Can. J. Math., Vol. XXXIII, No. 5, 1981, pp. 1097–1110

A CLASSIFICATION OF HOMOGENEOUS SURFACES

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Introduction. Throughout this paper a *surface* is a 2-dimensional (not necessarily compact) complex manifold. A surface X is *homogeneous* if a complex Lie group G of holomorphic transformations acts holomorphically and transitively on it. Concisely, X is homogeneous if it can be identified with the left coset space G/H, where H is a closed complex Lie subgroup of G. We emphasize that the assumption that G is a complex Lie group is an essential part of the definition. For example, the 2-dimensional ball $\mathbf{B}_2$ is certainly "homogeneous" in the sense that its automorphism group acts transitively. But it is impossible to realize $\mathbf{B}_2$ as a homogeneous space in the above sense.

The purpose of this paper is to give a detailed classification of the homogeneous surfaces. We give explicit descriptions of all possibilities. It turns out that except for the complement of the quadric in $\mathbf{P}_2$ (which has the affine quadric as universal cover) every non-compact homogeneous surface can be realized as a G-equivariant fiber space over a homogeneous Riemann surface, and it is useful to describe the 2-dimensional space in these terms.

The list of compact homogeneous surfaces has been known for some time (see [**13**]), and is easily stated: If X is a compact homogeneous surface, then it is either $\mathbf{P}_2$, $\mathbf{P}_1 \times \mathbf{P}_1$, a torus, a homogeneous Hopf surface, or the product of an elliptic curve with $\mathbf{P}_1$.

A particular type of homogeneous surface is one which has a compactification as a complex manifold to which the group action extends. More precisely, an *almost homogeneous surface* V is a compact surface whose automorphism group has an open orbit. This orbit turns out to be unique, and its complement is a proper analytic subvariety of V. In this sense the open orbit X has a nice compactification V. The almost homogeneous surfaces were classified by Potters [**12**]. Other than those which are homogeneous, V is one of the following: A Hopf surface with an abelian fundamental group; a topologically trivial $\mathbf{P}_1$-bundle over an elliptic curve; a Hirzebruch surface, possibly blown up at particular points. It has been noted [**4**] that a noncompact pseudoconcave homogeneous surface is nothing more than a Hirzebruch surface with its

Received February 21, 1980. The work of the first author was partially supported by NSF Grant No. 75.07086 while that of the second was supported by the Consiglio Nazionale delle Ricerche.

exceptional curve removed. This is not an immediate consequence of the definitions, because not all pseudoconcave surfaces can be compactified.

We will now give a rough outline of our classification. Let $X = G/H$ be a homogeneous surface. Let $G = R \rtimes S$ be a Levi-Malcev decomposition of G (i.e. R is the radical of G and S is a semi-simple part). We show that if R acts transitively on X, then, except in the case of trivial products, there is a solvable group M which also acts transitively on X, and additionally M has discrete isotropy. Thus, in this case $X = M/\Gamma$, where Γ is discrete. If M is abelian, then except for trivial cases, the only non-compact examples arise when Γ is a lattice of rank 3. These are topologically trivial $\mathbf{C}^*$-bundles over elliptic curves, and conversely any such bundle space is such an M/Γ. There is a unique non-abelian simply-connected group M of dimension 2. This group is easily described, and, except for trivial combinations of $\mathbf{C}$, $\mathbf{C}^*$, and elliptic curves, the resulting homogeneous space M/Γ is either a bundle of elliptic curves over $\mathbf{C}^*$ or a certain non trivial $\mathbf{C}^*$ bundle over $\mathbf{C}^*$. These bundles can be described using the detailed list of such Γ given in [**3**]. It is interesting to note that the former are not compactificable as almost homogeneous spaces.

If R does not act transitively, then, except for the case of a trivial $\mathbf{C}$- or $\mathbf{C}^*$-bundle over $\mathbf{P}_1$, some orbit of S is open. In this case X is one of the following: the affine quadric, $\mathbf{P}_2$ minus a quadric curve, a positive line bundle over $\mathbf{P}_1$, or any non-trivial $\mathbf{C}^*$-bundle over $\mathbf{P}_1$. Conversely, each of these is homogeneous.

Our paper is organized as follows: We gather the necessary definitions, preliminary facts, etc., in Section 1. In Section 2 we describe the group theoretically parallelizable case (i.e., M/Γ as above). The case in which the radical does not act transitively is treated in Section 3. In Section 4 we handle the solvable case. We summarize our results in the last section.

1. Preliminaries. If X is a complex manifold and G is a complex Lie group, then G is said to act *holomorphically* on X if there is a holomorphic map $G \times X \rightarrow X$, $(g, p) \mapsto g(p)$, so that $g(h(p)) = gh(p)$ and $e(p) = p$ for all $p \in X$, and for every $g \in G$ the map $p \rightarrow g(p)$ is an automorphism of X. In this paper we restrict our attention to connected surfaces X with a connected complex Lie group G acting holomorphically and transitively. For $p \in X$, the *isotropy group* H_p is defined as follows:

$$H_p := \{g \in G | g(p) = p\}.$$

The orbit map $G \rightarrow X$, $g \mapsto g(p)$, realizes G as the total space of a holomorphic fiber bundle with base X and fiber H_p. In this way, X is naturally identified with the left coset space G/H.

The *ineffectivity* I of the G-action on X is defined by

$$I := \{g \in G | g(p) = p \text{ all } p \in X\},$$

https://doi.org/10.4153/CJM-1981-084-x Published online by Cambridge University Press

and is a normal subgroup of G. If $I = \{e\}$ (resp. $I^0 = \{e\}$), we say that G acts *effectively* (resp. *almost effectively*) on X. The group G/I (resp. G/I^0) acts effectively (resp. almost effectively) on X. Thus, by replacing G with the quotient G/I, we may always assume that G acts effectively on X. We note that the universal covering $\tilde{G}$ of G also acts on X, and hence from now on by replacing G with $\tilde{G}$, we always assume that G is simply-connected and acts almost effectively on X.

Let G be as above then G possesses a unique maximal, connected, normal, solvable subgroup R which is called the *radical* of G. The group G is said to be *semi-simple* if $R = \{e\}$. The so-called Levi-Malcev Theorem (see [**2**]) asserts the existence of a connected, closed (not necessarily normal) semi-simple subgroup S of G so that $G = R \rtimes S$ (i.e. G is the semi-direct product of R with S). This is called a *Levi-Malcev* decomposition of G.

A Lie group G is said to act linearly on a subvariety X in $\mathbf{P}_n$ via the representation $\rho: G \rightarrow \mathrm{Aut}_{\mathcal{O}}\mathbf{P}_n$, if

$$\rho(G) \subset \{L \in \mathrm{Aut}_{\mathcal{O}}\mathbf{P}_n | L(X) \subset X\}.$$

If G is solvable, then $\rho(G)$ stabilizes a full flag of subspaces $P_n = L_n \supset L_{n-1} \supset \ldots \supset L_0$, where L_k is a linear, k-dimensional subspaces of $\mathbf{P}_n$ (i.e. $\rho(g)(L_k) \subset L_k$ for all $g \in G$). This is known as Lie's Flag Theorem.

If $\mathfrak{g}$ is the Lie algebra of G, then we have the adjoint representation ad: $G \rightarrow GL(\mathfrak{g})$. We assume that G and H are n- and k-dimensional respectively. Thus the Lie algebra $\mathfrak{h}$ of H is a subalgebra of $\mathfrak{g}$ and can therefore be considered as a point $\mathfrak{h}$ in the Grassman manifold $G_{k,n}$ of k-dimensional subspaces of $\mathfrak{g}$. Since $\mathrm{ad}(G) \subset GL(\mathfrak{g})$, we have the natural action of $\mathrm{ad}(G)$ on $G_{k,n}$. The $\mathrm{ad}(G)$-orbit of the point $\mathfrak{h}$ can be identified with G/N, where $N: = N_G(H^0)$ (i.e., the normalizer of the identity component of H in G). Of course $G_{k,n}$ can be realized as a submanifold of some $\mathbf{P}_m$ (e.g. via the Plücker embedding) so that the automorphisms of $G_{k,n}$ are restrictions of elements of $\mathrm{Aut}_{\mathcal{O}}\mathbf{P}_m$ which stabilize the embedded $G_{k,n}$. Thus we realize G acting linearly on G/N via the adjoint representation. We further note that $N_G(H^0) \supset H$, and consequently we have the normalizer fibration $G/H \rightarrow G/N$. There are two main advantages of this fibration.

1) G acts linearly on the base;

2) The fiber $N/H = N/H^0 / H/H^0$ is the quotient of a Lie group by a discrete group (i.e. group theoretically parallelizable).

If the base G/N is compact, then it is easily seen to be *rational* (i.e. the radical is contained in the normalizer, and it is realized as the quotient of a semi-simple part of G, S, by a parabolic subgroup). For more details about this and other discussion of the compact setting, we refer the reader to [**13**] or [**1**]. Although the above "definition" of a rational homogeneous space may sound somewhat mysterious, we only need these in dimensions 1 and 2 where they are $\mathbf{P}_1$ and $\mathbf{P}_1 \times \mathbf{P}_1$, $\mathbf{P}_2$ respectively.

2. The case of discrete isotropy. Throughout this section $X = G/H$, where G is 2-dimensional and H is discrete. Since $\pi_1(G) = 1$, it follows that either $G = (\mathbf{C}^2, +)$ or $G = \mathbf{C}^2$ where the group structure is defined by

$$(a, b)(a', b') = (a + a', e^a b' + b).$$

We will show that there is always a fibration $G/H \to G/J$ onto a homogeneous Riemann surface.

We begin by describing the case in which G is abelian. For our purposes the only interesting case is when H is a lattice of rank 3.

THEOREM 1. (Abelian) *Let $X = G/H$, where $G = (\mathbf{C}^2, +)$ and H is a lattice of rank 3, then X is naturally realizable as a topologically trivial, homogeneous $\mathbf{C}^*$-bundle over an elliptic curve. Conversely, every topologically trivial $\mathbf{C}^*$-bundle over an elliptic curve is such a G/H. This bundle is holomorphically trivial if and only if X possesses a non-constant analytic function.*

Proof. We identify G with $\mathbf{C}^2$ and $H = \langle (1, 0), (0, 1), (\alpha, \beta)\rangle_{\mathbf{Z}}$. Let $\mathbf{R}^3_H = \langle H\rangle_{\mathbf{R}}$ and let $\mathbf{C}_H$ be the maximal complex subspace of $\mathbf{R}^3_H$. Then

$$\mathbf{C}_H = \langle (\operatorname{Im} \alpha, \operatorname{Im} \beta)\rangle_{\mathbf{C}}.$$

We may assume that $e_1 := (1, 0)$ and $e_2 := (\operatorname{Im} \alpha, \operatorname{Im} \beta)$ are independent. In the basis $\{e_1, e_2\}$, we have

$$H = \langle (1, 0), (r_1, r_2), (s_1, s_2 + i)\rangle_{\mathbf{Z}},$$

where $r_j, s_j \in \mathbf{R}, j = 1, 2$. Let $A = \langle (1, 0)\rangle_{\mathbf{C}}$. Then

$$AH = \{(z, nr_2 + m(s_2 + i)) | z \in \mathbf{C}, n, m \in \mathbf{Z}\}$$

is a closed subgroup, and the fibration $G/H \to G/AH$ realizes X as a $\mathbf{C}^*$-bundle over an elliptic curve given by the lattice

$$\Gamma = \{nr_2 + m(s_2 + i) | n, m \in \mathbf{Z}\}$$

in the complex plane.

It is easy to check (see [**10**]) that the homogeneous $\mathbf{C}^*$-bundles over tori are necessarily topologically trivial. Furthermore, since topologically trivial bundles come from representations of the fundamental group of the base into the circle, such a bundle over an elliptic curve is always $\mathbf{C}^2$ modulo a lattice of rank 3.

Although the last statement in the theorem can be proved without reference to the group (see [**6**]), we find the following argument (which goes back to Remmert) more instructive. If $f \in \mathcal{O}(X)$, then, writing $H = \langle (1, 0), (0, 1), (\alpha, \beta) \rangle_{\mathbf{Z}}$, it follows that f has a Fourier-Series.

$$f(z) = \sum_{-\infty < n, m < \infty} a_{nm} \exp(2\pi i(nz_1 + mz_2)).$$

If $a_{nm} \neq 0$ for some $n, m \in \mathbf{Z}$, then, since $f(z_1 + \alpha, z_2 + \beta) = f(z_1, z_2)$, it follows that $n\alpha + m\beta = k \in \mathbf{Z}$. Thus $\chi: \mathbf{C}^2/H \rightarrow \mathbf{C}^*$, defined by

$$(z_1, z_2) \rightarrow \exp(2\pi i(nz_1 + mz_2)) \text{ on } \mathbf{C}^2,$$

is a character. Since the exact sequence

$$0 \rightarrow T \rightarrow \mathbf{C}^2/H \xrightarrow{\chi} \mathbf{C}^* \rightarrow 0$$

splits, the bundle is trivial. Thus, if X possesses a non-constant holomorphic function, then it is a product. The converse is obvious.

Remark. If X can be realized (even in the non-abelian case) as a G-equivariant $\mathbf{C}$-bundle over an elliptic curve T, then the bundle comes from a representation of $\pi_1(T)$ into the translation group of $\mathbf{C}$ (see [**8**]). If the bundle is non-trivial then, using the representation, one explicitly realizes X as $\mathbf{C}^2$ modulo a lattice of rank 2 (i.e. $\mathbf{C}^* \times \mathbf{C}^*$).

We now consider the non-abelian case. Since $\dim G = 2$, the following is a simplified version of a remark in [**7**]. We include the proof for the sake of completeness.

LEMMA. *Let G be the simply-connected, non-abelian complex Lie group of dimension* 2, *and let H be a discrete subgroup. Then there is a* 1-*dimensional closed subgroup J of G which contains H.*

Proof. If H is contained in the center $Z_G = \{(2\pi i n, 0) | n \in \mathbf{Z}\}$, then letting G' be the commutator subgroup of G, $Z_G . G' = : J$ suffices. Thus we may assume that H is not central. If H is abelian, then we consider for each $h \in H$ the map $\varphi_h: G \rightarrow G'$, $g \rightarrow ghg^{-1}h^{-1}$. Letting $Z_G(h)$ be the centralizer of h in G, we see that $\varphi_h^{-1}(e) = Z_G(h)$. Since $\dim G - \dim G' = 1$, it follows that

$$\dim_{\mathbf{C}} Z_G(h) \geqq 1 \text{ for all } h \in H.$$

But H is not central, and therefore some $Z_G(h) = : J$ is 1-dimensional. Obviously $J \supset H$.

It is now enough to consider the case in which $\Gamma : = H \cap G' \neq \{e\}$. Note that $G' = \mathbf{C}$, and define $\lambda: G \rightarrow \operatorname{Aut} G' = \operatorname{Aut} \mathbf{C}$, by $g \rightarrow \operatorname{int}_g$, where $\operatorname{int}_g(g') = g^{-1}g'g$. We observe that the automorphisms of G' which stabilize Γ form a discrete subgroup of $\operatorname{Aut} G'$. Thus $\lambda(H)$ is closed, and consequently $J : = HZ_G(G') = \lambda^{-1}(\lambda(H))$ is a closed, 1-dimensional subgroup.

THEOREM 2. (Non-abelian) *Let $X = G/H$, where G is the non-abelian, simply-connected complex Lie group of dimension 2, and where H is a discrete subgroup. If X is not a product of homogeneous Riemann surfaces, then $G/H \to G/G'H$ realizes X either as a bundle of elliptic curves over $\mathbf{C}^*$ or as a $\mathbf{C}^*$ bundle over $\mathbf{C}^*$. There is only one possibility for the latter case:*

$$\Gamma = \langle (\pi i, 0), (0, 2\pi i) \rangle_G.$$

In the former case H can be described in the following way.

Let $T_\tau = \{(0, n\omega_1 + m\omega_2) | n, m \in \mathbf{Z}\}$, where $\omega_1\omega_2^{-1} = \tau \in H$, let k be a fixed integer. Then H is one of the following:

1) $\langle (\pi i k, 0), T_\tau \rangle_G$, *with the further condition that k is odd,*

2) $\langle ((\pi/2)ik, 0), T_\tau \rangle_G$, *with the further condition that $\tau \equiv i (\mathrm{mod}\ SL_2(\mathbf{Z}))$, and k is odd;*

3) $\langle ((\pi i/3)k, 0), T_\tau \rangle_G$, *with the further condition that $\tau \equiv (1 + i\sqrt{3}/2)$ mod $(SL_2(\mathbf{Z}))$, and either $k \equiv \pm 1$ (mod 6) or $k \equiv \pm 2$ (mod 6).*

Proof. By the lemma, we have a fibration $X = G/H \to G/J$ whose base is 1-dimensional. We assume that X is not a product of homogeneous Riemann surfaces. Since G is non-abelian, X is not compact (Stokes' Theorem).

If H is abelian then it acts, up to a conjugation, as a group of translations, therefore, in the case in which rank H is either 1 or 2, X is a product.

The abelian subgroups of rank 3 are the following (see [**3**]):

(*) $H_\tau = \langle (2\pi i k, d), T_\tau \rangle$, where k, d, and T_τ are as in the statement of the theorem. Since $f(z, w) := \exp(2\pi i z)$ is H_τ-invariant, it follows from Theorem 1 that X is a product. This in fact proves that the bundle given by the lemma is trivial.

Hence the non-trivial bundles are given by the non-abelian discrete subgroups. The classification of these is exactly the list in the statement of the theorem (see [**3**]).

Remark. The non-trivial homogeneous elliptic curve bundles over $\mathbf{C}^*$ can not be compactified to almost homogeneous surfaces with the G-action extending. This follows in an elementary way from the classification of Potters' (see [**3**] for details).

3. The non-solvable case. The purpose of this section is to prove the theorem stated below. We begin with some notation. If $X = G/H$ and $G = R \rtimes S$ is a Levi-Malcev decomposition of G, then, providing Rp is closed for some $p \in X$, we may consider the *radical fibration*, $G/H \to G/RH$. If G is an algebraic group and H is an algebraic subgroup, we may consider a *maximal fibration* $G/H \to G/M$, where M is a maximal dimensional algebraic subgroup of G which contains H. We reserve this language for algebraic groups.

THEOREM. *Let $X = G/H$ be a non-compact homogeneous surface. Assume that the radical R of G does not act transitively on X. Let $G = R \rtimes S$ be a Levi-Malcev decomposition of G. Then, unless X is a holomorphically trivial $\mathbf{C}$- or $\mathbf{C}^*$-bundle over $\mathbf{P}_1$, some S-orbit is open, and X is one of the following homogeneous surfaces:*

1) *A non-trivial $\mathbf{C}^*$-bundle over $\mathbf{P}_1$, realized by the normalizer fibration $G/H \rightarrow G/N$;*

2) *A positive line bundle over $\mathbf{P}_1$, realized by the radical fibration $G/H \rightarrow G/RH$;*

3) *The affine quadric, which is an affine bundle over $\mathbf{P}_1$ realized by a maximal fibration $G/H \rightarrow G/M$.*

4) *The complement of the quadric curve in $\mathbf{P}_2$, in which case H is maximal, and G/H^0 is the affine quadric with $H/H^0 = \mathbf{Z}_2$.*

In all cases $S = SL_2(\mathbf{C})$, and in 3) and 4) $R = \{e\}$.

We note that the manifolds in 2) are just the Hirzebruch surfaces with their exceptional curves removed. Furthermore, the affine quadric is the only homogeneous affine bundle over $\mathbf{P}_1$ which is not a line bundle. It is of course a Stein submanifold of $\mathbf{C}^3$, and is realized as $SL_2(\mathbf{C})$ modulo diagonal matrices.

The proof of the theorem follows from a sequence of three lemmas. Recall that we always assume that G acts almost effectively on X and that $\pi_1(G) = 1$.

LEMMA 1. *Let $X = G/H$ be a non-compact, homogeneous surface, and assume that G is semi-simple. Then $G = SL_2(\mathbf{C})$, and H is an algebraic subgroup of G. If H is not maximal and M is a maximal proper algebraic subgroup of G which contains H, then M is parabolic, and the fibration*

$$G/H \rightarrow G/M$$

realizes X as either a non-trivial $\mathbf{C}^*$- or affine bundle over $\mathbf{P}_1$. In the latter case, X is the affine quadric, and H can be chosen to be the subgroup of diagonal matrices. Every non-trivial $\mathbf{C}^*$-bundle over $\mathbf{P}_1$ is homogeneous under the action of $SL_2(\mathbf{C})$, with isotropy

$$H_n: = \begin{pmatrix} \zeta_n & * \\ 0 & \zeta_n^{-1} \end{pmatrix},$$

where ζ_n is an n-th root of unity.

If H is maximal, then X is the complement of the quadric curve in $\mathbf{P}_2$, $H/H^0 = \mathbf{Z}_2$, and G/H^0 is the affine quadric.

Proof. Since there are no semi-simple groups of dimension two, the base G/N of the normalizer fibration is at least 1-dimensional. We note that the only 1-dimensional homogeneous space of a semi-simple group is $\mathbf{P}_1$. Thus, if $\dim_{\mathbf{C}} G/N = 1$, then the $G/H \rightarrow G/N$ realizes X as a bundle over $\mathbf{P}_1$, whose fiber is $\mathbf{C}^*$ or $\mathbf{C}$. We will describe these bundles later.

Realizing G as an algebraic group via the adjoint representation, we note that N is an algebraic subgroup. If $\dim_{\mathbf{C}} G/N = 2$, then, since H is an open subgroup of N, it is likewise an algebraic subgroup of G. In this case, we consider a maximal fibration $G/H \to G/M$, and note that either M is parabolic or G/M is Stein. (This is a consequence of the main theorem of [**11**] and, for example, Corollary 30.3 of [**9**].) If M is parabolic, then $G/M \cong \mathbf{P}_1$, because, if G/M were 2-dimensional, then $G/H \cong G/M$ (rational homogeneous spaces are simply-connected), and X would be compact.

If G/M is Stein, then it is 2-dimensional, because a semi-simple group can not act transitively on $\mathbf{C}$ or $\mathbf{C}^*$. Since G/H is a covering space of G/M, X is Stein, and, since N/H is finite it follows that G/N is Stein. The semi-simple group G acts linearly on G/N, and thus G/N is Zariski open in its closure V in $\mathbf{P}_n$. It is clear that G acts linearly on V. This action can be lifted to a minimal "equivariant desingularization" $\tilde{V}$ of V. (This is easy in dimension two, see [**4**].) Thus $\tilde{V}$ is an almost homogeneous compact surface, and G/N is an open subset of the open orbit of $\mathrm{Aut}_{\mathcal{O}} \tilde{V}$. Since G is semi-simple, the Albanese variety of $\tilde{V}$ is 0-dimensional. Furthermore $\tilde{V}$ is algebraic, and consequently it is a rational surface (see [**12**]).

Unless $\tilde{V} \cong \mathbf{P}_2$, the open orbit of $\mathrm{Aut}_{\mathcal{O}} \tilde{V}$ is a bundle over $\mathbf{P}_1$. (In fact $\tilde{V}$ is a Hirzebruch surface [**12**]). This violates the maximality of M. Thus it remains to consider the case when $\tilde{V} = \mathbf{P}_2$. We note that a Stein manifold of dimension greater than one has one "end" as a topological space. Thus $C := \mathbf{P}_2 \backslash (G/N)$ is a connected curve. If G should fix a point in $\mathbf{P}_2$, then we could blow it up, and obtain a Hirzebruch surface. This again violates the maximality of M. Thus C is a non-singular rational curve on which G acts transitively.

We note that C can not be a linear subspace of $\mathbf{P}_2$, because the semi-simple group would in this case have a fixed point $p \notin C$. Now let I be the ineffectivity of the G-action on C. Since I fixes every point of C and since C is not linear, I fixes every point of $\mathbf{P}_2$. Thus I is discrete. But $G/I \cong PSL_2(\mathbf{C})$, hence $G = SL_2(\mathbf{C})$.

Now, G/H^0 is also Stein, and thus $H^0 = L^{\mathbf{C}}$, where L is a 1-dimensional connected compact subgroup of $SL_2(\mathbf{C})$ (see [**11**]). Thus, by taking the appropriate conjugate, we may assume that H^0 is the subgroup of diagonal matrices. Thus G/H^0 is the affine quadric. But since H^0 is contained in a Borel subgroup, it is not maximal and thus H is not connected. An easy calculation shows that $N_G(H^0)/H^0 \cong \mathbf{Z}_2$. Hence $H = N_G(H^0)$, which is generated by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and the group H^0. In this case H is maximal, and X can be realized as

$$\mathbf{P}_2 \backslash \{[z_0:z_1:z_2] | z_0^2 + z_1^2 + z_2^2 = 0\}.$$

It remains to give more details in the case that H is not maximal. Recall that we have shown that in this case $G/H \to G/M$ realizes X as a bundle over $\mathbf{P}_1$ whose fiber is either $\mathbf{C}$ or $\mathbf{C}^*$. Since G is semi-simple, this can not be the trivial bundle. Now, a given $\mathbf{C}^*$-bundle over $\mathbf{P}_1$ is the principal bundle of some power of hyperplane section bundle $\mathscr{H}^n$, $n \in \mathbf{Z}\setminus\{0\}$. Since the Picard variety of $\mathbf{P}_1$ is trivial, the pullback of $\mathscr{H}^n$ by some $g \in SL_2(\mathbf{C})$ must be isomorphic to $\mathscr{H}^n$ itself. Thus the group of bundle equivariant automorphisms of $\mathscr{H}^n$ acts transitively on the base (i.e. $\mathbf{P}_1$). Call this complex Lie group $\hat{G}$. We may assume that $\pi_1(\hat{G}) = 1$, and let $\hat{G} = R \rtimes S$ be a Levi-Malcev decomposition. Thus S acts transitively on the base, and consequently $S = S' \times SL_2(\mathbf{C})$ where S' is some other semi-simple group, and S' is the ineffectivity of S-action on $\mathbf{P}_1$. Since S' acts on the fiber, we see that it is trivial, and thus $S = SL_2(\mathbf{C})$. The orbits of $SL_2(\mathbf{C})$ in $\mathscr{H}^n$ are either 1- or 2-dimensional, and, since $\mathbf{P}_1$ is simply-connected, any 1-dimensional orbit is a section. If $n > 0$, then we can compactify $\mathscr{H}^n$ by adding an ∞-section which may be blown down to a point. The resulting variety is algebraic and $SL_2(\mathbf{C})$ acts linearily on it, and fixes the point which corresponds to the ∞-section. Thus $SL_2(\mathbf{C})$ fixes a "complementary hyperplane" which cuts the variety in a curve, and hence $SL_2(\mathbf{C})$ has a 1-dimensional obrit in $\mathscr{H}^n$. If $n < 0$, then the 0-section of $\mathscr{H}^n$ is exceptional, and consequently is fixed by $SL_2(\mathbf{C})$. Since $\mathscr{H}^n$ is not trivial, there are no other 1-dimensional orbits. In summary, for all $n \in \mathbf{Z}\setminus\{0\}$, the group $SL_2(\mathbf{C})$ has one open orbit and one 1-dimensional orbit in $\mathscr{H}^n$. The open orbit is the associated $\mathbf{C}^*$-bundle space, and hence every $\mathbf{C}^*$-bundle over $\mathbf{P}_1$, is homogeneous under a $SL(\mathbf{C})$ action. One can easily check that the isotropy can be realized as in the statement of the lemma.

Note that the above argument shows that no semi-simple group acts transitively on the line bundle space $\mathscr{H}^n$, $n \in \mathbf{Z}$. So, in order to finish the proof, we need only to classify the homogeneous $\mathbf{C}$-bundles over $\mathbf{P}_1$ which are not line bundles. We observe that if $X = G/H \to G/M \cong \mathbf{P}_1$ is such a fibration (for arbitrary $G = R \rtimes S$), then S acts transitively on X. Otherwise, a 1-dimensional S-orbit would be a section. Hence, a classification for semi-simple groups is enough for the general case.

If $X = G/H \underset{\mathbf{C}}{\to} G/M = \mathbf{P}_1$ is as above and G is semi-simple, then $G = \bar{G} \times SL_2(\mathbf{C})$, where $\bar{G}$ is the ineffectivity of the G-action on $\mathbf{P}_1 = G/M$. But $\bar{G}$ is semi-simple and acts on the fibers (i.e. $\mathbf{C}$), and is therefore trivial. Thus $G = SL_2(\mathbf{C})$. From the homotopy sequence, $\pi_1(H) = \mathbf{Z}$. From this, an easy calculation shows that H must be conjugate to the group of diagonal matrices in $SL_2(\mathbf{C})$. Thus X is the affine quadric.

Remark. If $n < 0$, then the arguments above show that $\mathscr{H}^n$ is not homogeneous. We note that if $s \in \Gamma(\mathbf{P}_1, \mathscr{H}^n)$, then translation by s (i.e. $p \to p + s(\pi(p))$ is a well-defined automorphism. Since $SL_2(\mathbf{C})$ acts transitively on the complement of a "O-section", and since $\Gamma(\mathbf{P}_1, \mathscr{H}^n) \neq (0)$ for $n > 0$, it follows that $\mathscr{H}^n$ is homogeneous for $n > 0$.

LEMMA 2. *Let $X = G/H$ be a non-compact homogeneous surface. Let*

(*) $G/H \rightarrow G/N$

be the normalizer fibration, and assume that $G/N \cong \mathbf{P}_1$. Let $G = R \rtimes S$ be a Levi-Malcev decomposition of G. Then, unless (*) is a holomorphically trivial $\mathbf{C}$- or $\mathbf{C}^*$-bundle, S acts transitively on X.

Proof. If S does not act transitively, then it has at least one 1-dimensional orbit in X, and in this case (*) is a line bundle. This will lead to a contradiction. Let p be in a 1-dimensional S-orbit. We may assume that the bundle is non-trivial, and consequently S acts transitively on the complement of this orbit. We may assume that H is the isotropy group at p, and letting F be the fiber of (*) through p, N is just the stabilizer of F in G. Since N normalizes H, it follows that H fixes every point in F. But $H \supset S \cap H = S \cap N$, and the latter acts transitively on $F \backslash \{p\}$. This is the desired contradiction.

LEMMA 3. *Let $X = G/H$ be a non-compact homogeneous surface with normalizer fibration $G/H \rightarrow G/N$. Assume that the base G/N is 2-dimensional, and that the R-orbits in X are 1-dimensional. Let $G = R \rtimes S$ be a Levi-Malcev decomposition, and assume that S does not act transitively on X. Then, unless $X = \mathbf{C}^* \times \mathbf{P}_1$, $H = N$ and the radical fibration $G/H \rightarrow G/RH$ realizes X as a line bundle over $\mathbf{P}_1$.*

Proof. Since G is acting linearly on G/N in $\mathbf{P}_n$, there exists an R-stable flag, $\mathbf{P}_n = L_n \supset L_{n-1} \supset \ldots \supset L_0 = (p)$. If there exists a k so that $G/N \subset L_k \backslash L_{k-1} \cong \mathbf{C}^k$, then G/N is holomorphically separable. Since every 1-dimensional S-orbit is compact, and since the R-orbits are 1-dimensional, it follows that in this case S would act transitively on G/N. Then $V = L_k \cap G/N$ is a 1-dimensional, closed subvariety of G/N. Hence for $p \in V$, it follows that Rp is closed. Thus all R-orbits are closed, and we may consider the radical fibration $G/H \rightarrow G/RH$. We may assume $X \neq \mathbf{C}^* \times \mathbf{P}_1$. Therefore the fiber RH/H is $\mathbf{C}$, because S would act transitively on a non-trivial $\mathbf{C}^*$-bundle. Since $G/RH \cong \mathbf{P}_1$, it follows that $\pi_1(X) = 1$, and $H = N$. Furthermore, since S must have a 1-dimensional orbit in X, the radical fibration realizes X as a positive line bundle. (See the above remark.)

Proof of the theorem. We consider the various cases of the normalizer fibration. Since R doesn't act transitively on X, it follows that X is not group theoretically parallelizable. Hence the base G/N is either 1- or 2-dimensional. Lemma 3 and Lemma 1 handle the 2-dimensional case.

Suppose $G/N = \mathbf{C}, \mathbf{C}^*$. Then S fixes every point of G/N, and therefore acts on the fiber N/H. But $N/H \neq \mathbf{P}_1$. Hence S fixes every point of the

https://doi.org/10.4153/CJM-1981-084-x Published online by Cambridge University Press

fiber, and consequently $S = \{e\}$, contrary to assumption. Thus G/N is compact, and $G/N \cong \mathbf{P}_1$. Applying Lemma 2 and Lemma 1, the proof is finished.

4. The solvable case. The purpose of this section is to prove the following:

THEOREM. *Let $X = G/H$ be a non-compact homogeneous surface, and assume that G is solvable. Then X is either a product of homogeneous Riemann surfaces, or there exists a 2-dimensional solvable group $\hat{G}$ which acts transitively on X.*

(A detailed description of this group theoretically parallelizable situation is given in Section 2.)

The proof goes roughly as follows: If G' acts transitively, then the methods of [**5**] are sufficient. If $G' \subset H$, then X is an abelian group (an easy case). The main difficulties arise when the G'-orbits are 1-dimensional. But in this case G' is abelian (see Lemma 2). Using this information, and considering the fibration $G/H \to G/N_G(H \cap G')$, the proof is completed by elementary arguments.

We begin with two lemmas.

LEMMA 1. *If $X = G/H$ is a non-compact homogeneous surface, H is not discrete, G is nilpotent, then either $H^0 \supset G'$ or X is a product of homogeneous Riemann surfaces.*

Proof. We note that $N = N_G(H^0)$ is connected, and $\dim_{\mathbf{C}} N > \dim_{\mathbf{C}} H$ [**5**]. Thus G/N is simply-connected and is at most 1-dimensional. If $N = G$, then $H^0 \triangleleft G$. Since G/H^0 is both 2-dimensional and nilpotent, it is abelian. Thus $H^0 \supset G'$. If $\dim_{\mathbf{C}} G/N = 1$, then $G/N = \mathbf{C}$. In this case the bundle $G/H \to G/N$ is trivial and X is a product.

LEMMA 2. *Let $X = G/H$ be a non-compact homogeneous surface, and assume that G is solvable. If the orbits of the commutator subgroup G' are 1-dimensional, then G' is abelian.*

Proof. Let $p \in X$, and note that the orbit $G'p$ is either $\mathbf{C}$ or $\mathbf{C}^*$. Let G'' be the commutator subgroup of G'. If $\hat{H} := \{g \in G' | g(p) = p\}$, then, since G'' is connected and the ineffectivity is discrete, it is enough to show that $G'' \subset \hat{H}$. If $\hat{H}q = q$ for all $q \in G'p$, then $\hat{H}$ is ineffective and $G'/\hat{H}$ is an abelian group. Thus $\hat{H} \supset G''$. If $\hat{H}q$ is open for some $q \in G'p$, and $I := \{g \in G' | g(q) = q\}$, then every element of $I \cap \hat{H}$ fixes two points of $G'p$. Thus $I \cap \hat{H}$ is ineffective, and $G'/I \cap \hat{H}$ is a nilpotent Lie group of dimension 2. Since the only non-abelian group of this dimension is not nilpotent, $G'/I \cap \hat{H}$ is abelian, and therefore $G'' \subset I \cap \hat{H} \subset \hat{H}$.

Proof of the theorem. The proof is by induction on $\dim_{\mathbf{C}} G$. If $\dim_{\mathbf{C}} G = 2$, then let $\tilde{G} = G$. We now assume that $\dim_{\mathbf{C}} G = n > 2$ and consider the fibration

$$G/H \xrightarrow{\pi} G/N_G(H \cap G').$$

We only need the case where $G_p' = G'/G' \cap H$. Since the abelian case is clear, it follows from Lemma 1 that we may assume that the G'-orbits are 1-dimensional. Thus, by Lemma 2, G' is abelian and $G/N_G(H \cap G)$ is at most 1-dimensional. We complete the proof by considering two cases, depending on the dimension of the base.

Suppose that $\dim_{\mathbf{C}} G/N_G(H \cap G') = 1$. If $G' \cap H = H^0$, then $H^0 \lhd G$. In this case $\tilde{G}: = G/H^0$ is the desired group, and thus we assume that $G' \cap H$ is a proper subgroup of G'. We note that

$$N_G(G' \cap H)^0 = (G'H)^0.$$

Consider the exact sequence

$$0 \to G' \to G \xrightarrow{\varphi} G/G' = (\mathbf{C}^n, +) \to 0.$$

Thus $\varphi(H^0)$ is 1-codimensional. We pick a (closed, normal) complementary subgroup $B \subset G/G'$. Thus $\hat{G}: = \varphi^{-1}(B)$ is a closed, normal subgroup of G. Since the orbit of $\hat{G}$ of the point in G/H which corresponds to the coset H is open, and since $\hat{G} \lhd G$, it follows that $\hat{G}$ acts transitively on G/H. If $\dim_{\mathbf{C}} \hat{G} < \dim_{\mathbf{C}} G$, then the proof follows by induction. If $\hat{G} = G$, then G/G' is 1-dimensional. But in this case $\varphi(H^0) = \{0\}$. Thus $H^0 \subset G'$, and $N_G(H \cap G')^0 = G'$. Let $\mathfrak{g}$, $\mathfrak{g}'$ and $\mathfrak{h}$ be the Lie algebras of G, G', and H respectively. Let $\mathfrak{a} = \langle \mathbf{a} \rangle_{\mathbf{C}}$ be a 1-dimensional subspace of $\mathfrak{g}$ which has non-trivial image in $\mathfrak{g}/\mathfrak{g}'$. Define the map f_a: $\mathfrak{g}' \to \mathfrak{g}'$ by $x: \to [\mathbf{a}, x]$. Let x_0 be an eigenvector for f_a (i.e. $[\mathbf{a}, x_0] = \lambda x_0$). Then $\tilde{\mathfrak{g}}: = \langle \mathbf{a}, x_0 \rangle_{\mathbf{C}}$ is a Lie subalgebra of $\mathfrak{g}$ with corresponding (2-dimensional) group $\tilde{G}$.

Let $\mathfrak{h}$ be the Lie algebra of H^0. Since G acts almost effectively on X, it follows that $\bigcap_{g \in G} \mathrm{ad}(g)(\mathfrak{h}) = \{0\}$. Thus there exist $g \in G$ with $x_0 \notin \mathrm{ad}(g)(\mathfrak{h})$. Thus the 1-parameter group corresponding to x_0 acting on the point $q: = g(p)$ has 1-dimensional orbit in the fiber at q. Since $N_G(G' \cap H)^0 = G'$, and since $\mathbf{a}$ has non-trivial projection in $\mathfrak{g}/\mathfrak{g}'$, the 1-parameter group corresponding to $\mathbf{a}$ acts transitively on the base $G/N_G(G' \cap H)$. Thus $\tilde{G}q$ is open.

We must now do some detailed analysis in order to show that, when X is not realizable by this fibration as a product, $\tilde{G}$ acts transitively. If the base of $G/H \to G/N_G(H \cap G)$ is $\mathbf{C}$, then the bundle is trivial. Thus we may assume that the base is $\mathbf{C}^*$ or an elliptic curve.

We point out that if this fibration realizes X as a $\mathbf{C}$-bundle over an elliptic curve T, then either it is trivial or $\tilde{G}$ acts transitively. To show this we first note that if $\tilde{G}$ fixes a point in T, then it fixes every point in T. But for some $q \in X$, $\tilde{G}q$ is open. Thus $\tilde{G}$ acts transitively on T, and the restriction of the fibration realizes $\tilde{G}q$ as a $\mathbf{C}$- or $\mathbf{C}^*$-bundle over T. Thus it is enough to show in the latter case the original bundle is trivial. It is easy to check that $X\backslash\tilde{G}q$ is itself a homogeneous one to one cover of T, and is therefore a section which we consider as the O-section. Since every homogeneous $\mathbf{C}^*$-bundle over T is topologically trivial, the original fibration realizes X as a topologically trivial line bundle over T. Since X is homogeneous, this bundle is analytically trivial (we can move the O-section).

We now show that if the fiber of $G/H \rightarrow G/N_G(H \cap G')$ is either $\mathbf{C}^*$ or an elliptic curve, and the base B is likewise, then $\tilde{G}$ acts transitively. As above, we note that $\tilde{G}$ acts transitively on B. Letting q be as above, and F the fiber through q, it follows that the orbit of q via the stabilizer of F in $\tilde{G}$ is open in F. Since F is either $\mathbf{C}^*$ or an elliptic curve, this orbit is the entire fiber, and therefore $\tilde{G}q = X$.

Suppose that $G = N_G(H \cap G')$. We begin by showing that in this case $\dim_{\mathbf{C}} G' = 1$. Note first that $H \cap G' \lhd G$ and $H \cap G' \subset H$. Thus $H \cap G'$ is ineffective on X, and is consequently discrete. Since $G'p = G'/H \cap G'$ is 1-dimensional, it follows that $\dim_{\mathbf{C}} G' = 1$. It remains to construct $\tilde{G}$ in this case.

Since $\pi_1(G) = 1$, it follows that $G/G' = (\mathbf{C}^n, +)$. We note that $\dim_{\mathbf{C}} H = n - 1$. Letting $\varphi: G \rightarrow G/G'$ be the quotient map, we see that $\varphi(H^0)$ is a proper subgroup of $(\mathbf{C}^n, +)$. Let A be a 1-dimensional closed (normal) subgroup of G/G' which is transversal to $\varphi(H^0)$ at $\{0\}$. Then $\tilde{G}: = \varphi^{-1}(A)$ is a normal, closed subgroup of G. By construction $\tilde{G}p$ is open. Thus $\tilde{G}$ acts transitively on X.

5. Concluding remarks. Although the proof of the classification is complete, for the convenience of the reader we put the pieces together in one place. In Section 2 we classify the non-compact homogeneous surfaces $X = G/H$ when $\dim_{\mathbf{C}} G = 2$. (See Theorem 2.1, 2.2.) In Section 3 we provide a list of such $X = G/H$ when the radical of G does not act transitively (see Theorem 3.1). Finally, in Section 4 we point out that if X is not a product of homogeneous Riemann surfaces, and a solvable complex Lie group G acts transitively on X, then there is a 2-dimensional solvable group $\tilde{G}$ which also acts transitively. Thus we may refer to Section 2.

In summary, a complete list of non-compact homogeneous surfaces is the following: 1) Products of homogeneous Riemann surfaces; 2) Those surfaces which appear in Theorem 3.1; 3) Topologically trivial $\mathbf{C}^*$-bundles over elliptic curves (which are not analytically trivial); 4) Non-trivial elliptic curve bundles over $\mathbf{C}^*$ or a certain $\mathbf{C}^*$-bundle over $\mathbf{C}^*$ which is in fact a complexification of the Klein bottle. These are given by the non-abelian groups in Theorem 2.2.

In closing, we note that carrying out a similar project for 3-dimensional homogeneous manifolds would be much more difficult, because the group $SL_2(\mathbf{C})$ would play a big role in the case of discrete isotropy.

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