin{aligned}
a_{11} = \sigma - r - \omega, \sigma - r - \omega, \sigma - u - \rho, -u + 2k + 2\sigma - \rho - 1,\\
-r + 2k + 2\sigma + \rho - \omega - 1, -r + \sigma + \rho - \omega, \sigma, \sigma
\end{aligned}$$

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and

$$a_{12} = -u - v - \sigma - \rho, -u - v + 2k - \rho - 1, 0, -v - \sigma, 0, -v - \sigma, 0, -v - \sigma.$$

Next, define b_l ($l = 1, 4, 5, 7, 8, 11, 12$). First,

$$b_1 = 1, v + 1, 0, v, v + 1, v, v, v + 1.$$

The definitions of b_4, b_5, b_7, b_8 are similar to a_4, a_5, a_7, a_8, but all $2k + \sigma$ are replaced by $1 + \sigma$. Finally,

$$\begin{aligned} b_{11} = \sigma - r - \omega, \sigma - r - \omega, \sigma - u - \rho, -u + 2\sigma - \rho, \\ -r + 2\sigma + \rho - \omega, -r + \sigma + \rho - \omega, \sigma, \sigma, \end{aligned}$$

and

$$\begin{aligned} b_{12} = & -u - v + 2k - \sigma - \rho - 1, -u - v + 2k - \rho - 1, 0, -v + 2k - \sigma - 1, \\ & 0, -v + 2k - \sigma - 1, 0, -v + 2k - \sigma - 1. \end{aligned}$$

REMARK 6.2. When p, q, r, u, v are even, Formula (6.1) coincides with our previous result given in [11, Theorem 5.1]. On this occasion we correct some misprints in the statement of [11, Theorem 5.1]. On [**11**, line 8, p. 202], we should replace $\binom{2p+2q-2-\rho-\omega}{2q-1}$ by $\binom{2p+2q-2-\rho-\omega}{2p-1}$. On [**11**, lines 12 and 16, p. 203], we should replace $3^{-2v-\sigma}$ by $3^{-2v-\sigma-1+2k}$.

Proof of Theorem 6.1. The technique to prove this theorem is essentially the same as in our previous papers (see [11, Section 5]; also [9, Section 7], [12, Section 5]). Hence, it is enough to give a sketch of the proof here.

From [12, Lemma 5.3], we have

$$\begin{aligned} & \sum_{\substack{l\neq 0,\, m\geq 1 \\ l+m\neq 0 \\ l+2m\neq 0}} \frac{(-1)^{l+m} x^m e^{i(l+m)\theta}}{l^p m^s (l+m)^q} \\ & \quad - 2\sum_{j=0}^{p} \phi(p-j)\varepsilon_{p-j} \sum_{\xi=0}^{j} \binom{q-1+j-\xi}{q-1} (-1)^{j-\xi} \sum_{m=1}^{\infty} \frac{(-1)^m x^m e^{im\theta}}{m^{s+q+j-\xi}} \frac{(i\theta)^\xi}{\xi!} \\ & \quad + 2\sum_{j=0}^{q} \phi(q-j)\varepsilon_{q-j} \sum_{\xi=0}^{j} \binom{p-1+j-\xi}{p-1} (-1)^{p-1} \sum_{m=1}^{\infty} \frac{x^m}{m^{s+p+j-\xi}} \frac{(i\theta)^\xi}{\xi!} \\ & = -\frac{(-1)^{p+q}}{2^p} \sum_{m=1}^{\infty} \frac{x^m}{m^{s+p+q}} \end{aligned} \tag{6.2}$$

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for $p, q \in \mathbb{N}$, $\theta \in [-\pi, \pi]$, $s \in \mathbb{R}$ with $s > 1$ and $x \in \mathbb{C}$ with $|x| = 1$. By Lemma 5.2 with $d = r \in \mathbb{N}$, we have

$$\sum_{\substack{l\in\mathbb{Z},\, l\neq 0\\ m\geq 1\\ l+m\neq 0\\ l+2m\neq 0}} \frac{(-1)^l x^m e^{i(l+2m)\theta}}{l^p m^s (l+m)^q (l+2m)^r}$$
$$- 2\sum_{j=0}^{p} \phi(p-j)\,\lambda_{p-j} \sum_{\xi=0}^{j}\sum_{\omega=0}^{j-\xi} \binom{\omega+r-1}{\omega}(-1)^{\omega}$$
$$\times \binom{q-1+j-\xi-\omega}{b-1}(-1)^{j-\xi-\omega}\frac{1}{2^{r+\omega}}\sum_{m=1}^{\infty}\frac{x^m e^{2im\theta}}{m^{s+q+j-\xi+r}}\frac{(i\theta)^{\xi}}{\xi!}$$
$$+ 2\sum_{j=0}^{q} \phi(q-j)\,\lambda_{q-j} \sum_{\xi=0}^{j}\sum_{\omega=0}^{j-\xi} \binom{\omega+r-1}{\omega}(-1)^{\omega}$$
$$\times \binom{p-1+j-\xi-\omega}{p-1}(-1)^{p-1}\sum_{m=1}^{\infty}\frac{(-1)^m x^m e^{im\theta}}{m^{s+p+r+j-\xi}}\frac{(i\theta)^{\xi}}{\xi!}$$
$$+ 2\sum_{j=0}^{r} \phi(r-j)\,\lambda_{r-j} \sum_{\xi=0}^{j}\sum_{\omega=0}^{p-1} \binom{\omega+j-\xi}{\omega}(-1)^{\omega}$$
$$\times \binom{p+q-2-\omega}{q-1}(-1)^{p-1-\omega}\frac{1}{2^{j-\xi+\omega+1}}\sum_{m=1}^{\infty}\frac{x^m}{m^{s+p+q+j-\xi}}\frac{(i\theta)^{\xi}}{\xi!}$$
$$- 2\sum_{j=0}^{r} \phi(r-j)\,\lambda_{r-j} \sum_{\xi=0}^{j}\sum_{\omega=0}^{q-1} \binom{\omega+j-\xi}{\omega}(-1)^{\omega}$$
$$\times \binom{p+q-2-\omega}{p-1}(-1)^{p-1}\sum_{m=1}^{\infty}\frac{x^m}{m^{s+p+q+j-\xi}}\frac{(i\theta)^{\xi}}{\xi!} = 0$$

for $\theta \in [-\pi, \pi]$, $p, q, r \in \mathbb{N}$, $s \in \mathbb{R}$ with $s > 1$ and $x \in \mathbb{C}$ with $|x| \leq 1$. Here we replace x by $-xe^{i\theta}$ and move the terms corresponding to $l + 3m = 0$ of the first member on the left-hand side of the above equation to the right-hand side. Then we have

$$\sum_{\substack{l\in\mathbb{Z},\, l\neq 0\\ m\geq 1\\ l+m\neq 0\\ l+2m\neq 0\\ l+3m\neq 0}} \frac{(-1)^{l+m} x^m e^{i(l+3m)\theta}}{l^p m^s (l+m)^q (l+2m)^r}$$
$$- 2\sum_{j=0}^{p} \phi(p-j)\,\lambda_{p-j} \sum_{\xi=0}^{j}\sum_{\omega=0}^{j-\xi} \binom{\omega+r-1}{\omega}(-1)^{\omega}$$
$$\times \binom{q-1+j-\xi-\omega}{q-1}(-1)^{j-\xi-\omega}\frac{1}{2^{r+\omega}}\sum_{m=1}^{\infty}\frac{(-1)^m x^m e^{3im\theta}}{m^{s+q+r+j-\xi}}\frac{(i\theta)^{\xi}}{\xi!}$$
$$+ 2\sum_{j=0}^{q} \phi(q-j)\,\lambda_{q-j} \sum_{\xi=0}^{j}\sum_{\omega=0}^{j-\xi} \binom{\omega+r-1}{\omega}(-1)^{\omega}$$

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$$
\begin{aligned}
&\times \binom{p-1+j-\xi-\omega}{p-1}(-1)^{p-1}\sum_{m=1}^{\infty}\frac{x^m e^{2im\theta}}{m^{s+p+r+j-\xi}}\frac{(i\theta)^{\xi}}{\xi!}\\
&+2\sum_{j=0}^{r}\phi(r-j)\,\lambda_{r-j}\sum_{\xi=0}^{j}\sum_{\omega=0}^{p-1}\binom{\omega+j-\xi}{\omega}(-1)^{\omega}\\
&\times \binom{p+q-2-\omega}{q-1}(-1)^{p-1-\omega}\frac{1}{2^{j-\xi+\omega+1}}\sum_{m=1}^{\infty}\frac{(-1)^m x^m e^{im\theta}}{m^{s+p+q+j-\xi}}\frac{(i\theta)^{\xi}}{\xi!}\\
&+2\sum_{j=0}^{r}\phi(r-j)\,\lambda_{r-j}\sum_{\xi=0}^{j}\sum_{\omega=0}^{q-1}\binom{\omega+j-\xi}{\omega}(-1)^{\omega}\\
&\times \binom{p+q-2-\omega}{p-1}(-1)^{p-1}\sum_{m=1}^{\infty}\frac{(-1)^m x^m e^{im\theta}}{m^{s+p+q+j-\xi}}\frac{(i\theta)^{\xi}}{\xi!}\\
=&-\frac{(-1)^{p+q+r}}{3^{2p}2^{2q}}\sum_{m=1}^{\infty}\frac{x^m}{m^{s+p+q+r}}.
\end{aligned}
$$

We again apply Lemma 5.2 with $d=u\in\mathbb{N}$ to the above equation. Then we have

$$
\begin{aligned}
&\sum_{\substack{l\in\mathbb{Z},\, l\neq 0\\ m\geq 1\\ l+m\neq 0\\ l+2m\neq 0\\ l+3m\neq 0}}\frac{(-1)^{l+m}x^m e^{i(l+3m)\theta}}{l^p m^s (l+m)^q (l+2m)^r (l+3m)^u}\\
&\quad + J_1(\theta;x)+J_2(\theta;x)+J_3(\theta;x)+J_4(\theta;x)=0,
\end{aligned}
\tag{6.3}
$$

where

$$
\begin{aligned}
&J_1(\theta;x)\\
&=-2\sum_{j=0}^{p}\phi(p-j)\,\lambda_{p-j}\sum_{\xi=0}^{j}\sum_{\rho=0}^{j-\xi}\binom{\rho+u-1}{\rho}(-1)^{\rho}\sum_{\omega=0}^{j-\xi-\rho}\binom{\omega+r-1}{\omega}(-1)^{\omega}\\
&\quad\times 3^{-u-\rho}\binom{q-1+j-\xi-\rho-\omega}{q-1}\frac{(-1)^{j-\xi-\rho-\omega}}{2^{r+\omega}}\sum_{m=1}^{\infty}\frac{(-1)^m x^m e^{3im\theta}}{m^{s+q+r+u+j-\xi}}\frac{(i\theta)^{\xi}}{\xi!}\\
&\quad+2\sum_{j=0}^{u}\phi(u-j)\,\lambda_{u-j}\sum_{\xi=0}^{j}\sum_{\rho=0}^{p-1}\binom{\rho+j-\xi}{\rho}(-1)^{\rho}\sum_{\omega=0}^{p-1-\rho}\binom{\omega+r-1}{\omega}(-1)^{\omega}\\
&\quad\times 3^{-j+\xi-\rho-1}\binom{p+q-2-\rho-\omega}{q-1}\frac{(-1)^{p-1-\rho-\omega}}{2^{r+\omega}}\sum_{m=1}^{\infty}\frac{x^m}{m^{s+p+q+r+2j-\xi}}\frac{(i\theta)^{\xi}}{\xi!}.
\end{aligned}
$$

We can similarly write $J_2(\theta;x)$, $J_3(\theta;x)$ and $J_4(\theta;x)$, but these are omitted for the purpose of saving space.

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Next, setting $x = \pm ie^{-3i\theta/2}$ in (6.3) and moving the terms corresponding to $2l + 3m = 0$ of the first member on the left-hand side to the right-hand side, we have

$$\sum_{\substack{l\in\mathbb{Z},\, l\neq 0\\ m\geq 1\\ l+m\neq 0\\ l+2m\neq 0\\ l+3m\neq 0\\ 2l+3m\neq 0}} \frac{(-1)^{l+m}(\pm i)^m e^{i(2l+3m)\theta/2}}{l^p m^s (l+m)^q (l+2m)^r (l+3m)^u} + J_1(\theta;\pm ie^{-3i\theta/2}) + J_2(\theta;\pm ie^{-3i\theta/2}) + J_3(\theta;\pm ie^{-3i\theta/2}) + J_4(\theta;\pm ie^{-3i\theta/2}) = -\sum_{\substack{l,m=1\\ 2l=3m}}^{\infty} \frac{1}{(-l)^p m^s (-l+m)^q (-l+2m)^r (-l+3m)^u}. \tag{6.4}$$

Note that $(-1)^{l+m}(\pm i)^m = (\pm i)^{2l+3m}$. Hence, we can apply Lemma 5.3 with $d = v \in \mathbb{N}$ to (6.4) because we can see that the left-hand side of (6.4) is of the same form as the right-hand side of (5.5). Consequently, we obtain the equation given from (5.6). The first term on the left-hand side of the obtained equation is

$$\sum_{\substack{l\in\mathbb{Z},\, l\neq 0\\ m\geq 1\\ l+m\neq 0\\ l+2m\neq 0\\ l+3m\neq 0\\ 2l+3m\neq 0}} \frac{1}{l^p m^s (l+m)^q (l+2m)^r (l+3m)^u (2l+3m)^v}, \tag{6.5}$$

while the remaining terms on the left-hand side of the obtained equation can be expressed explicitly in terms of the Riemann zeta-function, which are $I_1 + \cdots + I_8$ in the statement of the theorem. On the other hand, we see that (6.5) is equal to

$$\begin{aligned}&\zeta_2(p,s,q,r,u,v;G_2) + (-1)^p\zeta_2(p,q,s,c,v,u;G_2) + (-1)^{p+q}\zeta_2(v,q,r,s,p,u;G_2)\\ &\quad + (-1)^{p+q+v}\zeta_2(v,r,q,s,u,p;G_2) + (-1)^{p+q+r+v}\zeta_2(u,r,s,q,v,a;G_2)\\ &\quad + (-1)^{p+q+r+u+v}\zeta_2(u,s,r,q,p,v;G_2).\end{aligned} \tag{6.6}$$

This can be shown by decomposing (6.5) by the same argument as in [9, Section 7]; or, since (6.5) coincides with $S(p,s,q,r,u,v),\mathbf{0};\{2\};G_2)$ (see (2.11)), (6.6) simply follows from (2.12). Thus we obtain the assertion of the theorem. □

Setting $(p,q,r,u,v) = (2a,b,2c-1,d,d)$ for $a,b,c,d \in \mathbb{N}$ in (6.1), we see that

$$\begin{aligned}&\zeta_2(2a,s,b,2c-1,d,d;G_2) + \zeta_2(2a,b,s,2c-1,d,d;G_2)\\ &\quad + (-1)^b\zeta_2(d,b,2c-1,s,2a,d;G_2) + (-1)^{b+d}\zeta_2(d,2c-1,b,s,d,2a;G_2)\\ &\quad - (-1)^{b+d}\zeta_2(d,2c-1,s,b,d,2a;G_2) - (-1)^b\zeta_2(d,s,2c-1,b,2a,d;G_2)\end{aligned} \tag{6.7}$$

is expressed in terms of $\zeta(s)$ and $\phi(s)$. As we noted above (see (6.6)), (6.7) coincides with

$$S((2a,s,b,2c-1,d,d),\mathbf{0};\{2\};G_2).$$

In particular when $s = b$, it is equal to $2\zeta_2(2a,b,b,2c-1,d,d);G_2)$ (see (3.2)). Therefore, we have the following.

COROLLARY 6.3. *For* $a, b, c, d \in \mathbb{N}$,

$$\zeta_2(2a, b, b, 2c-1, d, d; G_2) \in \mathbb{Q}[\{\zeta(j) \mid j \in \mathbb{N}_{\geq 2}\}]. \tag{6.8}$$

EXAMPLE 6.4. Putting $(p, q, r, u, v) = (2, 1, 1, 1, 1)$ in (6.1), we have

$$\begin{aligned}&\zeta_2(2, s, 1, 1, 1, 1; G_2) + \zeta_2(2, 1, s, 1, 1, 1; G_2) - \zeta_2(1, 1, 1, s, 2, 1; G_2)\\ &\quad + \zeta_2(1, 1, 1, s, 1, 2; G_2) - \zeta_2(1, 1, s, 1, 1, 2; G_2) + \zeta_2(1, s, 1, 1, 2, 1; G_2)\\ &\quad - \frac{1}{9}\zeta(2)\zeta(s+4) + \frac{109}{648}\zeta(s+6) = 0.\end{aligned}$$

Setting $s = 1$, we obtain a special case of (6.8), that is,

$$\zeta_2(2, 1, 1, 1, 1, 1; G_2) = \frac{1}{18}\zeta(2)\zeta(5) - \frac{109}{1296}\zeta(7), \tag{6.9}$$

which is (1.4) noted in Section 1. Similarly, we can compute

$$\zeta_2(4, 1, 1, 1, 1, 1; G_2) = \frac{1}{18}\zeta(4)\zeta(5) + \frac{145}{648}\zeta(2)\zeta(7) - \frac{19753}{46656}\zeta(9), \tag{6.10}$$

$$\zeta_2(2, 1, 1, 1, 2, 2; G_2) = -\frac{187}{324}\zeta(2)\zeta(7) + \frac{11149}{11664}\zeta(9), \tag{6.11}$$

$$\zeta_2(4, 2, 2, 1, 1, 1; G_2) = \frac{1}{18}\zeta(4)\zeta(7) + \frac{595}{648}\zeta(2)\zeta(9) - \frac{73201}{46656}\zeta(11), \tag{6.12}$$

$$\zeta_2(2, 1, 1, 5, 3, 3; G_2) = \frac{5}{4}\zeta(4)\zeta(11) + \frac{1043857}{23328}\zeta(2)\zeta(13) - \frac{41971423}{559872}\zeta(15), \tag{6.13}$$

$$\zeta_2(4, 2, 2, 1, 4, 4; G_2) = \frac{61441}{209952}\zeta(4)\zeta(13) + \frac{600677}{944784}\zeta(2)\zeta(15) - \frac{23172773}{17006112}\zeta(17), \tag{6.14}$$

$$\zeta_2(2, 4, 4, 3, 3, 3; G_2) = \frac{1}{8}\zeta(4)\zeta(15) + \frac{281221}{23328}\zeta(2)\zeta(17) - \frac{11177971}{559872}\zeta(19). \tag{6.15}$$

7. The parity result for the zeta-function of the root system G_2. We conclude this paper with a discussion on the parity result for the zeta-function of G_2.

It is well known that the double zeta values satisfy that

$$\sum_{m,n=1}^{\infty} \frac{1}{m^p(m+n)^q} \in \mathbb{Q}[\{\zeta(j+1) \mid j \in \mathbb{N}\}]$$

for $p, q \in \mathbb{N}$ with $q \geq 2$ and $2 \nmid (p+q)$, which was proved by Euler. The same situation holds for the zeta values of type A_2 (see [21]) and of type B_2 (see [22]):

$$\zeta_2(p, q, r; A_2),\ \ \zeta_2(t, u, v, w; B_2) \in \mathbb{Q}[\{\zeta(j+1) \mid j \in \mathbb{N}\}]$$

for $p, q, r, t, u, v, w \in \mathbb{N}$ with $2 \nmid (p+q+r)$ and $2 \nmid (t+u+v+w)$. These may be regarded as examples of 'parity results'. (In general, a 'parity result' means a property that some multiple zeta value whose weight and depth are of different parity can be written in terms of multiple zeta values of lower depth.) Does the same type of assertion hold for $\zeta_2(p, q, r, u, v, w; G_2)$? It seems that the answer is negative; in view of Example 4.1 (especially (4.25)), we find that the following modified statement is more plausible:

$$\zeta_2(p, q, r, u, v, w; G_2) \overset{?}{\in} \mathbb{Q}[\{\zeta(j+1),\ L(j, \chi_3) \mid j \in \mathbb{N}\}] \tag{7.1}$$

for $p, q, r, u, v, w \in \mathbb{N}$ with $2 \nmid (p + q + r + u + v + w)$.

In this direction, an interesting result was given in Okamoto's paper [19]. Inspired by the work of Nakamura [18] and Onodera [20], Okamoto proved (his Theorems 2.3 and 4.5) that the values of certain generalised double zeta-functions, including the case $\zeta_2(p, q, r, u, v, w; G_2)$ with $2 \nmid (p + q + r + u + v + w)$, can be expressed in terms of the Riemann zeta values and the values of Clausen-type functions, that is,

$$S_r(x) = \sum_{m \geq 1} \frac{\sin(2\pi mx)}{m^r} \quad \text{or} \quad C_r(x) = \sum_{m \geq 1} \frac{\cos(2\pi mx)}{m^r} \qquad (r \in \mathbb{N})$$

with $x = j/l \in \mathbb{Q}$ ($l \in \mathbb{N}$, $0 \leq j < l$). Moreover in his formula, in the case of G_2, only the cases $l = 1, 2, 3, 4, 6$ and 12 of Clausen-type functions appear. For $l = 1, 2, 3$ and 6, the values $S_r(j/l)$ and $C_r(j/l)$ can be written in terms of the values of $\zeta(s)$ and $L(s, \chi_3)$, similar to (4.23) and (4.24). Therefore, Okamoto's result implies that *if* $2 \nmid (p + q + r + u + v + w)$, *then the value* $\zeta_2(p, q, r, u, v, w; G_2)$ *can be written in terms of* $\zeta(s)$, $L(s, \chi_3)$, $S_r(j/l)$ *and* $C_r(j/l)$ *for* $l = 4, 12$ *and* $0 < j < l$ *with* $(j, l) = 1$. This may be regarded as a kind of 'generalised parity result'.

If we apply Okamoto's theorem [**19**] directly, we obtain a rather long expression of special values in terms of Clausen-type functions. But we have checked, using PARI/GP, that his expression actually agrees with our expression for (4.25), (6.9), (6.10), (6.11) and (6.12). To check other examples ((6.13), (6.14), (6.15)) we would require much more running time, so we did not check them.

Although only the values of $\zeta(s)$, $L(s, \chi_3)$ appear in all of our examples, we are not sure whether $S_r(j/l)$ or $C_r(j/l)$ ($l = 4, 12$; $0 < j < l$, $(j, l) = 1$) will appear or not (in other words, (7.1) would hold or not) in some other examples.

It seems that for the zeta-function of the root system G_2, the parity result is valid only in this generalised form. On the other hand, our Example 6.4 shows that sometimes the value $\zeta_2(p, q, r, u, v, w; G_2)$, with $2 \nmid (p + q + r + u + v + w)$, can be expressed only by the values of $\zeta(s)$. It is an interesting problem to determine when such a restricted form of the parity result holds.

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