

## Finite quasimple groups acting on rationally connected threefolds

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### *Abstract*

We show that the only finite quasi-simple non-abelian groups that can faithfully act on rationally connected threefolds are the following groups:  $\mathfrak{A}_5$ ,  $\mathrm{PSL}_2(\mathbf{F}_7)$ ,  $\mathfrak{A}_6$ ,  $\mathrm{SL}_2(\mathbf{F}_8)$ ,  $\mathfrak{A}_7$ ,  $\mathrm{PSp}_4(\mathbf{F}_3)$ ,  $\mathrm{SL}_2(\mathbf{F}_7)$ ,  $2.\mathfrak{A}_5$ ,  $2.\mathfrak{A}_6$ ,  $3.\mathfrak{A}_6$  or  $6.\mathfrak{A}_6$ . All of these groups with a possible exception of  $2.\mathfrak{A}_6$  and  $6.\mathfrak{A}_6$  indeed act on some rationally connected threefolds.

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### *1. Introduction*

The complex projective plane  $\mathbb{P}^2$  and projective space  $\mathbb{P}^3$  are among the most basic objects of geometry. They provide motivation for the study of two exceptionally complicated objects, the groups  $\mathrm{Cr}_2(\mathbb{C})$  and  $\mathrm{Cr}_3(\mathbb{C})$  of their birational transformations, known as the *plane Cremona group* and the *space Cremona group*, respectively. The group  $\mathrm{Cr}_2(\mathbb{C})$  has been studied intensively over the last two centuries, and many facts about it were established. The structure of the group  $\mathrm{Cr}_3(\mathbb{C})$  is much more complicated and mysterious. It still resists most attempts to study its global structure.

One approach to studying Cremona groups is by means of their finite subgroups. An almost complete classification of finite subgroups of the plane Cremona group  $\mathrm{Cr}_2(\mathbb{C})$  was

obtained Dolgachev and Iskovskikh in [DI09] (see also [Bla11, Tsy11, Tsy13] for further developments). For example, this classification implies that there are exactly three isomorphism classes of non-abelian simple finite subgroups of  $\mathrm{Cr}_2(\mathbb{C})$ , namely those of  $\mathfrak{A}_5$ ,  $\mathrm{PSL}_2(\mathbb{F}_7)$  and  $\mathfrak{A}_6$ .

Recent achievements in three-dimensional birational geometry allowed Prokhorov to prove

**THEOREM 1.1** ([Pro12, theorem 1.1]). *There are exactly six isomorphism classes of non-abelian simple finite subgroups of  $\mathrm{Cr}_3(\mathbb{C})$ , given by*

$$\mathfrak{A}_5, \mathrm{PSL}_2(\mathbb{F}_7), \mathfrak{A}_6, \mathrm{SL}_2(\mathbb{F}_8), \mathfrak{A}_7, \text{ and } \mathrm{PSp}_4(\mathbb{F}_3).$$

This classification became possible thanks to the general observation that a birational action of a finite group  $G$  on projective space can be regularized, that is, replaced by a regular action of this group on some more complicated rational threefold. Thus, instead of studying finite subgroups in the space Cremona group, one can consider a more general (and perhaps more natural) problem:

*Question 1.2.* What are the isomorphism classes of finite groups acting faithfully on rationally connected threefolds?

The classical technique to study this (hard) problem goes as follows. Let  $X$  be a rationally connected threefold faithfully acted on by a finite group  $G$ . Taking the  $G$ -equivariant resolution of singularities and applying the  $G$ -equivariant Minimal Model Program, we can replace  $X$  by a  $G$ -Mori fiber space. Thus, we may assume that  $X$  has terminal singularities, every  $G$ -invariant Weil divisor on  $X$  is  $\mathbb{Q}$ -Cartier, and there exists a  $G$ -equivariant surjective morphism

$$\phi: X \longrightarrow Z$$

whose general fibers are Fano varieties, and the morphism  $\phi$  is minimal in the following sense:  $\mathrm{rk} \mathrm{Pic}(X/Z)^G = 1$ . If  $Z$  is a point, then  $X$  is a Fano threefold, so that we say that  $X$  is a  $G\mathbb{Q}$ -Fano threefold. Similarly, if  $Z = \mathbb{P}^1$ , then  $X$  is fibered in del Pezzo surfaces, and we say that  $\phi$  is a  $G$ -del Pezzo fibration. Finally, if  $Z$  is a rational surface, then the general geometric fiber of  $\phi$  is  $\mathbb{P}^1$ , and  $\phi$  is said to be a  $G$ -conic bundle. In this case, we may assume that both  $X$  and  $Z$  are smooth due to a result of Avilov [Avi14]. A priori, the threefold  $X$  can be non-rational. However, if  $X$  is rational, then any birational map  $X \dashrightarrow \mathbb{P}^3$  induces an embedding

$$G \hookrightarrow \mathrm{Cr}_3(\mathbb{C}).$$

Vice versa, every finite subgroup of  $\mathrm{Cr}_3(\mathbb{C})$  arises in this way. Thus, keeping in mind that every smooth cubic threefold is non-rational, we see that Theorem 1.1 follows from the following (more explicit) result:

**THEOREM 1.3** ([Pro12, theorem 1.5]). *Let  $X$  be a Fano threefold with terminal singularities, and let  $G$  be a finite non-abelian simple subgroup in  $\mathrm{Aut}(X)$  such that  $\mathrm{rk} \mathrm{Cl}(X)^G = 1$ . Suppose also that  $G$  is not isomorphic to  $\mathfrak{A}_5$ ,  $\mathrm{PSL}_2(\mathbb{F}_7)$  or  $\mathfrak{A}_6$ . Then the following possibilities hold:*

- (i)  $G \simeq \mathfrak{A}_7$ , and  $X$  is the unique smooth intersection of a quadric and a cubic in  $\mathbb{P}^5$  that admits a faithful action of the group  $\mathfrak{A}_7$ ;
- (ii)  $G \simeq \mathfrak{A}_7$ , and  $X$  is  $\mathbb{P}^3$ ;
- (iii)  $G \simeq \text{PSp}_4(\mathbf{F}_3)$ , and  $X$  is  $\mathbb{P}^3$ ;
- (iv)  $G \simeq \text{PSp}_4(\mathbf{F}_3)$ , and  $X$  is the Burkhardt quartic in  $\mathbb{P}^4$ ;
- (v)  $G \simeq \text{SL}_2(\mathbf{F}_8)$ , and  $X$  is the unique smooth Fano threefold of Picard rank 1 and genus 7 that admits a faithful action of the group  $\text{SL}_2(\mathbf{F}_8)$ ;
- (vi)  $G \simeq \text{PSL}_2(\mathbf{F}_{11})$ , and  $X$  is the Klein cubic threefold in  $\mathbb{P}^4$ ;
- (vii)  $G \simeq \text{PSL}_2(\mathbf{F}_{11})$ , and  $X$  is the unique smooth Fano threefold of Picard rank 1 and genus 8 that admits a faithful action of the group  $\text{PSL}_2(\mathbf{F}_{11})$  (which is non-equivariantly birational to the Klein cubic threefold).

In this text, we extend the study of simple groups to quasi-simple groups.

*Definition 1.4.* A group is said to be *quasi-simple* if it is perfect, that is, it equals its commutator subgroup, and the quotient of the group by its center is a simple non-abelian group.

Taking the quotient by the center, which is again a rationally connected threefold, it follows from Theorem 1.3 that the only finite quasi-simple non-simple group that can (possibly) faithfully act on rationally connected threefolds are  $2.\mathfrak{A}_5$  or

$$\text{SL}_2(\mathbf{F}_7), \text{SL}_2(\mathbf{F}_{11}), \text{Sp}_4(\mathbf{F}_3), n.\mathfrak{A}_6, n.\mathfrak{A}_7 \text{ with } n = 2, 3, 6. \tag{1.4.1}$$

As the group  $2.\mathfrak{A}_5$  is a subgroup of  $\text{SL}_2(\mathbb{C})$ , there are many ways to embed it into  $\text{Cr}_2(\mathbb{C})$  (see [Tsy13]), and hence in  $\text{Cr}_3(\mathbb{C})$ . However, none of the groups of (1.4.1) embeds in  $\text{Cr}_2(\mathbb{C})$  (see Theorem 2.5). Some of them indeed act on rationally connected threefolds. The goal of this paper is to prove the following result:

**THEOREM 1.5.** *Every finite quasi-simple non-simple group that faithfully acts on a rationally connected threefold is isomorphic to one of the following groups*

$$\text{SL}_2(\mathbf{F}_7), 2.\mathfrak{A}_5, 2.\mathfrak{A}_6, 3.\mathfrak{A}_6, \text{ and } 6.\mathfrak{A}_6.$$

Moreover, the groups  $2.\mathfrak{A}_5$  and  $3.\mathfrak{A}_6$  act faithfully on rational threefolds, and the group  $\text{SL}_2(\mathbf{F}_7)$  acts faithfully on rationally connected threefolds.

Unfortunately, we do not know whether the groups  $2.\mathfrak{A}_6$  and  $6.\mathfrak{A}_6$  can act on a rationally connected threefold or not (see Appendix B for a discussion), and do not know if  $\text{SL}_2(\mathbf{F}_7)$  can act on a rational threefold.

We now give examples that prove the existence part of Theorem 1.5 (we omit the case of  $2.\mathfrak{A}_5$ , already explained above).

*Example 1.6.* Let  $G = 3.\mathfrak{A}_6$  act on  $V := \mathbb{C}^3$  and let  $\phi(x_1, x_2, x_3)$  be the invariant of degree 6 (unique up to scalar multiplication). Then we have the following induced actions:

- (i) on  $\mathbb{P}^3 = \mathbb{P}(V \oplus \mathbb{C})$ ,
- (ii) on the hypersurface  $X_6 \subset \mathbb{P}(1^3, 2, 2)$  given by  $\phi + y_1^3 + y_2^3 = 0$ ;

- (iii) on the hypersurface  $X_6 \subset \mathbb{P}(1^3, 2, 3)$  given by  $z^2 + y^3 + \phi = 0$ ;
- (iv) on the hypersurface  $X_6 \subset \mathbb{P}(1^4, 3)$  given by  $\phi + x_4^6 = y^2$ ;
- (v) on  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(d) \oplus \mathcal{O}_{\mathbb{P}^2})$  where  $d \geq 1$  is not a multiple of 3.

*Example 1.7.* The group  $\mathrm{SL}_2(\mathbf{F}_7)$  has an irreducible four-dimensional representation, which makes it act faithfully on  $\mathbb{P}^4$  and on  $\mathbb{P}(1, 1, 1, 1, 3)$ .

- (i) The weighted projective space  $\mathbb{P}(1, 1, 1, 1, 3)$  contains a  $\mathrm{SL}_2(\mathbf{F}_7)$ -invariant sextic hypersurface (see [Edg47, MS73]). This hypersurface, that we denote by  $X$ , is unique. In appropriate quasihomogeneous coordinates, the threefold  $X$  is given by

$$w^2 = 8x^6 - 20x^3yzt - 10x^2y^3z - 10x^2yt^3 - 10x^2z^3t - 10xy^3t^2 - 10xy^2z^3 - 10xz^2t^3 - y^5t - 15y^2z^2t^2 - yz^5 - zt^5,$$

where  $x, y, z, t$  are coordinates of weight 1, and  $w$  is a coordinate of weight 3. One can check that  $X$  is smooth, so that  $X$  is a smooth Fano threefold with  $\mathrm{Pic}(X) = \mathbb{Z} \cdot K_X$  and  $-K_X^3 = 2$ . Note that  $X$  is non-rational (see [Isk80b]).

- (ii) The group  $\mathrm{SL}_2(\mathbf{F}_7)$  also acts on the smooth quartic  $X_4 \subset \mathbb{P}^4$  given by  $y^4 = \phi_4$ , where  $\phi_4(x_1, x_2, x_3, x_4)$  is an invariant of degree 4. This variety is also non-rational [IM71].

The structure of the paper is as follows. In Section 2, we prove or recall some preliminary results we will use throughout the remainder of the paper. In Section 3, we begin with an analysis of Fano threefolds with at worst canonical Gorenstein singularities. In Section 4, we prove the main theorem except for the case of  $3.\mathfrak{A}_7$ , which is handled in Section 5.

The appendices are not needed for the proof of the main theorem, but are likely of independent interest. In Appendix A, we introduce the *Amitsur subgroup*: a new equivariant birational invariant inspired from arithmetic geometry. The Amitsur subgroup is used in Appendix B, where we prove that the group  $6.\mathfrak{A}_6$  does not act non-trivially on a conic bundle over a surface (see Theorem B.6).

## 2. Preliminaries

### 2.1. Notation

Throughout this paper the ground field is supposed to be the field of complex numbers  $\mathbb{C}$ . We employ the following standard notations used in the group theory:

- (i)  $\mu_n$  denotes the multiplicative group of order  $n$  (in  $\mathbb{C}^*$ );
- (ii)  $\mathfrak{A}_n$  denotes the alternating group of degree  $n$ ;
- (iii)  $\mathrm{SL}_n(\mathbf{F}_q)$  (resp.  $\mathrm{PSL}_n(\mathbf{F}_q)$ ) denotes the special linear group (resp. projective special linear group) over the finite field  $\mathbf{F}_q$ ;
- (iv)  $\mathrm{Sp}_n(\mathbf{F}_q)$  (resp.  $\mathrm{PSP}_n(\mathbf{F}_q)$ ) denotes the symplectic group (resp. projective symplectic group) over the finite field  $\mathbf{F}_q$ ;
- (v)  $n.G$  denotes a non-split central extension of  $G$  by  $\mu_n$ ;
- (vi)  $z(G)$  (resp.  $[G, G]$ ) denotes the center (resp. the commutator subgroup) of a group  $G$ .

All simple groups are supposed to be non-cyclic.

LEMMA 2.2. Let  $C$  be a smooth curve with a faithful action of a finite group  $G$  of genus  $g < |G|/4$ . Then

$$\frac{2g - 2}{|G|} + 2 = \sum_r c_r \left(1 - \frac{1}{r}\right),$$

where  $r$  varies over the orders of cyclic subgroups of  $G$ , and  $\{c_r\}$  are non-negative integers.

*Proof.* This is a standard consequence of the Riemann-Hurwitz formula for the quotient morphism  $C \rightarrow C/G$ :

$$2g - 2 = |G|(2g_q - 2) + \sum_p (e_p - 1),$$

where  $p$  varies over the branch points,  $e_p$  are the ramification indices, and  $g_q$  is the genus of the quotient. Recall that the stabilisers of all points must be cyclic, so we get a contribution of the form  $|G|(r - 1)/r = e_p - 1$  for each  $G$ -orbit (free orbits contributing 0). Solving for  $g$ , we see that  $g_q = 0$  or else  $g \geq \frac{1}{4}|G|$ .

LEMMA 2.3 (see, e.g., [Car57, p. 98]). Let  $X$  be an irreducible algebraic variety, let  $P$  be a point in  $X$ , and let  $G$  be a finite group in  $\text{Aut}(X)$  that fixes the point  $P$ . Then the natural linear action of  $G$  on the Zariski tangent space  $T_{P,X}$  is faithful.

THEOREM 2.4 ([Bli17]). Let  $G \subset \text{GL}_3(\mathbb{C})$  be a finite quasi-simple subgroup. Then  $G$  is isomorphic to one of the following groups:

$$2.\mathfrak{A}_5, \quad \mathfrak{A}_5, \quad 3.\mathfrak{A}_6, \quad \text{PSL}_2(\mathbf{F}_7).$$

THEOREM 2.5 ([DI09]). Let  $G \subset \text{Cr}_2(\mathbb{C})$  be a finite quasi-simple subgroup such that  $G/Z(G) \not\cong \mathfrak{A}_5$ . Then  $G$  is conjugate to one of the following actions:

- (i)  $\mathfrak{A}_6$  acting on  $\mathbb{P}^2$ ;
- (ii)  $\text{PSL}_2(\mathbf{F}_7)$  acting on  $\mathbb{P}^2$ ;
- (iii)  $\text{PSL}_2(\mathbf{F}_7)$  acting on a unique del Pezzo surface of degree 2.

In particular,  $G$  is simple.

LEMMA 2.6. Let  $G$  be a group isomorphic to one in the list (1.4.1).

- (i) If  $G \subset \text{Aut}(\mathbb{P}^3)$ , then  $G \simeq 3.\mathfrak{A}_6$  and the action is induced by the reducible representation  $V = V_1 \oplus V_3$  with  $\dim V_1 = 1$ ,  $\dim V_3 = 3$ .
- (ii) If  $G \subset \text{Aut}(X)$ , where  $X = X_d \subset \mathbb{P}^4$  is an irreducible hypersurface of degree  $d \leq 4$ , then  $G \simeq \text{SL}_2(\mathbf{F}_7)$ ,  $X$  is smooth quartic, and the action is induced by the reducible representation  $V = V_1 \oplus V_4$  with  $\dim V_1 = 1$ ,  $\dim V_4 = 4$ .
- (iii) If  $G \subset \text{Aut}(\mathbb{P}^5)$ , then there exists no  $G$ -invariant quadric  $Q \subset \mathbb{P}^5$  of corank  $\leq 2$ .
- (iv) If  $G \subset \text{Aut}(\mathbb{P}^5)$ , then there exists no  $G$ -invariant irreducible complete intersection of two quadrics.

*Proof.* The assertion (i) follows from Table 1.

*Table 1. Representations and invariants of low degrees.*

Group	Dim	Invariants in Low Degrees								
		2	3	4	5	6	7	8	9	10
PSL <sub>2</sub> (F <sub>7</sub> )	3			1		1		1		1
	6	1	2	3	4	8	10	15	22	30
	7	1	1	4	2	10	10	25	28	58
	8	1	2	3	5	15	19	44	72	120
SL <sub>2</sub> (F <sub>7</sub> )	4			1		1		3		2
	6			1		2		10		16
	8			2		10		44		106
PSL <sub>2</sub> (F <sub>11</sub> )	5		1		1	2	1	2	3	3
	10	1		4	1	16	10	54	56	176
	10	1	2	4	8	16	28	54	98	176
	11	1	1	3	4	20	24	78	134	300
	12	1	1	4	8	25	49	124	258	558
SL <sub>2</sub> (F <sub>11</sub> )	6			1		1		4		4
	10			1		6		44		124
	10			3		6		44		134
	12			4		15		124		516
2A <sub>6</sub>	5	1	1	2	2	4	3	6	6	9
	8	1	1	2	3	9	9	23	34	60
	9	1	2	4	7	14	23	46	80	140
	10	1		7	2	25	20	94	108	308
2.2A <sub>6</sub>	4							2		
	8		1		4		23		46	
	10			4		11		80		248
3.2A <sub>6</sub>	3					1				
	6		2			7			16	
	9		2			14			80	
	15		2			126			2234	
6.2A <sub>6</sub>	6					1				
	12					29				
2A <sub>7</sub>	6	1	1	2	2	4	4	6	7	10
	10			3		6	2	20	11	54
	14	1	2	4	8	21	42	105	233	506
	14	1	2	5	9	22	46	109	237	518
	15	1	1	5	4	25	45	150	320	826
	21	1	2	8	24	110	362	1284	4023	12046
	35	1	4	37	225	1582	8864	47098	223591	985678
2.2A <sub>7</sub>	4							1		
	14			1		8		94		438

Table 1. Continued.

Group	Dim	Invariants in Low Degrees								
		2	3	4	5	6	7	8	9	10
	20			3		64		919		7845
	20			7		68		929		7905
	36			38		1749		57807		1264859
3.2A <sub>7</sub>	6	1				3			5	
	15	1				23			314	
	15	3				33			404	
	21	2				110			4023	
	21					120			3806	
	24	2				208			11146	
6.2A <sub>7</sub>	6									
	24					177				
	36					1749				
PSp <sub>4</sub> (F <sub>3</sub> )	5			1		1		1		2
	6	1		1	1	2	1	3	2	4
	10			1		2		5	2	8
	15	1	1	3	6	13	21	48	90	180
	15	1	1	3	2	9	9	30	44	115
	20	1	2	5	10	26	56	151	380	980
	24	1	2	6	12	41	117	409	1268	4006
	30	1		9	7	108	267	1785	5816	26198
	30	2	5	15	89	361	1560	6526	25024	
	40		9	38	361	1987	12432	64242	318717	
	45		14	65	655	4365	29347	170291	925070	
	60	1	4	34	320	3316	30266	252784	1903375	13127051
	64	1	2	38	409	4706	46253	411176	3283749	23975553
	81	1	4	94	1258	18430	225424	2483426	24523546	220742112
Sp <sub>4</sub> (F <sub>3</sub> )	4									
	20		1			7		103		765
	20		2			9		106		783
	20					6		96		755
	36		3			184		5631		122657
	60		26			3148		252182		13116046
	60		29			3153		252155		13116416
	64		34			4579		411176		23967693
	80		79			16801		2256083		196172329

(ii) Regard  $\mathbb{P}^4$  as the projectivisation of a vector space  $V = \mathbb{C}^5$  and consider a lifting  $\tilde{G} \subset \text{SL}(V)$ , where  $\tilde{G}$  is quasi-simple. Since  $G$  is not simple,  $z(\tilde{G})$  is not a subgroup of scalar matrices, i.e. there exists a non-trivial decomposition  $V = V' \oplus V''$  of  $\tilde{G}$ -modules, where  $\dim V' > \dim V''$ . Then  $\dim V'' \leq 2$  and so the action of  $\tilde{G}$  on  $V''$  must be trivial and on  $V'$  it is faithful with  $\dim V' = 3$  or  $4$ . Then  $\tilde{G}$  has an invariant of degree  $\leq d$  on  $V'$ . From Table 1, the only possibility is  $\tilde{G} \simeq \text{SL}_2(\mathbb{F}_7)$  and  $d = 4$ .

(iii) and (iv) follow from Table 1.

LEMMA 2.7. *Let  $X$  be a threefold with terminal singularities and a faithful action of a group  $G$  from the list (1.4.1). Assume that  $X$  has a  $G$ -fixed point  $P$ . Then one of the following holds:*

- (i)  $P \in X$  is smooth and  $G \simeq 3.A_6$ ;
- (ii)  $P \in X$  is of type  $\frac{1}{2}(1, 1, 1)$  and  $G \simeq 3.A_6$ .

*Proof.* First, consider the case where  $P \in X$  is Gorenstein. The group  $G$  faithfully acts on the tangent space  $T_{P,X}$ . If  $P \in X$  is smooth, then  $\dim T_{P,X} = 3$  and  $G \simeq 3.A_6$  by Theorem 2.4. If  $P \in X$  is singular, then there exists an analytic equivariant embedding  $(X, P) \subset (T, 0)$ , where  $T \simeq \mathbb{C}^4$  and the action on  $T$  is linear. Let  $\phi(x_1, \dots, x_4) = 0$  be the (invariant) equation of  $X$  in  $T$ . Write  $\phi = \sum \phi_d$ , where  $\phi_d$  is homogeneous of degree  $d$ . By the classification of terminal singularities [Rei87], we conclude  $\phi_2 \neq 0$ . If moreover  $G \hookrightarrow GL_4(\mathbb{C}) = GL(T)$  is irreducible, then the group  $G/z(G)$  faithfully acts on  $\mathbb{P}(T) = \mathbb{P}^3$ . In this case  $\phi_2 = 0$  defines an invariant quadric  $Q \subset \mathbb{P}^3$  which must be smooth. Thus  $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$  and then the simple group  $G/z(G)$  embeds into  $PGL_2$ ; impossible for  $G$  in the list (1.4.1). Let  $G \hookrightarrow GL_4(\mathbb{C}) = GL(T)$  be reducible. We have a decomposition  $T = T' \oplus T''$ , where  $T'$  is irreducible faithful with  $\dim T' < 4$ . If  $\dim T' = 3$ , then  $G \simeq 3.A_6$ . Again  $3.A_6$  has no invariants of degree  $\leq 3$  on  $T'$ , so  $\phi_2 = x_4^2$  and  $\phi_3 = \lambda x_4^3$ . This contradicts the classification of terminal singularities. Hence  $\dim T' = 2$  and  $G \simeq 2.A_5$ . Again we have a contradiction.

Consider the case where  $(X, P)$  is a singularity of index  $r > 1$ . Let  $\pi : (X^\sharp, P^\sharp) \rightarrow (X, P)$  be the index one cover and let  $G^\sharp \subset \text{Aut}(X^\sharp, P^\sharp)$  be the natural lifting of  $G$ . We have an exact sequence

$$1 \longrightarrow \mu_r \longrightarrow G^\sharp \xrightarrow{\nu} G \longrightarrow 1.$$

Since  $G$  is a quasi-simple group and  $\text{Aut}(\mu_r)$  is abelian, this is a central extension. Let  $Z^\sharp \subset G^\sharp$  be the preimage of  $z(G)$ . Since  $z(G)$  is cyclic,  $Z^\sharp$  is an abelian group with two generators and one of these generators is of order 2 or 3. In this situation, the automorphism group  $\text{Aut}(Z^\sharp)$  is solvable. Hence  $Z^\sharp$  coincides with the center of  $G^\sharp$ . Thus either  $G^\sharp = \mu_r \times G$  or  $Z^\sharp \simeq \mu_6$  and  $G/z(G) \simeq A_6$  or  $A_7$ . In both cases there exists a quasi-simple subgroup  $G' \subset G^\sharp$  such that  $\nu(G') = G$ . By the above  $G' \simeq 3.A_6$  and  $P^\sharp \in X^\sharp$  is smooth. Since the representation of  $G'$  on  $T_{P^\sharp, X^\sharp}$  is irreducible,  $r = 2$ .

2.8. Varieties of minimal degree

THEOREM 2.8.1 (F. Enriques, see e.g. [EH87]). *Let  $Y = Y_d \subset \mathbb{P}^N$  be an irreducible subvariety of degree  $d$  and dimension  $n$  which is not contained in a hyperplane. Then  $d \geq N - n + 1$  and the equality holds if and only if  $Y$  is one of the following:*

- (i)  $Y = \mathbb{P}^N$ ;
- (ii)  $Y = Y_2 \subset \mathbb{P}^N$ , a smooth quadric;
- (iii) a rational scroll  $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ , where  $\mathcal{E}$  is an ample rank  $n$  vector bundle on  $\mathbb{P}^1$ , embedded by the linear system  $|\mathcal{O}(1)|$ ;
- (iv) a Veronese surface  $F_4 \subset \mathbb{P}^5$ ;
- (v) a cone over one of the varieties from (ii), (iii) or (iv).



2.9. Group actions on K3 surfaces

**THEOREM 2.9.1** ([Muk88]) *Let a finite quasi-simple group  $G$  faithfully act on a K3 surface  $S$ . Then  $G \simeq \mathfrak{A}_5, \mathfrak{A}_6$  or  $\mathrm{PSL}_2(\mathbf{F}_7)$ . In particular,  $G$  is simple. Moreover, if  $G \simeq \mathfrak{A}_6$  or  $\mathrm{PSL}_2(\mathbf{F}_7)$ , then  $\mathrm{rk} \mathrm{Pic}(S) = 20$  and  $\mathrm{rk} \mathrm{Pic}(S)^G = 1$ .*

**COROLLARY 2.9.2.** *Let  $S$  be a K3 surface with at worst Du Val singularities. Suppose  $S$  admits a faithful action of a quasi-simple group  $G$ , where  $G \not\simeq \mathfrak{A}_5$ . Then  $S$  is smooth.*

2.10. Linearisations

Let  $X$  be a proper complex variety with a faithful action of a finite group  $G$ . Let  $\mathcal{E}$  be a vector bundle on  $X$ . We say that  $\mathcal{E}$  is  $G$ -invariant if there exist isomorphisms  $\phi_g : g^*\mathcal{E} \rightarrow \mathcal{E}$  for every  $g \in G$ . We say that  $\mathcal{E}$  is  $G$ -linearisable if  $\mathcal{E}$  is  $G$ -invariant and one can select the isomorphisms  $\{\phi_g\}_{g \in G}$  such that  $\phi_{gh} = \phi_h \circ h^*(\phi_g)$  for all  $g, h \in G$ . Equivalently, this means that  $G$  acts on the total space of  $\mathcal{E}$  linearly on the fibers and the projection to  $X$  is equivariant. The particular choice of action on  $\mathcal{E}$  is called a *linearization*.

**PROPOSITION 2.11.** *If  $X$  is a smooth  $G$ -variety, then the canonical line bundle has a canonical linearization.*

*Proof.* The points of the total space of the tangent bundle  $TX$  are of the form  $(x, t)$  where  $x \in X$  and  $t \in T_x X$ . For  $g \in G$ ,  $g(x, t) := (g(x), dg(t))$  defines an action on  $TX$  which is linear on the fibers. Thus  $TX$  is linearizable, and so is the canonical bundle.

**LEMMA 2.12** ([DI09, lemma 5.11]) *Let  $f : X \rightarrow Y$  be a double cover of smooth varieties whose branch divisor  $B \subset Y$  is given by an invertible sheaf  $\mathcal{L}$  together with a section  $s_B \in H^0(Y, \mathcal{L}^{\otimes 2})$  (see [Wav68]) and let  $\tau$  is the Galois involution. Suppose a group  $G$  acts on  $Y$  leaving invariant  $B$ . Then there exists a subgroup  $G' \subset \mathrm{Aut}(X)$  fitting into the exact sequence*

$$1 \longrightarrow \langle \tau \rangle \longrightarrow G' \xrightarrow{\delta} G \longrightarrow 1,$$

where  $\delta$  is induced by  $G' \rightarrow \mathrm{Aut}(Y)$ . The sequence splits if and only if  $\mathcal{L}$  admits a  $G$ -linearisation and in the corresponding representation of  $G$  in  $H^0(Y, \mathcal{L}^{\otimes 2})$  the section  $s_B$  is  $G$ -invariant.

3. Gorenstein Fano threefolds

3.1. Special Fano threefolds

Recall that a  $G\mathbb{Q}$ -Fano variety is a variety  $X$  with only terminal  $\mathbb{Q}$ -factorial singularities equipped with an action of a group  $G$  such that the anticanonical class  $-K_X$  is ample and the invariant part  $\mathrm{Pic}(X)^G$  of the Picard group is of rank 1. We say that  $X$  is a  $G$ -Fano variety if additionally the singularities of  $X$  are Gorenstein.

**PROPOSITION 3.1.1.** *Let  $G$  be a group from the list (1.4.1) and let  $X$  be a  $G$ -Fano threefold (with only terminal Gorenstein singularities). Then  $\mathrm{rk} \mathrm{Pic}(X) = 1$ .*

*Proof.* The lattice  $\text{Pic}(X)$  is equipped with a pairing

$$(D_1, D_2) = -K_X \cdot D_1 \cdot D_2.$$

which is  $G$ -invariant and non-degenerate (by the Hodge index theorem). Hence the group  $G$  acts on the orthogonal complement  $W := K_X^\perp$  of  $K_X$  in  $\text{Pic}(X)$ . According to [Pro13b] we have  $\text{rk Pic}(X) \leq 4$  and so  $\text{rk } W \leq 3$ . Since  $\text{rk Pic}(X)^G = 1$ , we have  $W^G = 0$ . But the groups from the list (1.4.1) have no rational representations of dimension  $\leq 3$ , a contradiction.

Fano threefolds with Fano index  $\iota(X) = 2$  are also called *del Pezzo threefolds*.

**PROPOSITION 3.1.2.** *Let  $X$  be a Fano threefold with at worst canonical Gorenstein singularities. Assume that  $\iota(X) = 2$  and  $(-K_X/2)^3 \leq 4$ . Furthermore, assume that  $\text{Aut}(X)$  contains a subgroup  $G$  from the list (1.4.1). Then  $(X, G)$  is as in Example 1.6(iii).*

*Proof.* Let  $A = -K_X/2$  and  $d(X) := A^3$ . Consider the possibilities for  $d(X)$  case by case. We use the classification of del Pezzo threefolds [Fuj90], [Shi89].

Case  $d(X) = 1$ . Here  $\text{Bs}|A| = \{P\}$  and  $G$  faithfully acts on  $T_{P,X}$ . Hence  $G \simeq 3.A_6$  by Theorem 2.4. Let  $\tilde{G}$  be the universal central extension of  $G$  (see, e.g., [Asc00, section 33]). Then the action of  $G$  on  $X$  lifts to an action of  $\tilde{G}$  on  $H^0(X, tA)$  for any  $t$ . Hence  $\tilde{G}$  acts on the graded algebra

$$R(X, A) := \bigoplus_{t \geq 0} H^0(X, tA).$$

In our case  $R(X, A)$  is generated by its elements  $x_1, x_2, x_3, y, z$  with  $\deg x_i = 1, \deg y = 2, \deg z = 3$  and a unique relation of degree 6.

There exists a natural isomorphism  $H^0(X, A) \simeq T_{P,X}$ . The subspace

$$S^2(H^0(X, A)) \subset H^0(X, -K_X)$$

is invariant. Since  $\dim H^0(X, -K_X) = 7$  and  $\dim S^2(H^0(X, A)) = 6$ , the element  $y \in H^0(X, -K_X)$  can be taken to be a relative invariant of  $\tilde{G}$ . Similarly, the subspace of the 10-dimensional space  $H^0(X, 3A)$  generated by  $x_1, x_2, x_3, y$  is invariant and of codimension 1. Hence the element  $z \in H^0(X, 3A)$  can be taken to be a relative invariant of  $\tilde{G}$ . Since the action on  $x_1, x_2, x_3$  has no invariants of degree  $< 6$  and has a unique invariant  $\phi_6$  of degree 6,  $X \subset \mathbb{P}(1^3, 2, 3)$  is given by the equation

$$z^2 + y^3 + \phi_6$$

and so we are in the situation of Example 1.6(iii).

Case  $d(X) = 2$ . In this case the map given by the linear system  $|A|$  is a finite morphism  $\Phi_{|A|} : X \rightarrow \mathbb{P}^3$  of degree 2 branched over a quartic  $B \subset \mathbb{P}^3$ . The group  $G$  acts non-trivially on  $\mathbb{P}^3$ . Therefore,  $G/z(G)$  is either  $\text{PSL}_2(\mathbb{F}_7)$  or  $A_6$ . In the case  $G/z(G) = A_6$ , the group  $G$  has no invariant quartic. Hence  $G/z(G) = \text{PSL}_2(\mathbb{F}_7)$  and  $G \simeq \text{SL}_2(\mathbb{F}_7)$ .

Similar to the above considered case  $R(X, A)$  is generated by  $x_1, \dots, x_4, y$  with  $\deg x_i = 1, \deg y = 2$  with a unique relation of degree 4. We may take  $y$  to be a relative invariant for  $\tilde{G}$ . Hence  $X \subset \mathbb{P}(1^4, 2)$  can be given by the equation

$$y^2 + y\phi_2 + \phi_4 = 0$$

Since the action on  $x_1, \dots, x_4$  has no invariants of degree  $\leq 2$ ,  $\phi_2 = 0$ . Now we see that  $z(G)$  acts trivially on  $X$  (cf. Lemma 2.12), a contradiction.

Case  $d(X) = 3$ . This case does not occur by Lemma 2.6(ii).

Case  $d(X) = 4$ . In this case  $X$  is an intersection of two quadrics  $Q_1$  and  $Q_2$  in  $\mathbb{P}^5$ , which is impossible by Lemma 2.6(iv).

**COROLLARY 3.1.3.** *Let  $G$  be a group from the list (1.4.1) and let  $X$  be a  $G$ -Fano threefold with  $\iota(X) = 2$ . Then  $X$  is as in Example 1.6(iii).*

*Proof.* By Proposition 3.1.2 we may assume that  $(-K_X/2)^3 \geq 5$  and by Proposition 3.1.1 we have  $\text{rk Pic}(X) = 1$ . Hence by [Pro13a, theorem 1.7] and [CS16, proposition 7.1.10] we have  $(-K_X/2)^3 = 5$ ,  $X$  is smooth, and  $\text{Aut}(X) \simeq \text{PSL}_2(\mathbb{C})$ . On the other hand,  $\text{PSL}_2(\mathbb{C})$  does not contain finite non-solvable groups different from  $\mathfrak{A}_5$ , a contradiction.

**LEMMA 3.2** (cf. [Pro12, lemma 5.2]) *Let  $X$  be a Fano threefold with at worst canonical Gorenstein singularities. Assume that  $\text{Aut}(X)$  contains a subgroup  $G$  as in the list (1.4.1). Then the linear system  $|-K_X|$  is base point free.*

*Proof.* Assume that  $\dim \text{Bs}|-K_X| = 0$ , then by [Shi89]  $\text{Bs}|-K_X|$  is a single point, say  $P$ , and  $X$  has at  $P$  terminal Gorenstein singularity of type  $cA_1$ . By Lemma 2.7 this is impossible.

Thus  $\dim \text{Bs}|-K_X| > 0$ , then by [Shi89]  $\text{Bs}|-K_X|$  is a smooth rational curve  $C$  contained in the smooth locus of  $X$ . The action of  $G$  on  $C$  must be trivial and we obtain a contradiction as above.

**LEMMA 3.3.** *Let  $X$  be a Fano threefold with at worst canonical Gorenstein singularities. Assume that  $\text{Aut}(X)$  contains a subgroup  $G$  from the list (1.4.1). If the linear system  $|-K_X|$  is not very ample, then  $(X, G)$  is as in Examples 1.7(i), 1.6(iii) or 1.6(iv).*

*Proof.* (cf. [Pro12, lemma 5.3]). Assume that the linear system  $|-K_X|$  defines a morphism  $\varphi : X \rightarrow \mathbb{P}^{g+1}$  which is not an embedding. Let  $Y = \varphi(X)$  and let  $\tilde{G}$  be the image of  $G$  in  $\text{Aut}(X)$ . Then  $\varphi$  is a double cover and  $Y \subset \mathbb{P}^{g+1}$  is a subvariety of degree  $g - 1$  (see [Isk80a] and [PCS05]). Hence either  $\tilde{G} \simeq G$  or  $\tilde{G}$  is the quotient of  $G$  by a subgroup of order 2. Let  $H$  be the class of a hyperplane section of  $Y$  and let  $B \subset Y$  be the branch divisor. Then

$$-K_X = \varphi^*H, \quad K_X = \varphi^*\left(K_Y + \frac{1}{2}B\right), \quad K_Y + H + \frac{1}{2}B = 0. \tag{3.3.1}$$

Apply Theorem 2.8.1. The case where  $Y$  is a quadric (case 2.8.1(ii)) does not occur by Lemma 2.6(ii) and [Pro12, lemma 3.6].

If  $Y \simeq \mathbb{P}^3$  (the case 2.8.1(i)), then the morphism  $\varphi : X \rightarrow \mathbb{P}^3$  is a double cover with branch divisor  $B \subset \mathbb{P}^3$  of degree 6 by (3.3.1). The groups  $\mathfrak{A}_7$  and  $\text{PSp}_4(\mathbb{F}_3)$  have no non-trivial invariant hypersurfaces of degree 6. If  $\tilde{G} \simeq \text{PSL}_2(\mathbb{F}_7)$ , then we get Example 1.7(i). Likewise, if  $\tilde{G} \simeq \mathfrak{A}_6$ , we get the case 1.6(iv).

If  $Y$  is a cone over the Veronese surface (case 2.8.1(iv)), then  $Y \simeq \mathbb{P}(1, 1, 1, 2)$  and  $\mathcal{O}_{\mathbb{P}}(B) = \mathcal{O}_{\mathbb{P}}(6)$  by (3.3.1). Hence  $\iota(X) = 2$  and  $X$  is a del Pezzo threefold of degree 1. This case was considered in Proposition 3.1.2.

Finally consider the case where  $Y$  is either a rational scroll, a cone over a rational scroll, or a cone over a rational normal curve. Then  $Y$  is the image of  $\hat{Y} := \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ , where  $\mathcal{E}$  is a

nef rank  $n$  vector bundle on  $\mathbb{P}^1$ , under the map defined by the linear system  $|\mathcal{O}(1)|$ . Thus  $\nu : \hat{Y} \rightarrow Y$  is the “minimal” resolution of singularities which is given by the blowup of the maximal ideal of  $\text{Sing}(Y)$ . In particular,  $\nu$  is  $G$ -equivariant. Then we have the following equivariant commutative diagram

$$\begin{array}{ccc}
 \hat{X} & \xrightarrow{\hat{\varphi}} & \hat{Y} \\
 \eta \downarrow & & \downarrow \nu \\
 X & \xrightarrow{\varphi} & Y
 \end{array}
 \begin{array}{c}
 \searrow \pi \\
 \mathbb{P}^1,
 \end{array}$$

where  $\hat{X}$  is the normalization of the fiber product. The morphism  $\eta$  is a crepant contraction and  $\hat{X}$  has at worst canonical Gorenstein singularities [PCS05, lemma 3.6]. The group  $\bar{G}$  trivially acts on  $\mathbb{P}^1$ , so it non-trivially acts on each fiber  $F \simeq \mathbb{P}^2$  of  $\pi$ . Write  $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2) \oplus \mathcal{O}_{\mathbb{P}^1}(d_3)$ , where  $d_3 \geq d_2 \geq d_1 \geq 0$ . If  $d_1 < d_2$ , then the surjection  $\mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}^1}(d_2) \oplus \mathcal{O}_{\mathbb{P}^1}(d_3)$  defines an invariant subscroll  $\hat{Y}_2 \subset \hat{Y}$  such that  $Y_2 \cap F$  is a line. According to Theorem 2.4 this is impossible. Thus we may assume that  $d_1 = d_2$  and, similarly,  $d_2 = d_3$ . So,  $\hat{Y} \simeq \mathbb{P}^2 \times \mathbb{P}^1 \simeq Y$ . Note that  $B$  intersects  $F$  along a quartic curve which must be invariant by (3.3.1). Then the only possibility is that  $B$  is a divisor of bidegree  $(4, 0)$  or a (reducible) divisor of bidegree  $(4, 2)$ , where  $\bar{G} \simeq \text{PSL}_2(\mathbb{F}_7)$ . In the former case  $X$  is the product of  $\mathbb{P}^1$  and del Pezzo surface of degree 2. In the latter case  $X$  is described in [Kry16, example 1.8] as the threefold  $X_1$ . In both cases the group  $G$  splits by Lemma 2.12, a contradiction.

*Remark 3.3.2.* Assume that  $-K_X$  is very ample. Then, by Proposition 2.11, our group  $G$  acts faithfully on the space  $H^0(X, -K_X)^\vee$  so that the induced action on its projectivization  $\mathbb{P}(H^0(X, -K_X)^\vee) = \mathbb{P}^{g+1}$  is also faithful. This implies that the representation  $H^0(X, -K_X)^\vee$  of  $G$  is reducible.

**LEMMA 3.4.** *Let  $X$  be a Fano threefold with at worst canonical Gorenstein singularities. Assume that  $\text{Aut}(X)$  contains a subgroup  $G$  as in the list (1.4.1). Assume that the linear system  $|-K_X|$  is very ample but the image  $X = X_{2g-2} \subset \mathbb{P}^{g+1}$  is not an intersection of quadrics. Then  $(X, G)$  is as in Example 1.7(ii).*

*Proof.* By our assumption  $g \geq 3$ . If  $g = 3$ , then  $X = X_4 \subset \mathbb{P}^4$  is a quartic. Inspecting the list (1.4.1) one can see that the only possibility is  $G \simeq \text{SL}_2(\mathbb{F}_7)$ , which implies that  $(X, G)$  is as in Example 1.7(i).

Now assume that  $g > 3$ . Since  $X = X_{2g-2} \subset \mathbb{P}^{g+1}$  is projectively normal [Isk80a], the restriction map

$$H^0(\mathbb{P}^{g+1}, \mathcal{O}_{\mathbb{P}^{g+1}}(2)) \longrightarrow H^0(X, \mathcal{O}_X(2))$$

is surjective. This allows us to compute that the number of linear independent quadrics passing through  $X$  is equal to

$$\frac{1}{2}(g - 2)(g - 3) > 0.$$

Let  $Y \subset \mathbb{P}^{g+1}$  be the intersection of all quadrics containing  $X$ . It is known that  $Y$  is a reduced irreducible variety of minimal degree (see [Isk80a] and [PCS05]). Thus  $Y$  is described by Theorem 2.8.1.

If  $g = 4$ , then  $Y$  is a (unique) quadric passing through  $X$  and  $X$  is cut out on  $Y$  by a cubic, say  $Z$ . We may assume that  $Z$  is  $G$ -invariant. Thus our group  $G$  has invariants of degrees 2 and 3. Hence,  $z(G)$  is of order 6 and so  $G \supset 6.A_6$ . But then  $G$  has no faithful reducible representations of dimension 6 by Table 1.

Thus it remains to consider the case where  $Y$  is either a rational scroll, a cone over a rational scroll, or a cone over a rational normal curve. Arguing as in the proof of Lemma 3.3 and using [PCSO5, lemma 4.7], we conclude that there exists a  $G$ -equivariant crepant extraction  $\eta: \hat{X} \rightarrow X$  and a degree 3 del Pezzo fibration  $\hat{X} \rightarrow \mathbb{P}^1$ . This gives us a contradiction (see Theorem 2.5).

LEMMA 3.5. *Let  $X$  be a Fano threefold with at worst canonical singularities and  $G \subset \text{Aut}(X)$  be a finite group contained in the list (1.4.1). Assume that there exists a  $G$ -invariant divisor  $S \in |-K_X|$  such that the pair  $(X, S)$  is not plt and either  $S$  is irreducible or  $(X, S)$  is not lc. Then  $G$  has a fixed point  $P \in S$  such that  $(X, S)$  is not plt at  $P$ .*

*Proof.* Let  $c$  be the log canonical threshold of  $(X, S)$ , that is, the pair  $(X, cS)$  is maximally lc. Then  $c \leq 1$  and  $-(K_X + cS)$  is nef.

Consider the case  $c < 1$ . Then  $-(K_X + cS)$  is ample. Let  $\Lambda \subset X$  be the locus of lc singularities of  $(X, cS)$ . By Shokurov’s connectedness principle (see [Sho93] and [Kol92, chapter 17]),  $\Lambda$  is connected (and clearly  $G$ -invariant). If  $\dim \Lambda = 0$ , then  $\Lambda$  must be an invariant point. Suppose  $\dim \Lambda = 1$ . If there exists a zero-dimensional center of lc singularities, then replacing  $cS$  with small invariant perturbation  $(c - \epsilon)S + \Delta$  we get a zero-dimensional locus of lc singularities and may argue as above (see [Pro12, claim 4.7.1]). Otherwise  $\Lambda$  must be a minimal center of lc singularities and by Kawamata subadjunction theorem [Kaw98, theorem 1]  $\Lambda$  is a smooth rational curve, because  $-(K_X + cS)$  is ample. Since  $G$  cannot act non-trivially on  $\mathbb{P}^1$ , the action of  $G$  on  $\Lambda$  is trivial.

Now consider the case  $c = 1$  and  $S$  is irreducible. Then  $(X, S)$  is lc. Let  $\nu: S' \rightarrow S$  be the normalisation. Write

$$0 \sim \nu^*(K_X + S)|_S = K_{S'} + D',$$

where  $D'$  is the different, an effective integral Weil divisor on  $S'$  such that the pair  $(S', D')$  is lc (see [Sho93, section 3], [Kol92, chapter 16], and [Kaw07]). The group  $G$  acts naturally on  $S'$  and  $\nu$  is  $G$ -equivariant. Now consider the minimal resolution  $\mu: \tilde{S} \rightarrow S'$  and let  $\tilde{D}$  be a uniquely defined divisor such that

$$K_{\tilde{S}} + \tilde{D} = \mu^*(K_{S'} + D') \sim 0, \quad \mu_*\tilde{D} = D'.$$

is usually called the log crepant pull-back of  $D'$ . Here  $\tilde{D}$  is again an effective reduced divisor. Run  $G$ -equivariant MMP on  $\tilde{S}$ . Clearly, the whole  $\tilde{D}$  cannot be contracted. We get a model  $(S_{\min}, D_{\min})$  such that  $(S_{\min}, D_{\min})$  is lc,  $(K_{S_{\min}} + D_{\min}) \sim 0$ , and  $D_{\min} \neq 0$ . Assume that  $S_{\min}$  has an equivariant conic bundle structure  $\pi: S_{\min} \rightarrow B$ . Then  $D_{\min}$  has one or two horizontal components which must be  $G$ -invariant. By adjunction any horizontal component of  $D_{\min}$  is either rational or elliptic curve. Such a curve does not admit a non-trivial action of  $G$ , so the action of  $G$  on the corresponding component  $\tilde{D}_1 \subset \tilde{D}$  and  $\nu(\mu(\tilde{D}_1))$  must be trivial. Similarly, if  $S_{\min}$  is a del Pezzo surface with  $\text{rk Pic}(S_{\min})^G = 1$ , then  $K_{S_{\min}}^2 = 2$  or 9 by Theorem 2.5

and so  $\tilde{D}$  has at most 3 components. Arguing as above we get that the action of  $G$  on some component  $\tilde{D}_1 \subset \tilde{D}$  is trivial.

4. Proof of main result

In this section, we prove Theorem 1.5 omitting the proof that  $3\mathcal{A}_7$  cannot act faithfully on a rationally connected threefold. The case of  $3\mathcal{A}_7$  will be dealt with in Section 5 later.

4.1. Singularities of quotients

First, we need two auxiliary local results.

LEMMA 4.1.1. *Let  $(X \ni P)$  be a threefold terminal singularity of index 1. Suppose that a group  $A$  of order 2 acts on  $(X \ni P)$  so that either the action is free in codimension 1 or the fixed point locus is a  $\mathbb{Q}$ -Cartier divisor. Then the quotient  $(X \ni P)/A$  is canonical.*

*Proof.* Denote  $(Y \ni Q) := (X \ni P)/A$ . If  $A$  acts freely in codimension one, then the assertion is a consequence of [KSB88, proposition 6.12]. Let  $\text{Fix}(A, X)$  contain a divisor, say  $D$ . We have an  $A$ -equivariant embedding  $(X \ni P) \subset (\mathbb{C}^4 \ni 0)$  and we may assume that the action on  $\mathbb{C}^4$  is diagonalizable. If this action is of type  $\frac{1}{2}(1, 0, 0, 0)$ , then  $\mathbb{C}^4/A$  is smooth and so  $Y$  is Gorenstein. Since the quotient singularities are always rational [KM98, proposition 5.13], this implies that  $(Y \ni Q)$  is canonical [KM98, corollary 5.24]. Thus we may assume that the action is of type  $\frac{1}{2}(1, 1, 0, 0)$  and the equation of  $X$  is of the form

$$\phi = x_1\phi_1(x_1, \dots, x_4) + x_2\phi_2(x_1, \dots, x_4) = 0.$$

But then the fixed point locus is not a  $\mathbb{Q}$ -Cartier divisor.

LEMMA 4.1.2. *Let  $(X \ni P)$  be a threefold terminal cyclic quotient singularity of index 2 acted upon by a finite group  $G_P$  such that  $z(G_P)$  contains a subgroup  $A \simeq \mu_2$ . Suppose that the center of any extension of  $G_P$  by  $\mu_2$  does not contain an element of order 4. Then the quotient  $(X \ni P)/A$  is canonical.*

*Proof.* Let  $\pi : (X^\sharp \ni P^\sharp) \rightarrow (X \ni P)$  be the index-one cover, so that  $(X \ni P) = (X^\sharp \ni P^\sharp)/\mu_2$  where the action of  $\mu_2$  is free outside  $P^\sharp$  [KM98, definition 5.19]. By our assumption  $(X^\sharp \ni P^\sharp) \simeq (\mathbb{C}^3, 0)$  and the action of  $A$  is of type  $\frac{1}{2}(1, 1, 1)$ . The action of  $G_P$  lifts to an action of  $G_P^\sharp$  on  $(X^\sharp \ni P^\sharp)$ , where  $G_P^\sharp$  is an extension of  $G_P$  by  $\mu_2$ . Let  $A^\sharp \subset G_P^\sharp$  be the preimage of  $A$  (a group of order 4).

If  $A^\sharp \simeq \mu_2 \times \mu_2$ , then the elements of this group act as follows:  $\frac{1}{2}(1, 1, 1)$ ,  $\frac{1}{2}(1, 1, 0)$ ,  $\frac{1}{2}(0, 0, 1)$ . It is easy to see that in this case the quotient is canonical Gorenstein.

Assume that the extension  $A^\sharp \simeq \mu_4$ . Then the action on  $\mathbb{C}^3$  is of type  $\frac{1}{4}(1, 1, 1)$  or  $\frac{1}{4}(1, 1, -1)$ . By our assumption the former case does not occur. In the latter case the quotient is terminal.

4.2.  $G$ -birationally superrigid Fano threefolds

Second, we need the following global result.

THEOREM 4.3. *Let  $X$  be a Fano threefold with terminal Gorenstein singularities, and let  $G$  be a finite subgroup in  $\text{Aut}(X)$ . Suppose that  $X$  and  $G$  fit one of the following seven cases:*

- (i)  $G \simeq \mathfrak{A}_7$ , and  $X$  is the unique smooth intersection of a quadric and a cubic in  $\mathbb{P}^5$  that admits a faithful action of the group  $\mathfrak{A}_7$ ;
- (ii)  $G \simeq \mathfrak{A}_7$  and  $X = \mathbb{P}^3$ ;
- (iii)  $G \simeq \mathrm{PSp}_4(\mathbf{F}_3)$  and  $X = \mathbb{P}^3$ ;
- (iv)  $G \simeq \mathrm{PSp}_4(\mathbf{F}_3)$  and  $X$  is the Burkhardt quartic in  $\mathbb{P}^4$ ;
- (v)  $G \simeq \mathrm{PSL}_2(\mathbf{F}_{11})$  and  $X$  is the Klein cubic threefold in  $\mathbb{P}^4$ ;
- (vi)  $G \simeq \mathrm{PSL}_2(\mathbf{F}_{11})$  and  $X$  is the unique smooth Fano threefold of Picard rank 1 and genus 8 that admits a faithful action of the group  $\mathrm{PSL}_2(\mathbf{F}_{11})$ .

Let  $\rho: X \dashrightarrow V$  be a  $G$ -birational map such that  $V$  is a Fano variety with at most canonical singularities. Then  $\rho$  is biregular.

The proof of this result is based on the following technical result, which originated in [CS12, CS14, CS16, CS17].

PROPOSITION 4.4. *Let  $X$  be a Fano threefold with terminal Gorenstein singularities, and let  $G$  be a finite subgroup in  $\mathrm{Aut}(X)$ . Write  $-K_X \sim nH$ , where  $H$  is a Cartier divisor on  $X$ , and  $n = \iota(X)$  is the Fano index of the threefold  $X$ . Let  $\mathcal{M}$  be a linear system on the threefold  $X$  that does not have fixed components, and let  $\lambda$  be a positive rational number such that*

$$\lambda \mathcal{M} \sim_{\mathbb{Q}} -K_X.$$

Suppose that  $(X, \lambda \mathcal{M})$  does not have terminal singularities. Then one of the following (non-exclusive) possibilities holds:

- (i) there exists a  $G$ -orbit  $\Sigma \subset X$  such that

$$|\Sigma| = h^0(\mathcal{O}_X((n+1)H)) - h^0(\mathcal{O}_X((n+1)H) \otimes \mathcal{I}_\Sigma),$$

where  $\mathcal{I}_\Sigma$  is the ideal sheaf of  $\Sigma$ ;

- (ii) there exists a  $G$ -irreducible reduced curve  $C$  that consists of  $r \geq 1$  pairwise disjoint smooth isomorphic irreducible components  $C_1, \dots, C_r$  such that  $2g - 2 \leq nd$ ,  $rd \leq H^3 n^2$  and

$$r((n+1)d - g + 1) = h^0(\mathcal{O}_X((n+1)H)) - h^0(\mathcal{O}_X((n+1)H) \otimes \mathcal{I}_C),$$

where  $d = H \cdot C_i$ ,  $g$  is the genus of any curve  $C_i$ , and  $\mathcal{I}_C$  is the ideal sheaf of  $C$ .

*Proof.* Since  $-K_X$  is Cartier, the log pair  $(X, 2\lambda \mathcal{M})$  does not have klt singularities by [CS14, lemma 2.2]. Choose  $\mu \leq 2\lambda$  such that  $(X, \mu \mathcal{M})$  is strictly lc. Let  $Z$  be a minimal center of lc singularities of the log pair  $(X, \mu \mathcal{M})$ , see [Kaw97, Kaw98] for a precise definition. Then  $\theta(Z)$  is also a minimal center of lc singularities of this log pair for every  $\theta \in G$ . Moreover, we have

$$Z \cap \theta(Z) \neq \emptyset \iff Z = \theta(Z)$$

by [Kaw97, proposition 1.5].

Since  $\mathcal{M}$  does not have fixed components, the center  $Z$  is either a curve or a point. Observe that

$$\mu\mathcal{M} \sim_{\mathbb{Q}} \frac{\mu}{\lambda}nH,$$

where  $\mu/\lambda \leq 2$ . It would be easier to work with  $(X, \mu\mathcal{M})$  if it did not have centers of lc singularities that are different from  $\theta(Z)$  for  $\theta \in G$ . This is possible to achieve if we replace the boundary  $\mu\mathcal{M}$  by (a slightly more complicated) effective boundary  $B_X$  such that

$$B_X \sim_{\mathbb{Q}} \left(\frac{\mu}{\lambda}n + \epsilon\right)H$$

for some positive rational number  $\epsilon$  that can be chosen arbitrary small. This is known as the Kawamata–Shokurov trick or the perturbation trick (see [CS16, lemma 2.4.10] and the proofs of [Kaw97, theorem 1.10] and [Kaw98, theorem 1]). By construction, we may assume that

$$\frac{\mu}{\lambda}n + \epsilon \leq 2n + \epsilon < 2n + 1. \tag{4.4.1}$$

Note that the coefficients of  $B_X$  depend on  $\epsilon$ . But we can chose  $\epsilon$  as small as we wish, so that the number  $\mu n/\lambda + \epsilon$  can be as close to  $2n$  as we need.

Let  $\Sigma$  be the union of all log canonical centers  $\theta(Z)$  for  $\theta \in G$ . Then  $\Sigma$  is either a  $G$ -orbit or a disjoint union of irreducible isomorphic curves, which are transitively permuted by  $G$ . In both cases, we have an exact sequence of vector spaces

$$\begin{aligned} 0 \longrightarrow H^0(\mathcal{O}_X((n+1)H) \otimes \mathcal{I}_{\Sigma}) &\longrightarrow H^0(\mathcal{O}_X((n+1)H)) \\ &\longrightarrow H^0(\mathcal{O}_{\Sigma} \otimes \mathcal{O}_X((n+1)H)) \longrightarrow H^1(\mathcal{O}_X((n+1)H) \otimes \mathcal{I}_{\Sigma}), \end{aligned} \tag{4.4.2}$$

where  $\mathcal{I}_{\Sigma}$  is an ideal sheaf of the locus  $\Sigma$ , and  $\mathcal{O}_{\Sigma}$  is its structure sheaf. Note that  $\mathcal{I}_{\Sigma}$  is the multiplier ideal sheaf of the log pair  $(X, B_X)$ . Since

$$K_X + B_X \sim_{\mathbb{Q}} \left(\left(\frac{\mu}{\lambda} - 1\right)n + \epsilon\right)H,$$

we can apply Nadel’s vanishing (see [Laz04, theorem 9.4.17]) to deduce that

$$h^1(\mathcal{O}_X((n+1)H) \otimes \mathcal{I}_{\Sigma}) = 0.$$

In particular, if  $Z$  is a point, it follows from (4.4.2) that

$$|\Sigma| = h^0(\mathcal{O}_X((n+1)H)) - h^0(\mathcal{O}_X((n+1)H) \otimes \mathcal{I}_{\Sigma}).$$

To complete the proof of the proposition, we may assume that  $\Sigma$  is disjoint union of irreducible isomorphic curves  $C_1 = Z, C_2, \dots, C_r$ , which are transitively permuted by  $G$ . In particular, if  $r = 1$ , then  $\Sigma = C_1 = Z$  is a  $G$ -invariant irreducible curve in  $X$ .

Let  $d = H \cdot C_i$ . Then  $rd \leq H^3 n^2$ . This immediately follows from Corti’s [Cor00, theorem 3.1]. Namely, observe that  $(X, \mu\mathcal{M})$  is not klt at general points of every curve  $C_i$ . Let  $M$  and  $M'$  be general surfaces in  $\mathcal{M}$ . Then, applying [Cor00, theorem 3.1] to the log pair  $(X, \mu\mathcal{M})$  at general point of the curve  $C_i$ , we obtain

$$\text{mult}_{C_i}(M \cdot M') \geq \frac{4}{\mu^2}.$$



Then

$$\frac{n^2}{\lambda^2} H^3 = H \cdot M \cdot M' \geq \sum_{i=1}^r H \cdot C_i \text{mult}_{C_i}(M \cdot M') \geq rd \frac{4}{\mu^2},$$

so that  $rd \leq H^3 n^2 \mu^2 / 4 \lambda^2 \leq H^3 n^2$  as claimed.

By Kawamata’s subadjunction [Kaw98, theorem 1], each curve  $C_i$  is smooth. Let  $g$  be its genus. Moreover, for every ample  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor  $A$  on  $X$ , it follows from [Kaw98, theorem 1] that

$$(K_X + B_X + A) \Big|_{C_i} \sim_{\mathbb{Q}} K_{C_i} + B_{C_i}$$

for some effective  $\mathbb{Q}$ -divisor  $B_{C_i}$  on the curve  $C_i$ . Computing the degrees of the left hand side and the right-hand side in this  $\mathbb{Q}$ -linear equivalence, we see that  $2g - 2 \leq nd$ . In particular, the divisor  $(n + 1)H|_{C_i}$  is non-special on  $C_i$ , so that

$$h^0(\mathcal{O}_{C_i}((n + 1)H|_{C_i})) = (n + 1)d - g + 1$$

by the Riemann–Roch formula. Now using (4.4.2), we get

$$r((n + 1)d - g + 1) = h^0(\mathcal{O}_X((n + 1)H)) - h^0(\mathcal{O}_X((n + 1)H) \otimes \mathcal{I}_{\Sigma}),$$

which complete the proof of the proposition.

*Remark 4.5.* In the notations and assumptions of Proposition 4.4, there exists a central extension  $\tilde{G}$  of the group  $G$  such that the line bundle  $H$  is  $\tilde{G}$ -linearisable. Recall the Riemann–Roch theorem for a divisor  $D$  on a smooth threefold  $X$ :

$$\chi(\mathcal{O}_X(D)) = \frac{1}{6} D^3 - \frac{1}{4} D^2 \cdot K_X + \frac{1}{12} D \cdot K_X^2 + \frac{1}{12} D \cdot c_2(X) + \chi(\mathcal{O}_X).$$

Thus, the vector space  $H^0(\mathcal{O}_X((n + 1)H))$  is a representation of the group  $\tilde{G}$  of dimension

$$h^0(\mathcal{O}_X((n + 1)H)) = \frac{(n + 1)(2n + 1)(3n + 2)}{12} H^3 + \frac{2n + 2}{n} + 1.$$

Then the exact sequence (4.4.2) in the proof of Proposition 4.4 is an exact sequence of  $\tilde{G}$ -representations.

*Proof of Theorem 4.3.* Suppose that there exists a non-biregular  $G$ -birational map  $\rho: X \dashrightarrow V$  such that  $V$  is a Fano variety with at most canonical singularities. Applying [CS16, theorem 3.2.1], we see that there exists a  $G$ -invariant linear system  $\mathcal{M}$  on the threefold  $X$  such that  $\mathcal{M}$  does not have fixed components, and the singularities of the log pair  $(X, \lambda \mathcal{M})$  are not terminal, where  $\lambda$  is a positive rational number such that  $\lambda \mathcal{M} \sim_{\mathbb{Q}} -K_X$ . We will obtain a contradiction using Proposition 4.4 and Remark 4.5.

Let  $m = (n + 1)(n + 2)/12H^3 + 2/n$ . Note that the divisor  $H$  is very ample in each of our cases, and  $h^0(\mathcal{O}_X(H)) = m + 1$ . Thus, we may identify  $X$  with its image in  $\mathbb{P}^m$ . Then  $G$  is a subgroup in  $\text{PGL}_{m+1}(\mathbb{C})$ . Let  $\tilde{G}$  be a finite subgroup in  $\text{GL}_{m+1}(\mathbb{C})$  that maps surjectively to  $G$  by the natural projection. We may assume that  $\tilde{G}$  is the smallest group with this property.

The vector space  $H^0(\mathcal{O}_X(H))$  is an irreducible representation of the group  $\tilde{G}$ . From Table 1, this is immediate in all cases except 4.3(vi). In this remaining case,  $G \simeq \text{PSL}_2(\mathbf{F}_{11})$

and  $X$  the unique smooth Fano threefold of Picard rank 1 and genus 8 that admits a faithful action of the group  $\mathrm{PSL}_2(\mathbf{F}_{11})$  (see [Pro12, example 2.9]). The space  $H^0(\mathcal{O}_X(H))$  is isomorphic to the representation  $\bigwedge^2 V$ , where  $V$  is a 5-dimensional faithful representation of  $G$ . Recall that

$$\chi_{\bigwedge^2 V}(g) = \frac{1}{2} \left( \chi_V(g)^2 - \chi_V(g^2) \right),$$

where  $g \in G$  and  $\chi_{\bigwedge^2 V}$  (resp.  $\chi_V$ ) is the character of  $\bigwedge^2 V$  (resp.  $V$ ). Evaluating  $g$  at any element of order 2 in  $\mathrm{PSL}_2(\mathbf{F}_{11})$ , we conclude that  $\bigwedge^2 V$  is irreducible from the character table [CCN+85].

Since  $H^0(\mathcal{O}_X(H))$  is an irreducible representation of the group  $\tilde{G}$ , the threefold  $X$  does not contain  $G$ -invariant subvarieties contained in a proper linear subspace of  $\mathbb{P}^m$ .

Applying Proposition 4.4, we see that either  $X$  contains a  $G$ -orbit  $\Sigma$  such that

$$m < |\Sigma| = h^0(\mathcal{O}_X((n+1)H)) - h^0(\mathcal{O}_X((n+1)H) \otimes \mathcal{I}_\Sigma),$$

where  $\mathcal{I}_\Sigma$  is the ideal sheaf of  $\Sigma$ , or there exists a  $G$ -irreducible reduced curve  $C$  that is a disjoint union of smooth irreducible curves  $C_1, \dots, C_r$  of genus  $g$  and degree  $d = H \cdot C_i$  such that  $rd \leq H^3 n^2$ ,  $2g - 2 \leq nd$  and

$$r((n+1)d - g + 1) = h^0(\mathcal{O}_X((n+1)H)) - h^0(\mathcal{O}_X((n+1)H) \otimes \mathcal{I}_C),$$

where  $\mathcal{I}_C$  is the ideal sheaf of the curve  $C$ . In the former case, by Remark 4.5, the number  $|\Sigma|$  is the dimension of some  $\tilde{G}$ -subrepresentation in  $H^0(\mathcal{O}_X((n+1)H))$ . Likewise, in the latter case, the number  $r((n+1)d - g + 1)$  is also the dimension of some  $\tilde{G}$ -subrepresentation in  $H^0(\mathcal{O}_X((n+1)H))$ . Moreover, if  $r = 1$ , then the natural homomorphism  $G \rightarrow \mathrm{Aut}(C)$  is injective, because  $C$  is not contained in a hyperplane in this case. Thus, if  $r = 1$ , then

$$84(g - 1) \geq |G| \tag{4.5.1}$$

by Hurwitz’s automorphisms theorem.

**Case 4.3(i).** Here  $G \simeq \mathfrak{A}_7$  and  $X$  is the unique smooth intersection of a quadric and a cubic in  $\mathbb{P}^5$  that admits a faithful action of the group  $\mathfrak{A}_7$ . We have  $n = 1$ ,  $H^3 = 6$ ,  $m = 5$ ,  $\tilde{G} \simeq \mathfrak{A}_7$  and  $h^0(\mathcal{O}_X(2H)) = 20$ . Note that  $H^0(\mathcal{O}_X(H))$  is the irreducible  $\mathfrak{A}_7$ -representation obtained as the quotient of the standard permutation representation by the trivial representation.

Suppose that there exists a  $G$ -orbit  $\Sigma$  in  $X$  such that  $5 < |\Sigma| \leq 20$ . Using Table 2, we see that  $|\Sigma|$  is either 7 or 15. In the case  $|\Sigma| = 15$ , a stabilizer of a point in  $\Sigma$  is isomorphic to  $\mathrm{PSL}_2(\mathbf{F}_7)$ . The restriction of the representation  $H^0(\mathcal{O}_X(H))$  to  $\mathrm{PSL}_2(\mathbf{F}_7)$  is the quotient of a transitive permutation representation so it has no trivial subrepresentations. Since  $\mathrm{PSL}_2(\mathbf{F}_7)$  is simple, it therefore has no one-dimensional subrepresentations. The action cannot fix a point so this case is impossible. In the case  $|\Sigma| = 7$ , the stabiliser is isomorphic to  $\mathfrak{A}_6$ . In this case, there is a fixed point in the ambient space  $\mathbb{P}^5$  corresponding to the fixed point of the natural permutation action. One checks that this point does not lie on  $X$  by explicitly checking the defining equations, which are just elementary symmetric functions.

Thus, the threefold  $X$  contains a  $G$ -irreducible reduced curve  $C$  that is a disjoint union of smooth irreducible curves  $C_1, \dots, C_r$  of genus  $g$  and degree  $d$  such that  $rd \leq 6$  and  $2g - 2 \leq d$ . As above, this shows that  $r = 1$ , so that  $d \leq 6$  and  $g \leq 4$ , which contradicts (4.5.1).

Table 2. Conjugacy classes of maximal subgroups with indices.

$\mathfrak{A}_5$		$\mathfrak{A}_6$		$\mathfrak{A}_7$	
$\mathfrak{A}_4$	5	$\mathfrak{A}_5$	6	$\mathfrak{A}_6$	7
$D_{10}$	6	$\mathfrak{A}_5$	6	$\text{PSL}_2(\mathbf{F}_7)$	15
$\mathfrak{S}_3$	10	$\mu_3^2 \times \mu_4$	10	$\text{PSL}_2(\mathbf{F}_7)$	15
		$\mathfrak{S}_4$	15	$\mathfrak{S}_5$	21
		$\mathfrak{S}_4$	15	$(\mathfrak{A}_4 \times \mu_3) \times \mu_2$	35
$\text{PSL}_2(\mathbf{F}_7)$		$\text{PSL}_2(\mathbf{F}_{11})$		$\text{PSp}_4(\mathbf{F}_3)$	
$\mathfrak{S}_4$	7	$\mathfrak{A}_5$	11	$\mu_2^4 \times \mathfrak{A}_5$	27
$\mathfrak{S}_4$	7	$\mathfrak{A}_5$	11	$\mathfrak{S}_6$	36
$\mu_7 \times \mu_3$	8	$\mu_{11} \times \mu_5$	12	$\text{SU}_3(\mathbf{F}_2) \times \mu_3$	40
		$D_{12}$	55	$\mu_3^3 \times \mathfrak{S}_4$	40
				$\mu_2 \cdot (\mathfrak{A}_4 \wr \mu_2)$	45

Case 4.3(ii). Here  $G \simeq \mathfrak{A}_7$  and  $X = \mathbb{P}^3$ . We have  $n = 4$ ,  $H^3 = 1$ ,  $m = 3$ ,  $\tilde{G} \simeq 2.\mathfrak{A}_7$  and  $h^0(\mathcal{O}_X(5H)) = 56$ . Note that  $H^0(\mathcal{O}_X(H))$  is an irreducible four-dimensional representation of the group  $\tilde{G}$ .

Suppose that  $\mathbb{P}^3$  contains a  $G$ -orbit  $\Sigma$  such that  $3 < |\Sigma| \leq 56$ . Going through the list of subgroups in  $G$  of index  $\leq 56$ , we see that

$$|\Sigma| \in \{7, 15, 21, 35, 42\}.$$

Let  $G_P$  be the stabiliser of a point  $P \in \Sigma$ . Using Table 2, we conclude  $G_P$  is isomorphic to one of the following groups:  $\mathfrak{A}_6$ ,  $\text{PSL}_2(\mathbf{F}_7)$ ,  $\mathfrak{S}_5$ ,  $(\mathfrak{A}_4 \times \mu_3) \times \mu_2$ , or  $\mathfrak{A}_5$ . Let  $\tilde{G}_P$  be a subgroup in  $\tilde{G}$  that is mapped to  $G_P$ . We claim that the restriction of the representation  $V = H^0(\mathcal{O}_X(H))$  to  $\tilde{G}_P$  does not contain one-dimensional subrepresentations, which contradicts the fact that  $G_P$  fixes the point  $P \in \mathbb{P}^3$ . Let  $\chi$  be the 4-dimensional representation of  $\tilde{G}$ ; we will consider restricted characters of  $\chi$  (see [CCN+85]). We have  $\chi(g) = -(1 \pm \sqrt{-7})/2$  when  $g \in \tilde{G}$  has order 7; if  $G_P \simeq \text{PSL}_2(\mathbf{F}_7)$  then the only possibility is that  $\tilde{G}_P \simeq \text{PSL}_2(\mathbf{F}_7)$  and  $V|_{\tilde{G}_P}$  is irreducible. We have  $\chi(g) = -1$  when  $g$  has order 5, which means that  $V|_{\tilde{G}_P}$  is irreducible if  $G_P \cong \mathfrak{A}_5$  (and a fortiori  $\mathfrak{S}_5$  and  $\mathfrak{A}_6$ ). It remains to consider  $G_P \cong (\mathfrak{A}_4 \times \mu_3) \times \mu_2$ , which contains a 3-Sylow subgroup  $H \subseteq G$ . For some elements  $g$  of order 3, we have  $\chi(g) = -2$  meaning that  $\chi|_H$  does not have any trivial subrepresentations. Since the group  $G_P$  has no non-trivial maps to  $\mu_3$ , we conclude that there are no one-dimensional subrepresentations. With the claim proved, we see this case is impossible.

Thus, there is a  $G$ -irreducible reduced curve  $C$  in  $\mathbb{P}^3$  with the following properties:  $C$  is union of smooth irreducible curves  $C_1, \dots, C_r$  of genus  $g$  and degree  $d$ ,  $rd \leq 16$ ,  $2g - 2 \leq 4d$  and

$$r(5d - g + 1) \leq 56.$$

As above, we see that  $r \in \{1, 7, 15\}$ . If  $r = 15$ , then  $d = 1$ , so that  $g = 0$  and

$$90 = r(5d - g + 1) \leq 56,$$

which is absurd. Likewise, if  $r = 7$ , then  $d \leq 2$ , so that  $g = 0$  and

$$35d + 7 = 7(5d + 1) = r(5d - g + 1) \leq 56,$$

so that  $d = 1$ . In this case, the stabilizer of the line  $C_1$  is isomorphic to  $\mathfrak{A}_6$ , which is impossible, since the restriction of the representation  $H^0(\mathcal{O}_X(H))$  to the subgroup  $2.\mathfrak{A}_6$  is irreducible. Thus, we see that  $r = 1$ , so that  $C$  is irreducible. Using (4.5.1), we see that  $g \in \{31, 32, 33\}$ . By Lemma 2.2, one of the expressions

$$\frac{169}{2^2 \cdot 3 \cdot 7}, \quad \frac{5071}{2^3 \cdot 3^2 \cdot 5 \cdot 7}, \quad \frac{634}{3^2 \cdot 5 \cdot 7}$$

is a non-negative integer combination of expressions of the form  $1 - 1/r$  for

$$r = 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14.$$

Since  $1/2 + 2/3 + 6/7$  is larger than these expressions, this is impossible.

**Case 4.3(iii).** Here  $G \simeq \text{PSp}_4(\mathbf{F}_3)$  and  $X = \mathbb{P}^3$ . We have  $n = 4$ ,  $H^3 = 1$ ,  $m = 3$ ,  $\tilde{G} \simeq \text{Sp}_4(\mathbf{F}_3)$  and  $h^0(\mathcal{O}_X(5H)) = 56$ . We will see that  $H^0(\mathcal{O}_X(5H))$  is a direct sum of irreducible representations of  $\tilde{G}$  of dimensions 20 and 36. Indeed,  $h^0(\mathcal{O}_X(H))$  is an irreducible 4-dimensional representation  $V$  of  $\tilde{G}$  with character  $\chi$ . Note that every summand of  $S^5V$  must be a faithful representation of  $\tilde{G}$  since 5 is coprime to 2. Via the Newton identities, we have the standard formula for the 5th symmetric power

$$S^5\chi(g) = \frac{1}{120} \left[ \chi(g)^5 + 10\chi(g^2)\chi(g)^3 + 15\chi(g^2)^2\chi(g) + 20\chi(g^3)\chi(g)^2 + 20\chi(g^3)\chi(g^2) + 30\chi(g^4)\chi(g) + 24\chi(g^5) \right],$$

where  $g$  is an element of  $\tilde{G}$ . From the character table [CCN+85], we see that  $\chi(g) = -1$  for any element  $g$  of order 5. We compute that  $S^5\chi(g) = 1$  and conclude from the character table that the only possibility is a sum of characters of degree 20 and 36 as desired.

Suppose that  $\mathbb{P}^3$  contains a  $G$ -orbit  $\Sigma$  such that  $|\Sigma|$  is the dimension of some  $\tilde{G}$ -subrepresentation in  $H^0(\mathcal{O}_X(5H))$ . Then

$$|\Sigma| \in \{20, 36, 56\}.$$

Using Table 3, we see that  $|\Sigma| = 36$ . Let  $G_P$  be the stabiliser of a point  $P \in \Sigma$ . Then  $G_P \simeq \mathfrak{S}_6$ . The group  $G$  contains one such subgroup up to conjugation. Let  $\tilde{G}_P$  be a subgroup in  $\tilde{G}$  that is mapped to  $G_P$ . Then  $\tilde{G}_P \simeq 2.\mathfrak{S}_6$ , and the restriction of the representation  $H^0(\mathcal{O}_X(H))$  to  $\tilde{G}_P$  does not contain one-dimensional subrepresentations. This contradicts the fact that  $G_P$  fixes the point  $P \in \mathbb{P}^3$ .

Thus, there is a  $G$ -irreducible reduced curve  $C$  in  $\mathbb{P}^3$  that is a union of smooth irreducible curves  $C_1, \dots, C_r$  of genus  $g$  and degree  $d$  such that  $rd \leq 16$ ,  $2g - 2 \leq 4d$  and

$$r(5d - g + 1) \leq 56.$$

Arguing as above, we see that  $r = 1$ , so that  $d \leq 16$  and  $g \leq 33$ , which is impossible by (4.5.1).

**Case 4.3(iv).** Here  $G \simeq \text{PSp}_4(\mathbf{F}_3)$  and  $X$  is the Burkhardt quartic in  $\mathbb{P}^4$ . We have  $n = 1$ ,  $H^3 = 4$ ,  $m = 4$ ,  $\tilde{G} \simeq G$  and  $h^0(\mathcal{O}_X(2H)) = 15$ . Note that  $\mathbb{P}^3$  does not contain a  $G$ -orbit  $\Sigma$  such that  $4 < |\Sigma| \leq 15$ , because  $G$  does not contain subgroups of such index. Thus, there is a  $G$ -irreducible reduced curve  $C$  in  $\mathbb{P}^3$  that is union of smooth irreducible curves  $C_1, \dots, C_r$

of genus  $g$  and degree  $d$  such that  $rd \leq 4$  and  $2g - 2 \leq d$ . Since there are no subgroups of index 2, 3, or 4, we have  $r = 1$ . Thus  $d \leq 4$  and  $g \leq 3$ , which contradicts (4.5.1).

Case 4.3(v). Here  $G \simeq \text{PSL}_2(\mathbf{F}_{11})$  and  $X$  is the Klein cubic threefold in  $\mathbb{P}^4$ . We have  $n = 2$ ,  $H^3 = 3$ ,  $m = 4$ ,  $\tilde{G} \simeq G$  and  $h^0(\mathcal{O}_X(3H)) = 34$ . If  $\chi$  is the character of the representation  $V = h^0(\mathcal{O}_X(3H))$  of  $G$ , then the character of the third symmetric power is given by

$$S^3\chi(g) = \frac{1}{6} \left[ \chi(g)^3 + 3\chi(g^2)\chi(g) + 2\chi(g^3) \right]$$

for  $g \in G$ . The trivial character occurs exactly once in  $S^3V$  by Table 1. From the character table [CCN+85], for any element  $g \in G$  of order 3 we have  $\chi(g) = -1$  and thus  $S^3\chi(g) = 2$ . Note that all irreducible characters  $\rho$  satisfy  $\rho(g) \in \mathbb{R}$ , but only the trivial and 10-dimensional ones have  $\rho(g) > 0$ . Since there is only one trivial subrepresentation, this forces the existence of at least one 10-dimensional irreducible subrepresentation. The possible irreducible characters have degrees 1, 5, 10, 11, 12, thus  $1 + 10 + 12 + 12$  is the only possibility for  $S^3V$ . For any  $h \in G$  of order 5,  $\chi(h) = 0$  and  $S^3\chi(h) = 0$  while  $\rho(h) \neq 0$  for the 12-dimensional irreducible representations. We conclude that the vector space  $H^0(\mathcal{O}_X(3H))$  splits as a sum of two non-isomorphic twelve-dimensional representations, and one ten-dimensional representation.

Suppose that  $X$  contains a  $G$ -orbit  $\Sigma$  such that  $|\Sigma|$  is the dimension of some  $G$ -subrepresentation in  $H^0(\mathcal{O}_X(3H))$ . Going through the list of subgroups in  $G$  of index  $\leq 34$ , we see that  $|\Sigma| = 12$ . Let  $G_P$  be the stabilizer of a point  $P \in \Sigma$ . Then  $G_P \simeq \mu_{11} \rtimes \mu_5$ , and the restriction of the representation  $H^0(\mathcal{O}_X(H))$  to  $G_P$  is irreducible. This contradicts to the fact that  $G_P$  fixes a point in  $\mathbb{P}^4$ .

Thus, there is a  $G$ -irreducible reduced curve  $C$  in  $\mathbb{P}^4$  that is union of smooth irreducible curves  $C_1, \dots, C_r$  of genus  $g$  and degree  $d$  such that  $rd \leq 12$ ,  $2g - 2 \leq 2d$  and  $r(3d - g + 1) \leq 34$ . Going through the list of subgroups in  $G$  of index  $\leq 12$ , we see that  $r \in \{1, 11, 12\}$ . If  $r = 12$  or  $r = 11$ , then  $d = 1$ , so that  $g = 0$ , which gives

$$11 \times 4 \leq 4r = r(3d + 1) = r(3d - g + 1) \leq 34,$$

which is absurd. Thus, we have  $r = 1$ , so that  $C$  is irreducible. Then  $d \leq 12$  and  $g \leq 13$ . Using (4.5.1), we see that  $g \geq 9$ , so that  $g \in \{9, 10, 11, 12, 13\}$ . Since  $|G| = 25920 = 2^6 \cdot 3^4 \cdot 5$  and  $2g - 2$  is never divisible by  $2^5$  or  $3^3$ , the expression on the left hand side from Lemma 2.2 has  $2^2 \cdot 3^2$  in the denominator when written in lowest terms. However, it is a non-negative integer combination of expressions of the form  $1 - \frac{1}{r}$  for  $r = 1, 2, 3, 4, 5, 6, 9, 12$ . Thus, this case is impossible.

Case 4.3(vi). Here  $G \simeq \text{PSL}_2(\mathbf{F}_{11})$  and  $X$  is the unique smooth Fano threefold of Picard rank 1 and genus 8 that admits a faithful action of the group  $\text{PSL}_2(\mathbf{F}_{11})$ . We have  $n = 1$ ,  $H^3 = 14$ ,  $m = 9$ ,  $\tilde{G} \simeq G$  and  $h^0(\mathcal{O}_X(3H)) = 40$ . Going through the list of subgroups in  $G$  of index  $\leq 40$ , we see that either  $G_P \simeq \mathfrak{A}_5$  or  $G_P \simeq \mu_{11} \rtimes \mu_5$ . Since  $X$  is smooth, the embedded tangent space at  $P$  is 3-dimensional and has a faithful  $G_P$  action. This means that the restriction of the representation  $H^0(\mathcal{O}_X(H))$  to  $G_P$  has a trivial representation and a 3-dimensional faithful subrepresentation. This is impossible if  $G_P \simeq \mu_{11} \rtimes \mu_5$ , so  $G_P \simeq \mathfrak{A}_5$ . From the character table [CCN+85] of  $\text{PSL}_2(\mathbf{F}_{11})$ , we see that any faithful irreducible character  $\chi$  of degree

$\leq 10$  must have  $\chi(g) = 0$  for all  $g \in G$  of order 5. However, we have  $\rho(g) = (1 \pm \sqrt{5})/2$  for an irreducible character  $\rho$  of  $\mathfrak{A}_5$  of degree 3. Thus there must be two 3-dimensional faithful  $\mathfrak{A}_5$ -subrepresentations of  $H^0(\mathcal{O}_X(H))$  along with a trivial subrepresentation. The remaining character must be a character  $\sigma$  of  $\mathfrak{A}_5$  of degree 2 such that  $\sigma(g) = -2$ . No such characters exist, so this case is impossible.

Thus, there is a  $G$ -irreducible reduced curve  $C$  in  $X$  that is a union of smooth irreducible curves  $C_1, \dots, C_r$  of genus  $g$  and degree  $d$  such that  $rd \leq 14$ ,  $2g - 2 \leq d$  and  $r(2d - g + 1) \leq 40$ . Arguing as above, we see that  $r \in \{1, 11, 12\}$ . As above, denote by  $G_1$  the stabiliser of the curve  $C_1$ . If  $r = 12$ , then  $d = 1$ , so that  $C_1$  is a line, which implies that the restriction of the representation  $H^0(\mathcal{O}_X(H))$  to  $G_1$  contains a two-dimensional subrepresentation. But we already checked that this is not the case, so that  $r \neq 12$ . Similarly, we see that  $r \neq 11$ , because  $G_1 \simeq \mathfrak{A}_5$  in this case, and the restriction of the representation  $H^0(\mathcal{O}_X(H))$  to  $G_1$  does not have two-dimensional subrepresentations either. Hence, we see that  $r = 1$ , so that  $C$  is irreducible. Then  $d \leq 14$  and  $g \leq 8$ , which is impossible by (4.5.1). This completes the proof of Theorem 4.3.

4.6. *The proof*

Now we are ready to prove:

PROPOSITION 4.7. *Let  $X$  be a rationally connected threefold. Then the group  $\text{Aut}(X)$  does not contain a subgroup isomorphic to  $\text{SL}_2(\mathbf{F}_{11})$ ,  $\text{Sp}_4(\mathbf{F}_3)$ ,  $2.\mathfrak{A}_7$  or  $6.\mathfrak{A}_7$ .*

*Proof.* Let  $X$  be a rationally connected threefold. Let  $G$  be a subgroup in  $\text{Aut}(X)$ . Suppose that  $G$  is one of the following groups  $\text{SL}_2(\mathbf{F}_{11})$ ,  $\text{Sp}_4(\mathbf{F}_3)$ ,  $2.\mathfrak{A}_7$  or  $6.\mathfrak{A}_7$ . We seek a contradiction. We may assume that  $G \not\cong 6.\mathfrak{A}_7$  because in this case taking the quotient by a subgroup of order 3 in the center reduces the problem to  $2.\mathfrak{A}_7$ . Thus  $z(G) \simeq \mu_2$ . We may assume that  $X$  has a structure of  $G$ -Mori fiber space  $\pi : X \rightarrow S$ . By Theorem 2.5 the base  $S$  is a point, i.e.  $X$  is a  $G\mathbb{Q}$ -Fano threefold.

Let  $Y = X/z(G)$ , let  $\pi : X \rightarrow Y$  be the quotient map and let  $\bar{G} := G/z(G)$ . Then  $Y$  has canonical singularities by Lemmas 4.1.1, 4.1.2 and Claim 4.7.2 below. Using Theorem 1.3, we see that  $Y$  is  $\bar{G}$ -birational to one of the Fano threefolds listed in Theorem 4.3 and by this theorem  $Y$ , in fact, is  $\bar{G}$ -isomorphic to one of the varieties in the list. In all our cases  $Y$  is a  $\bar{G}$ -Fano with at worst isolated ordinary double points (in fact,  $Y$  is smooth except for the case 4.3(iv)). Moreover,  $\text{Cl}(Y)^{\bar{G}} = \text{Pic}(Y)$  by [CPS16, lemma 2.2]. The Hurwitz formula gives

$$K_X = \pi^* \left( K_Y + \frac{1}{2}B \right), \tag{4.7.1}$$

where  $B$  is the branch divisor. Thus  $B$  is non-zero  $\bar{G}$ -invariant and there exists a Cartier divisor  $D$  such that  $2D \sim B$ . In particular, the Fano index of  $Y$  is even. Therefore, we are left with the cases 4.3(ii), 4.3(iii), 4.3(v). But in the case 4.3(ii) we have  $Y \simeq \mathbb{P}^3$ ,  $\bar{G} \simeq \mathfrak{A}_7$  and  $\text{deg } B \leq 6$  by (5.7.1) which is impossible because the minimal degree of invariants in this case is at least 8 (see Table 1). Likewise we obtain a contradiction in the cases 4.3(iii) and 4.3(v).

CLAIM 4.7.2. *Any  $z(G)$ -fixed point of  $X$  satisfies the conditions of Lemma 4.1.1 or Lemma 4.1.2.*

*Proof.* Note that the fixed point locus of  $z(G)$  on  $X$  is  $G$ -invariant and so its divisorial part must be a  $\mathbb{Q}$ -Cartier divisor. Hence any Gorenstein point of  $X$  satisfies conditions of Lemma 4.1.1. Assume that  $X$  is a non-Gorenstein Fano threefold. Let  $P \in X$  be a non-Gorenstein point and let  $G_P \subset G$  be its stabiliser.

Arguing as in the proof of [Pro12, lemma 6.1] or Lemma 5.5 below one can show that  $P \in X$  is a cyclic quotient singularity of type  $\frac{1}{2}(1, 1, 1)$ . As in Lemma 4.1.2, consider the index-one cover  $\pi : (X^\sharp \ni P^\sharp) \rightarrow (X \ni P)$  and the lifting  $G_P^\sharp$  to  $\text{Aut}(X^\sharp \ni P^\sharp)$ . Since the length of the orbit of  $P$  is at most 15 (see [Pro12, lemma 6.1] or Lemma 5.5), we see that there are only the following possibilities [CCN+85], [GAP18]:

- (i)  $G \simeq 2.A_7, G_P \simeq 2.A_6;$
- (ii)  $G \simeq 2.A_7, G_P \simeq \text{SL}_2(\mathbb{F}_7);$
- (iii)  $G \simeq \text{SL}_2(\mathbb{F}_{11}), G_P \simeq 2.A_5;$
- (iv)  $G \simeq \text{SL}_2(\mathbb{F}_{11}), G_P \simeq \mu_{22} \rtimes \mu_5.$

Now one can see that in the cases (i)-(iii) any extension  $G_P^\sharp$  of  $G_P$  by  $\mu_2$  splits. Consider the case (iv). Assume that the center of  $G_P^\sharp$  contains an element  $z$  of order 4. Then the kernel of the homomorphism  $G_P^\sharp \rightarrow \mu_5$  must be a cyclic group  $\mu_{44}$ . But then  $G_P^\sharp$  has no faithful 3-dimensional representation, a contradiction.

### 5. $3.A_7$

The aim of this section is to prove the following.

**PROPOSITION 5.1.** *Let  $X$  be a rationally connected threefold. Then the group  $\text{Aut}(X)$  does not contain a subgroup isomorphic to  $3.A_7$ .*

Let  $G = 3.A_7$ . Assume that  $G \subset \text{Aut}(X)$  where  $X$  is a rationally connected threefold. We may assume that  $X$  has the structure of a  $G$ -Mori fiber space  $\pi : X \rightarrow S$ . By Theorem 2.5, the base  $S$  is a point, i.e.  $X$  is a  $G\mathbb{Q}$ -Fano threefold. We distinguish two cases: 5.2 and 5.3.

#### 5.2. Actions on Gorenstein Fano threefolds

First we consider the case where  $K_X$  is Cartier, i.e. the singularities of  $X$  are at worst terminal Gorenstein. By Propositions 3.1.1 and 3.1.2 we have  $\text{Pic}(X) = \mathbb{Z} \cdot K_X$ . Let  $g = g(X)$  be the genus of  $X$ . Thus  $(-K_X)^3 = 2g - 2$ . By Lemmas 3.2 and 3.3 the linear system  $| -K_X |$  defines an embedding to  $\mathbb{P}^{g+1}$ . By [Pro16] we have  $g \neq 12$ . Recall that any Fano threefold  $X$  with terminal Gorenstein singularities admits a smoothing, i.e. a deformation  $\mathfrak{X} \rightarrow \mathfrak{D} \ni 0$  over a disk  $\mathfrak{D} \subset \mathbb{C}$  such that the central fiber  $\mathfrak{X}_0$  is isomorphic to  $X$  and a general fiber is smooth [Nam97]. The numerical invariants such as the degree, the Picard number, and the Fano index are constant in such a family  $\mathfrak{X}/\mathfrak{D}$ . Now by the classification of smooth Fano threefolds [Isk80a] or [IP99] we conclude that  $g \leq 10$ . By Lemma 2.7 the group  $G$  has no fixed points.

**CLAIM 5.2.1.**  *$X$  has no  $G$ -invariant hyperplane sections.*

*Proof.* Assume that there exists a  $G$ -invariant divisor  $S \in | -K_X |$ . By Theorem 2.9.1 the pair  $(X, S)$  is not plt. By Lemma 3.5 the surface  $S$  is reducible and reduced. Since  $\text{Pic}(X) =$

$\mathbb{Z} \cdot [S]$ , the linear system  $|-K_X| = |S|$  has no fixed components. By Bertini’s theorem this  $G$ -invariant surface  $S \in |-K_X|$  is unique. Write  $S = \sum_{i=1}^m S_i$ . Then

$$\sum (-K_X)^2 \cdot S_i = 2g - 2 \leq 18.$$

So the cardinality of the orbit of  $S_1$  equals 7 or 14 by Table 2. In both cases  $g = 8$  so that  $\dim H^0(X, \mathcal{O}_X(-K_X)) = 10$ . We get a contradiction with Table 1.

Let  $V := H^0(X, \mathcal{O}_X(-K_X))^\vee$ . The group  $G$  faithfully acts on  $V$  and  $\mathbb{P}(V)$ . Hence the representation  $V$  of  $G$  is reducible. Since  $\dim V = g + 2 \leq 12$ , we have  $V = V' \oplus V''$  (as a  $G$ -module), where  $V'$  and  $V''$  are irreducible representations with  $\dim V'' = \dim V' = 6$ . Thus  $g = 10$ .

LEMMA 5.2.2. *Let  $U$  be an irreducible 6-dimensional representation of  $3\mathfrak{A}_7$  and let  $Q \subset S^2U^\vee$  be a 6-dimensional subrepresentation. Then the base locus of  $Q$  on  $\mathbb{P}(U)$  is empty.*

*Proof.* Observe from Table 1, that all 6-dimensional irreducible representations  $U$  have a unique invariant cubic in  $\mathbb{P}(U)$  defined by a polynomial  $f$ . Then the quadratic polynomials  $\partial f / \partial x_i$  generate a 6-dimensional irreducible subrepresentation  $Q \subseteq S^2U^\vee$  which is isomorphic to  $U$ . The complement  $U'$  of  $Q$  in  $S^2U^\vee$  is 15-dimensional. If  $U$  is faithful, then so must be  $U'$  and we conclude immediately from the character table [CCN+85] that  $S^2U^\vee \simeq U \oplus U_{15}$ , where  $U_{15}$  is a faithful irreducible representation of dimension 15. If  $U$  is not faithful, then from Table 1 we see that  $U'$  contains a unique trivial subrepresentation. In this case,  $S^2U^\vee \simeq \mathbb{C} \oplus U \oplus W_{14}$ , where  $W_{14}$  is a non-faithful irreducible representation of dimension 14. From the character table, the only possibility is that  $U'$  is a direct sum of an irreducible 14-dimensional representation and a trivial representation. In either case,  $Q$  is the the unique 6-dimensional irreducible subrepresentation. One can check that the hypersurface  $f = 0$  in  $\mathbb{P}(U)$  is smooth. Hence quadrics from  $Q$  have no common zeros in  $\mathbb{P}(U)$ .

Remark 5.2.3. Let  $X_3$  a cubic fourfold in  $\mathbb{P}^5$  that admits a faithful action of the group  $\mathfrak{A}_7$ . Then  $X_3$  is one of two hypersurfaces that (implicitly) appear in the proof of Lemma 5.2.2. For one of them, the action of  $\mathfrak{A}_7$  is given by the standard irreducible six-dimensional representation of the group  $\mathfrak{A}_7$ . For the other, the action is given by an irreducible faithful six-dimensional representation of the group  $3\mathfrak{A}_7$ . In the former case, one has  $\alpha_{\mathfrak{A}_7}(X_3) \leq 2/3$ , because  $\mathbb{P}^4$  contains an  $\mathfrak{A}_7$ -invariant quadric hypersurface. Here,  $\alpha_{\mathfrak{A}_7}(X_3)$  is the  $\mathfrak{A}_7$ -invariant  $\alpha$ -invariant of Tian defined in [Tia87]. However, in the latter case, one has  $\alpha_{\mathfrak{A}_7}(X_3) \geq 1$ . Indeed, suppose that  $\alpha_{\mathfrak{A}_7}(X_3) < 1$ . Then there exists an effective  $\mathfrak{A}_7$ -invariant divisor  $D$  on  $X_3$  such that  $D \sim_{\mathbb{Q}} -K_{X_3}$  and  $(X_3, D)$  is not lc. This follows from the algebraic formula for  $\alpha_{\mathfrak{A}_7}(X_3)$  given in [CS08, appendix A]. Choose positive rational number  $\mu < 1$  such that  $(X_3, \mu D)$  is strictly lc. Let  $Z$  be a minimal center of lc singularities of the log pair  $(X_3, \mu D)$ . Then  $\dim(Z) \leq 2$ , because  $\mathbb{P}^6$  does not contain  $\mathfrak{A}_7$ -invariant hyperplanes and quadric hypersurfaces. Using the perturbation trick (see the proofs of [Kaw97, theorem 1.10] and [Kaw98, theorem 1]), we may assume that all log canonical centers of the log pair  $(X_3, \mu D)$  are of the form  $\theta(Z)$  for some  $\theta \in G$ . Moreover, by [Kaw97, proposition 1.5], either  $Z \cap \theta(Z) = \emptyset$  or  $Z = \theta(Z)$  for every  $\theta \in G$ . On the other hand, it follows from [Laz04, theorem 9.4.17] that the union of all log canonical centers of the log pair  $(X_3, \mu D)$  is connected, which implies that  $Z$  is  $\mathfrak{A}_7$ -invariant. In particular, since  $\mathbb{P}^6$  does not have  $\mathfrak{A}_7$ -fixed points, the center  $Z$  is not a point, and  $\mathfrak{A}_7$  acts faithfully on  $Z$ . However, Kawamata’s



subadjunction [Kaw98, theorem 1] implies that  $Z$  is a normal Fano type subvariety, so that either it is a smooth rational curve or a rational surface with at most quotient singularities. This is impossible, because  $\mathfrak{A}_7$  is not contained in  $\text{Cr}_2(\mathbb{C})$  by [DI09]. Thus, we see that  $\alpha_{\mathfrak{A}_7}(X_3) \geq 1$  in the case when the action of  $\mathfrak{A}_7$  on  $\mathbb{P}^6$  is given by an irreducible faithful six-dimensional representation of the group  $3.\mathfrak{A}_7$ . In particular, this hypersurface admits a Kähler–Einstein metric by [Tia87]. Now, we know from [Liu22] that all smooth cubic fourfolds are Kähler–Einstein. However, when we originally wrote this paper, the only known examples of Kähler–Einstein smooth cubic fourfolds were described in [AGP06], and our  $\mathfrak{A}_7$ -invariant cubic fourfold was not one of them: in appropriate homogeneous coordinates on  $\mathbb{P}^6$  it is given by

$$\begin{aligned} &x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 + x_1x_2x_3 + x_1x_2x_4 \\ &\quad + x_1x_2x_5 + x_1x_4x_5 + x_1x_5x_6 + x_2x_4x_5 + x_2x_4x_6 + x_3x_4x_5 \\ &\quad + \omega^2x_1x_3x_4 + \omega^2x_1x_4x_6 + \omega^2x_2x_3x_5 + \omega^2x_2x_5x_6 + \omega x_1x_2x_6 + \omega x_1x_3x_5 \\ &\quad + \omega x_1x_3x_6 + \omega x_2x_3x_4 + \omega x_2x_3x_6 + \omega x_3x_4x_6 + \omega x_3x_5x_6 + \omega x_4x_5x_6 = 0, \end{aligned}$$

where  $\omega$  is a primitive cubic root of unity. This implies that it does not contain planes, while Kähler–Einstein smooth cubic fourfolds found in [AGP06] always contain many planes.

Let  $Q := H^0(X, \mathcal{I}_X(2)) \subset S^2V^\vee$  be the space of quadrics passing through  $X$ . Then  $\dim Q = 28$ . Consider the decomposition

$$S^2V = (S^2V') \oplus (S^2V'') \oplus (V' \otimes V''). \tag{5.2.4}$$

Since  $X \not\subset \mathbb{P}(V')$  and  $X \not\subset \mathbb{P}(V'')$  we have

$$Q \cap S^2V' \neq 0, \quad Q \cap S^2V'' \neq 0.$$

If both representations  $V'$  and  $V''$  are faithful, then  $V' \not\cong V''$  and (5.2.4) has the form

$$\begin{array}{ccccc} S^2V = & S^2V' & \oplus & S^2V'' & \oplus & V' \otimes V'' \\ & \wr & & \wr & & \wr \\ & V'' \oplus U_{15} & & V' \oplus U'_{15} & & \mathbb{C} \oplus W_{14} \oplus W_{21}, \end{array}$$

where  $U_{15}, U'_{15}$  are different faithful 15-dimensional irreducible representations, and  $W_k, k = 14, 21$  are irreducible representations of  $\mathfrak{A}_7$  with  $\dim W_k = k$ . The symmetric powers follow from the proof of Lemma 5.2.2. Since  $V'$  and  $V''$  are dual, the trace shows there is a trivial subrepresentation of  $V' \otimes V''$ . If  $\chi$  is the character of the product  $V' \otimes V''$ , we observe that  $\chi(g) = 0$  for all non-central  $g$  of order 3 and 4. Since the trivial representation is already a summand, we require (non-faithful) subrepresentations with characters whose values are not  $\geq 0$  for such  $g$ . Looking at the character table  $V' \otimes V'' \simeq \mathbb{C} \oplus W_{14} \oplus W_{21}$  is the only possibility.

If  $V''$  is not faithful, then similarly (5.2.4) has the form

$$\begin{array}{ccccc} S^2V = & S^2V' & \oplus & S^2V'' & \oplus & V' \otimes V'' \\ & \wr & & \wr & & \wr \\ & U_6 \oplus U_{15} & & \mathbb{C} \oplus W_6 \oplus W_{14} & & U'_{15} \oplus U_{21}. \end{array}$$

In this case, that  $V' \otimes V'' \simeq U'_{15} \oplus U_{21}$  is the only possibility can be seen by considering the values of faithful characters on the trivial element and an involution.

In both these decompositions all the irreducible summands are pairwise non-isomorphic. Counting dimensions one can see that either  $Q \cap S^2V'$  or  $Q \cap S^2V''$  contains a 6-dimensional subrepresentation. Suppose that this holds, for example, for  $Q \cap S^2V'$ . This implies that  $X \subset \mathbb{P}(V'')$ , a contradiction.

5.3. *Actions on non-Gorenstein Fano threefolds*

Now we consider the case where  $K_X$  is not Cartier.

LEMMA 5.3.1 ([Kaw92, Rei87]) *Let  $X$  be a (terminal)  $\mathbb{Q}$ -Fano threefold whose non-Gorenstein singularities are exactly  $N$  cyclic quotient points of type  $\frac{1}{2}(1, 1, 1)$ . Then we have*

$$-K_X \cdot c_2 = 24 - \frac{3N}{2}, \tag{5.3.2}$$

$$\dim |-K_X| = \frac{1}{2}(-K_X)^3 - \frac{1}{4}N + 2, \tag{5.3.3}$$

$$\dim |-2K_X| = \frac{5}{2}(-K_X)^3 - \frac{1}{4}N + 4. \tag{5.3.4}$$

ASSUMPTION 5.4. Let  $G = 3.\mathfrak{A}_7$  and let  $X$  be a non-Gorenstein  $G\mathbb{Q}$ -Fano threefold. Let  $\text{Sing}'(X)$  be the set of non-Gorenstein points and let  $N$  be its cardinality.

LEMMA 5.5. *The following assertions hold:*

- (i)  $G$  has no fixed points on  $X$ ;
- (ii)  $G$  acts transitively on  $\text{Sing}'(X)$ ;
- (iii) every non-Gorenstein point  $P \in X$  is cyclic quotient singularity of type  $\frac{1}{2}(1, 1, 1)$ ;
- (iv) for the stabiliser  $G_P$  of  $P \in \text{Sing}'(X)$  there are the following possibilities:
  - (a)  $G_P \simeq \mathfrak{A}_6$ ,  $N = 7$  or  $14$ ;
  - (b)  $G_P \simeq \text{PSL}_2(\mathbb{F}_7) \times \mu_3$ ,  $N = 15$ .

*Proof.* Take a point  $P \in \text{Sing}'(X)$ . Let  $r$  be the index of  $P$ , let  $\Omega$  be its orbit, and let and  $n := |\Omega|$ . Bogomolov–Miyaoaka inequality [Kaw92, KMMT00] gives us

$$0 < -K_X \cdot c_2 = 24 - \sum (r_i - 1/r_i) = 24 - 3N/2 \leq 24 - 3n/2.$$

Then using the list of maximal subgroups from Table 2 one can obtain the following possibilities:

- (i)  $G_P \simeq 3.\mathfrak{A}_6$ ,  $n = 7$ ;
- (ii)  $G_P \simeq \text{PSL}_2(\mathbb{F}_7) \times \mu_3$ ,  $n = 15$ .

Then one can proceed similarly to [Pro12, lemma 6.1].

COROLLARY 5.6. *Let  $\sigma : X_P \rightarrow X$  be the blowup of  $P \in \text{Sing}'(X)$  and let  $E_P = \sigma^{-1}(P)$  be the exceptional divisor. Then  $X_P$  is smooth along  $E_P$ ,  $E_P \simeq \mathbb{P}^2$ ,  $\mathcal{O}_{E_P}(E_P) \simeq \mathcal{O}_{\mathbb{P}^2}(-2)$ , and the action of  $G_P$  on  $E_P$  has no fixed points.*

5·7. First we consider the case  $|-K_X| = \emptyset$ .

LEMMA 5·7·1. *In the above notation,  $N = 14$  or  $15$ ,*

$$(-K_X)^3 = \frac{1}{2}N - 6 \geq 1, \tag{5·7·2}$$

$$\dim |-2K_X| = N - 11 \geq 3. \tag{5·7·3}$$

LEMMA 5·7·4. *The linear system  $|-2K_X|$  has no fixed components.*

*Proof.* As in [Pro12, claim 6·8·1] one can show that the Weil divisor class group  $\text{Cl}(X)$  is torsion free. Thus  $\text{Cl}(X)^G \simeq \mathbb{Z}$ . Let  $A$  be the ample generator of this group. Write  $-K_X = aA$ . Since  $K_X$  is not Cartier,  $a$  is odd. On the other hand,  $a^3A^3 = (-K_X)^3 = 1$  or  $3/2$ . Since  $aA^3 \in \mathbb{Z}/2$ , this implies  $a = 1$ , i.e.  $\text{Cl}(X)^G \simeq \mathbb{Z} \cdot K_X$ . Since  $|-K_X| = \emptyset$  the assertion follows.

LEMMA 5·8. *Let  $\mathcal{M} = |-2K_X|$ . Then the log pair  $(X, 3\mathcal{M}/2)$  is lc.*

*Proof.* Suppose that  $(X, 3\mathcal{M}/2)$  is not lc. We seek a contradiction. Choose  $\mu < 3/2$  such that  $(X, \mu\mathcal{M})$  is strictly lc. Let  $Z$  be a minimal center of lc singularities of the log pair  $(X, \mu\mathcal{M})$ . Then  $Z$  is either a point or a curve, because the base locus of  $|-2K_X|$  does not have surfaces by Lemma 5·7·4.

Observe that  $\theta(Z)$  is also a minimal center of lc singularities of this log pair for every  $\theta \in G$ . Moreover, for every  $\theta \in G$ , either  $Z \cap \theta(Z) = \emptyset$  or  $Z = \theta(Z)$  by [Kaw97, proposition 1·5]. Using the perturbation trick (see [CS16, lemma 2·4·10] or the proofs of [Kaw97, theorem 1·10] and [Kaw98, theorem 1]), for every sufficiently small  $\epsilon > 0$ , we can replace the boundary  $\mu\mathcal{M}$  by an effective boundary  $B_X$  such that

$$B_X \sim_{\mathbb{Q}} -2(\mu + \epsilon)K_X,$$

the log pair  $(X, B_X)$  is strictly lc, and all its (not necessarily minimal) centers of lc singularities are the subvarieties  $\theta(Z)$  for  $\theta \in G$ . In particular, there are minimal log canonical centers of the log pair  $(X, B_X)$ . We may assume that  $\mu + \epsilon < 3/2$ , since  $\mu < 3/2$ .

Let  $\Sigma$  be the union of all log canonical centers  $\theta(Z)$  for  $\theta \in G$ . Then  $\Sigma$  is either a  $G$ -orbit or a disjoint union of irreducible isomorphic curves, which are transitively permuted by  $G$ . In the latter case, each such curve is smooth by Kawamata’s [Kaw98, theorem 1]. Let  $\mathcal{I}_{\Sigma}$  be the ideal sheaf of the locus  $\Sigma$ . Then

$$h^1(\mathcal{O}_X(-2K_X) \otimes \mathcal{I}_{\Sigma}) = 0,$$

by [Laz04, theorem 9·4·17] or [Kol97, theorem 2·16], because  $-2K_X - (K_X + B_X)$  is ample. In particular, if  $Z$  is a point, we see that

$$|\Sigma| = h^0(\mathcal{O}_{\Sigma} \otimes \mathcal{O}_X(-2K_X)) = h^0(\mathcal{O}_X(-2K_X)) - h^0(\mathcal{O}_X(-2K_X) \otimes \mathcal{I}_{\Sigma}) \leq 5,$$

which must be a point since  $3\mathfrak{A}_7$  does not have nontrivial subgroups of index  $\leq 5$ . This is impossible by Lemma 2·7.

Thus, we see that  $\Sigma$  is a disjoint union of irreducible isomorphic smooth curves. Denote them by  $C_1 = Z, C_2, \dots, C_r$ . Let  $d = -2K_X \cdot C_i$  and let  $g$  be the genus of the curve  $C_1$ . By Kawamata’s subadjunction [Kaw98, theorem 1], for every ample  $\mathbb{Q}$ -divisor  $A$  on  $X$ , we have

$$(K_X + B_X + A) \Big|_{C_i} \sim_{\mathbb{Q}} K_{C_i} + B_{C_i}$$

for some effective  $\mathbb{Q}$ -divisor  $B_{C_i}$  on the curve  $C_i$ . This gives  $d > 2g - 2$ , so that  $d \geq 2g - 1$ . Thus, we have

$$r(d - g + 1) = h^0(\mathcal{O}_\Sigma \otimes \mathcal{O}_X(-2K_X)) = h^0(\mathcal{O}_X(-2K_X)) - h^0(\mathcal{O}_X(-2K_X) \otimes \mathcal{I}_\Sigma) \leq 5$$

by the Riemann–Roch formula applied to each curve  $C_i$ . Now, using  $r(d - g + 1) \leq 5$  and  $d \geq 2g - 1$ , we deduce that  $r = 1$  and  $g \leq 5$ . This implies that  $Z = C_1$  is pointwise fixed by  $G$ , since  $G$  cannot act non-trivially on a smooth curve of genus  $\leq 5$ . This is a contradiction, since  $G$  does not fix a point in  $X$  by Lemma 2.7.

By (5.7.3) we have that the dimension of  $H^0(X, \mathcal{O}_X(-2K_X))$  is at most 5. Hence the action of  $G = 3.\mathfrak{A}_7$  on this space is trivial.

Let  $S$  be a general surface in  $|-2K_X|$ . We claim that  $S$  is normal. Indeed, it follows from Lemma 5.8 that  $(X, \mathcal{M})$  is lc. Then, by [Kol97, theorem 4.8], the log pair  $(X, S)$  is also lc, so that  $S$  has lc singularities by [Kol97, theorem 7.5]. In particular, the surface  $S$  is normal. Take another general surface  $S' \in |-2K_X|$  and consider the invariant curve  $S \cap S'$ . Write

$$S \cap S' = \sum m_i C_i.$$

Put  $d := -2K_X \cdot C_i$ . Since  $-2K_X$  is an ample Cartier divisor, the numbers  $d_i$  are integral and positive. Then by (5.7.2)

$$\sum m_i d_i = (-2K_X) \cdot (S \cap S') = (-2K_X)^3 = 4(N - 12) = 8 \quad \text{or} \quad 12.$$

From Table 2, we see that  $G = 3.\mathfrak{A}_7$  has at least one invariant component, say  $C_1$ . We have

$$-2K_S \cdot C_1 = \left(\sum m_i C_i\right) \cdot C_1 \leq (-2K_X) \cdot \left(\sum m_i C_i\right) \leq (-2K_X)^3 \leq 12.$$

In particular,  $C_1^2 \leq 12$  and  $K_S \cdot C_1 \leq 6$ . Then by the genus formula

$$2p_a(C_1) - 2 = (K_S + C_1) \cdot C_1 \leq 18, \quad p_a(C_1) \leq 10.$$

But the according to the Hurwitz bound the action of  $G = 3.\mathfrak{A}_7$  on  $C_1$  must be trivial. This contradicts Lemma 2.7. Thus the case  $|-K_X| = \emptyset$  does not occur.

5.9. Consider the case  $\dim |-K_X| = 0$ . Then  $(-K_X)^3 = N/2 - 4 \leq 7/2$ . Let  $S \in |-K_X|$  be the unique anticanonical member. By Theorem 2.9.1 the singularities of  $S$  are worse than Du Val. Since  $G$  has no fixed points, by Lemma 3.5 the pair  $(X, S)$  is lc and  $S$  is reducible:  $S = \sum_{i=1}^m S_i$ . Then  $\sum (-2K_X)^2 \cdot S_i \leq 14$ . So the cardinality of the orbit of  $S_1$  equals 7. Let  $v : S' \rightarrow S_1$  be the normalisation. Then by the adjunction  $K_{S'} + D' = v^*(K_X + S)|_{S_1} \sim 0$  and the pair  $(S', D')$  is lc. Since  $D' \neq 0$ , the surface  $S'$  is either rational or birationally equivalent to a ruled surface over an elliptic curve. On the other hand, The pair  $(S', D')$  has a faithful action of the stabilizer  $3.\mathfrak{A}_6 \subset 3.\mathfrak{A}_7$ . This is impossible.

5.10. Now we consider the case  $\dim |-K_X| > 0$ .

LEMMA 5.10.1 ([Pro12, lemma 6.6]) *The pair  $(X, |-K_X|)$  is canonical and therefore a general member  $S \in |-K_X|$  is a K3 surface with Du Val singularities.*

Let  $\sigma : Y \rightarrow X$  be the blowup of all non-Gorenstein points and let  $E = \sum E_i$  be the exceptional divisor. Thus  $Y$  has at worst terminal Gorenstein singularities and it is smooth near  $E$ . Since  $(X, |-K_X|)$  is canonical, the linear system  $|-K_Y|$  is the birational transform of  $|-K_X|$ . Put

$$g := \dim |-K_X| - 1 = \frac{1}{2}(-K_X)^3 - \frac{1}{4}N + 1,$$

LEMMA 5.10.2 ([Pro12, lemma 6.7]) *The image of the ( $G$ -equivariant) rational map  $\Phi : X \rightarrow \mathbb{P}^{g+1}$  given by the linear system  $|-K_X|$  is three-dimensional.*

*Proof.* Suppose that  $\dim \Phi(X) < 3$ . Since  $X$  is rationally connected,  $G$  acts trivially on  $\Phi(X)$  and on  $\mathbb{P}^{g+1}$ , this contradicts Theorem 2.9.1.

LEMMA 5.10.3. *The divisor  $-K_Y$  is nef and big.*

*Proof.* Assume that  $-K_Y$  is not nef. Then  $-K_Y \cdot C' < 0$  for some curve  $C'$ . Let  $C$  be the  $G$ -orbit of  $C'$ . Then  $-K_Y \cdot C < 0$  and  $C$  is  $G$ -invariant. Note that  $C \cap E_i$  is contained in the base locus of the restricted linear system  $|-K_Y||_{E_i}$  which is a linear system of lines. Thus  $C \cap E_i$  is a  $G_{E_i}$ -invariant point on  $E_i$ . This contradicts Corollary 5.6. Thus  $-K_Y$  is nef. By Lemma 5.10.2 it is big.

LEMMA 5.10.4. *The linear system  $|-K_Y|$  is base point free and defines a crepant birational morphism*

$$\Phi : Y \longrightarrow \bar{Y} \subset \mathbb{P}^{g+1}$$

*whose image  $\bar{Y}$  is a Fano threefold with at worst canonical Gorenstein singularities. Moreover,  $\bar{Y}$  is an intersection of quadrics.*

*Proof.* Follows from Lemmas 3.2, 3.3, and 3.4.

Let  $\Pi_i := \Phi(E_i)$ . Then  $\Pi_1, \dots, \Pi_N$  are planes in  $\mathbb{P}^{g+1}$ . Fix a plane, say  $\Pi_1$  and let  $G_1 \subset G$  be its stabiliser. Suppose that  $\Pi_1 \cap \Pi_i := l$  is a line for some  $i$ . Then the  $G_1$ -orbit of  $l$  is given on  $\Pi_1 \simeq \mathbb{P}^2$  by an invariant polynomial, say  $\phi$ , which is a product of linear terms. By [Coh76, p. 412] one can see that  $\deg \phi \geq 45$  if  $G_1 \simeq 3.A_6$  and  $\deg \phi \geq 21$  if  $G_1 \simeq \text{PSL}_2(\mathbb{F}_7) \times \mu_3$ . But this implies that  $N \geq 21$ , a contradiction.

If  $\Pi_1 \cap \Pi_i := p$  is a point for some  $i$ , then we can argue as above because by duality the  $G_1$ -orbit of  $p$  has at least 21 elements.

Therefore, the planes  $\Pi_1, \dots, \Pi_N$  are disjoint. Then  $\Phi$  is an isomorphism and  $Y$  is a Fano threefold with terminal Gorenstein singularities and  $\text{rk Pic}(Y) \geq 8$  because the divisors  $E_i$  are linear independent elements of  $\text{Pic}(Y)$ . Moreover,  $Y$  is  $G\mathbb{Q}$ -factorial and  $\text{rk Pic}(Y)^G = 2$ . There exists an  $G$ -extremal Mori contraction  $\varphi : Y \rightarrow Z$  which is different from  $\sigma : Y \rightarrow X$ . Now  $\varphi$  is birational and not small. But then the  $\varphi$ -exceptional divisor  $D$  meets  $E$  and so none of the components of  $D$  are contracted to points. Therefore,  $Z$  is a Fano threefold with  $G\mathbb{Q}$ -factorial terminal Gorenstein singularities and  $\text{rk Pic}(Y)^G = 1$ . This contradicts the above considered case 5.2.

### Appendix A. Amitsur Subgroup

Here we review linearizations of line bundles and define a useful equivariant birational invariant. Much of this simply mirrors known results in the arithmetic setting, but our proofs

have a more geometric flavor. First, we review some facts about linearization of line bundles; see [Dol99, sections 1 and 2] for a more thorough discussion.

Let  $X$  be a proper complex variety with a faithful action of a finite group  $G$ . One defines a morphism of  $G$ -linearised line bundles to be a morphism of line bundles such that the map on the total spaces is equivariant. We denote the group of isomorphism classes of  $G$ -linearised line bundles by  $\text{Pic}(X, G)$ . Note that a line bundle  $\mathcal{L}$  is  $G$ -invariant if and only if  $[\mathcal{L}] \in \text{Pic}(X)^G$ . There is an evident group homomorphism  $\text{Pic}(X, G) \rightarrow \text{Pic}(X)^G$  obtained by forgetting the linearisation.

Given a  $G$ -invariant line bundle  $\mathcal{L}$ , one constructs a cohomology class  $\delta(\mathcal{L}) \in H^2(G, \mathbb{C}^\times)$  as follows. Select an arbitrary isomorphism  $\phi_g : g^*\mathcal{L} \rightarrow \mathcal{L}$  for each  $g \in G$ . Recall that any automorphism of a line bundle corresponds to multiplication by a non-zero scalar since  $X$  is proper. Define a function  $c : G \times G \rightarrow \mathbb{C}^\times$  via

$$c(g, h) := \phi(gh) (\phi_h \circ h^*(\phi_g))^{-1}$$

for all  $g, h \in G$ . One checks that  $c$  is a 2-cocycle and its cohomology class is independent of the isomorphism class of the line bundle.

We have the following exact sequence of abelian groups:

$$1 \longrightarrow \text{Hom}(G, \mathbb{C}^\times) \longrightarrow \text{Pic}(X, G) \longrightarrow \text{Pic}(X)^G \longrightarrow H^2(G, \mathbb{C}^\times).$$

We define the *Amitsur subgroup* as the group

$$\text{Am}(X, G) := \text{im}(\text{Pic}(X)^G \rightarrow H^2(G, \mathbb{C}^\times)).$$

This is the name used for the arithmetic version in [Lie17].

Note that  $\text{Am}(X, G)$  is a contravariant functor in  $X$  via pullback of line bundles. In fact, it is actually a birational invariant of smooth projective  $G$ -varieties. This is well known in the arithmetic case (see, for example, [CKM08, section 5]).

**THEOREM A.1.** *If  $X$  and  $Y$  are smooth projective  $G$ -varieties that are  $G$ -equivariantly birationally equivalent, then  $\text{Am}(X, G) = \text{Am}(Y, G)$ .*

*Proof.* First, assume that the theorem holds in the case where  $X \rightarrow Y$  is a blow-up of a smooth  $G$ -invariant subvariety.

Let  $f : X \dashrightarrow Y$  be a  $G$ -equivariant birational map. By [RY02], we may resolve indeterminacies and obtain a sequence of equivariant blow-ups  $Z \rightarrow X$  with smooth  $G$ -invariant centers and an equivariant birational morphism  $Z \rightarrow Y$ . By functoriality, we have an inclusion  $\text{Am}(Y, G) \subseteq \text{Am}(Z, G)$ . By assumption, we have equality  $\text{Am}(Z, G) = \text{Am}(X, G)$ . Thus  $\text{Am}(Y, G)$  is naturally a subset of  $\text{Am}(X, G)$ . Repeating the same argument for  $f^{-1}$  shows that the two sets are equal.

We now may assume that  $\pi : X \rightarrow Y$  is a blow up of a smooth  $G$ -invariant subvariety  $C$ . Let  $E$  be the exceptional divisor on  $X$ . We have  $\text{Pic}(X)^G = \pi^*\text{Pic}(Y)^G \oplus \mathbb{Z}[\mathcal{O}_X(E)]$  and so  $\text{Am}(X, G) = \text{Am}(Y, G) + \mathbb{Z}\delta([\mathcal{O}_X(E)])$ . To complete the proof, we show that  $\mathcal{O}_X(E)$  is  $G$ -linearisable.

Let  $\mathcal{L}$  be a very ample line bundle on  $X$  giving an embedding  $X \subseteq \mathbb{P}^n$  for some  $n$ . Since  $H^2(G, \mathbb{C}^\times)$  is torsion, by replacing  $\mathcal{L}$  by a sufficiently divisible power  $\mathcal{L}^{\otimes n}$  we may assume

that  $\mathcal{L}$  is  $G$ -linearisable and that  $C = X \cap L$  where  $L$  is a linear subspace of  $\mathbb{P}^n$ . We obtain  $E$  as the pullback of the exceptional divisor of the blow-up of  $\mathbb{P}^n$  along  $L$ .

Thus, the theorem reduces to showing that the exceptional divisor  $E$  on the blow-up  $X$  of a linear subspace  $L$  of  $\mathbb{P}^n$  with dimension  $m$ , is  $G$ -linearizable where  $G$  has a linearizable action on  $\mathbb{P}^n$ . This follows from Lemma A.2 below.

LEMMA A.2. *Let  $X$  be the blow-up of  $\mathbb{P}^n$  along a linear subspace  $L$  of dimension  $m$ . Let  $G$  be a finite group acting faithfully on  $X$ . Then the line bundle associated to the exceptional divisor is  $G$ -linearizable.*

*Proof.* In this case,  $X$  is a toric variety. The Picard group is generated by the class  $H$  of the pullback of a hyperplane in  $\mathbb{P}^n$  and and class  $E$  of the exceptional divisor. The Cox ring  $R$  may be described as the polynomial ring

$$R := \mathbb{C}[x_0, \dots, x_m, y_1, \dots, y_{n-m}, z]$$

along with a grading and irrelevant ideal described as follows. The grading is determined by its values on the generators:  $\deg(x_i) = H$ ,  $\deg(y_i) = H - E$  and  $\deg(z) = E$ . The irrelevant ideal is the radical ideal of  $R$  whose corresponding subvariety  $C$  is the union of the subspaces defined by  $x_0 = \dots = x_m = 0$  and  $y_1 = \dots = y_{n-m} = z = 0$ . Let  $S$  be the Neron-Severi torus dual to  $\text{Pic}(X)$ . For a multihomogeneous element  $m \in R$  with grading  $aH + bE$ , then  $(\lambda, \mu) \in S$  acts on  $S$  via  $(\lambda, \mu) \cdot (m) = \lambda^a \mu^b m$ . The variety  $X$  is obtained as the quotient of  $\text{Spec}(R) \setminus C$  by  $S$ .

From section 4 of [Cox95], the group of invertible elements among the graded ring of endomorphisms of  $R$  form a group  $\widetilde{\text{Aut}}(X)$  normalising  $S$  with quotient  $\text{Aut}(X)$ . The group  $\widetilde{\text{Aut}}(X)$  is isomorphic to  $U \rtimes (\text{GL}_{m+1}(\mathbb{C}) \times \text{GL}_{n-m}(\mathbb{C}) \times \mathbb{C}^\times)$  where  $\mathbb{C}^\times$  acts by scalar multiplication on  $z$ ,  $\text{GL}_{m+1}$  acts linearly on  $x_0, \dots, x_m$ ,  $\text{GL}_{n-m}$  acts linearly on  $y_1, \dots, y_{n-m}$  and elements  $u \in U$  are all of the form  $\text{id} + n$  where  $n$  is a linear map from the span of  $x_0, \dots, x_m$  to  $zy_1, \dots, zy_{n-m}$ . Since  $G$  is finite, we may assume that  $G$  has a preimage in  $\widetilde{G}$  in the subgroup  $\text{SL}_{m+1}(\mathbb{C}) \times \text{SL}_{n-m}(\mathbb{C})$  of  $\widetilde{\text{Aut}}(X)$ . Thus  $\widetilde{G} \cap S = (\mu_d, 1)$  where  $d = \text{gcd}(m + 1, n - m)$ .

Recall that the canonical bundle on  $\mathbb{P}^n$  is always linearizable, so to show that  $E$  is linearizable it suffices to show that that  $G$  has an action on the global sections of the very ample line bundle  $\mathcal{O}_X(E + d(n + 1)H)$ . From [Cox95], the vector space  $H^0(X, \mathcal{O}_X(E + d(n + 1)H))$  is isomorphic to vector subspace of  $R$  with grading  $[E + d(n + 1)H]$ . This is spanned by monomials  $x_i^a y_j^b z^c$  where  $a + b = d(n + 1)$  and  $c = b + 1$ . Now  $s = (\lambda, \mu) \in S$  acts via  $s(m) = \lambda^{d(n+1)} \mu m$  on every such monomial, thus  $\widetilde{G} \cap S$  acts trivially. We conclude the action  $\widetilde{G}$  factors through  $G$  as desired.

Remark A.3. Note that there is a much shorter proof of Theorem A.1 when  $X$  and  $Y$  are surfaces (which is actually the only case we need). One reduces to the case where  $\pi : X \rightarrow Y$  is a blow-up of a  $G$ -orbit of points. We only need to show that the exceptional divisor  $E$  is  $G$ -linearisable as before. This is immediate since  $K_X = \pi^*K_Y + E$  and both  $K_X$  and  $K_Y$  are  $G$ -linearisable. This proof fails in higher dimensions since the exceptional divisor may appear with multiplicity.

We have the following due to proposition 2.2 of [Dol99]:

PROPOSITION A.4. *If  $X$  is a smooth curve with a faithful action of  $G$ , then  $\text{Am}(X, G) = H^2(G, \mathbb{C}^\times)$  for all subgroups  $G \subseteq \text{Aut}(X)$ .*

Since the canonical bundle is always linearizable, we have the following:

PROPOSITION A.5. *If  $X$  is a smooth Fano variety of index 1 with  $\text{rk Pic}(X)^G = 1$ , then  $\text{Am}(X, G)$  is trivial for all subgroups  $G \subseteq \text{Aut}(X)$ .*

LEMMA A.6. *If  $f : X \rightarrow Y$  is a  $G$ -equivariant morphism, then the morphism  $f^* : \text{Am}(Y, G) \rightarrow \text{Am}(X, G)$  is injective. If moreover,  $f$  has a  $G$ -equivariant section, then  $\text{Am}(Y, G)$  is a direct summand of  $\text{Am}(X, G)$ .*

*Proof.* Let  $\mathcal{L}$  be an element of  $\text{Pic}(Y)^G$  and suppose  $E$  is the extension of  $G$  that acts on the total space of  $\mathcal{L}$ . Then  $E$  also acts on  $f^*\mathcal{L}$ . If  $f^*\mathcal{L}$  is  $G$ -linearisable, then  $E$  splits; thus  $\mathcal{L}$  must also be linearisable. Now suppose  $s : Y \rightarrow X$  is a section. By definition  $f \circ s = \text{id}_Y$ , so the map induced by functoriality  $\text{Am}(Y, G) \rightarrow \text{Am}(X, G) \rightarrow \text{Am}(Y, G)$  is the identity.

It is easy to determine the possible values of the invariant for rational surfaces (c.f. proposition 5.3 of [CKM08] for the arithmetic case).

PROPOSITION A.7. *Suppose  $X$  is a rational surface with  $G \subseteq \text{Aut}(X)$ . We have the following possibilities:*

- (i)  $X$  is  $G$ -equivariantly birationally equivalent to  $\mathbb{P}^2$  and  $\text{Am}(X, G)$  is isomorphic to  $\mathbb{Z}/3\mathbb{Z}$  or  $0$ ;
- (ii)  $X$  is  $G$ -equivariantly birationally equivalent to  $\mathbb{P}^1 \times \mathbb{P}^1$ , and  $\text{Am}(X, G)$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{Z}/2\mathbb{Z}$  or  $0$ ;
- (iii)  $X$  is  $G$ -equivariantly birationally equivalent to  $\mathbf{F}_n$ . If  $n$  is odd then  $\text{Am}(X, G) = 0$ . If  $n \geq 2$  is even, then  $\text{Am}(X, G)$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$  or  $0$ ;
- (iv) if  $X$  is  $G$ -equivariantly birationally equivalent to a minimal conic bundle surface with singular fibers, then  $\text{Am}(X, G)$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$  or  $0$ ;
- (v) otherwise,  $X$  is isomorphic to a del Pezzo surface of degree  $\leq 6$  and  $\text{Am}(X, G)$  is trivial.

*Proof.* It suffices to assume  $X$  is a  $G$ -minimal surface.

If  $X$  is a  $G$ -minimal del Pezzo surface, then either  $X = \mathbb{P}^2$ ,  $X = \mathbb{P}^1 \times \mathbb{P}^1$ , or it has Fano index 1. In the last case,  $\text{Am}(X, G) = 0$  by Proposition A.5.

For  $\mathbb{P}^2$ , the index is 3 and there exist non-linearizable line bundles. Thus  $\text{Am}(X, G) = 0$  or  $\mathbb{Z}/3\mathbb{Z}$ .

For  $\mathbb{P}^1 \times \mathbb{P}^1$ , we consider two cases. First, assume the fibers are interchanged by  $G$ . Then  $\text{Pic}(X)^G \simeq \mathbb{Z}$  and  $-K_X$  has index 2. Not every group is linearizable, so we have  $\text{Am}(X, G) = 0$  or  $\mathbb{Z}/2\mathbb{Z}$ .

Now suppose the fibers are not interchanged. Taking  $G = G_1 \times G_2$  with each  $G_i$  acting on each  $\mathbb{P}^1$  separately, one may have  $\mathcal{O}(1, 0)$  and  $\mathcal{O}(0, 1)$  be non-linearizable with distinct classes in  $H^2(G, \mathbb{C}^\times)$ . Thus  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  and all its subgroups are a possibility for  $\text{Am}(X, G)$ .



Suppose  $X$  is a ruled surface  $\mathbf{F}_n$ . Since we have a  $G$ -invariant section, by Lemma A.6  $\text{Am}(X, G) = \text{Am}(\mathbb{P}^1, G)$  so it only depends on the group  $G$ . From [DI09], we see that the reductive part of  $\text{Aut}(X)$  is isomorphic to  $\mathbb{C}^\times \times \text{PSL}_2(\mathbb{C})$  if  $n$  is even and  $\mathbb{C}^\times \times \text{SL}_2(\mathbb{C})$  if  $n$  is odd. The subgroup  $\mathbb{C}^\times$  acts trivially on the base, so  $\text{Am}(X, G) = 0$  if  $n$  is odd, but can be  $\text{Am}(X, G) = \mathbb{Z}/2\mathbb{Z}$  if  $n$  is even.

If  $X \rightarrow \mathbb{P}^1$  is a minimal  $G$ -conic bundle, then  $\text{Pic}(X)^G$  is generated by  $-K_X$  and  $\pi^*\text{Pic}(\mathbb{P}^1, G)$ . Since  $-K_X$  is always linearisable,  $\text{Am}(X)$  is either trivial or  $\mathbb{Z}/2\mathbb{Z}$ . Note that, for example,  $\mathbb{Z}/2\mathbb{Z} \times \mathfrak{A}_5$  acts on an exceptional conic bundle (see proposition 5.3 of [DI09]). Thus  $\text{Am}(X)$  can be non-trivial.

Note that the Amitsur subgroup can distinguish between equivariant birational equivalence classes quickly that might be more involved using other methods.

*Example A.8.* The action of  $\mathfrak{A}_5$  on  $\mathbb{P}^2$  has trivial Amitsur subgroup  $\text{Am}(\mathbb{P}^2, \mathfrak{A}_5)$  since  $\mathfrak{A}_5 \subseteq \text{PGL}_3(\mathbb{C})$  lifts to  $\text{GL}_3(\mathbb{C})$ . The action of  $\mathfrak{A}_5$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  is not linearisable since  $\mathfrak{A}_5 \subseteq \text{PGL}_2(\mathbb{C})$  does not lift to  $\text{GL}_2(\mathbb{C})$ . Thus  $\text{Am}(\mathbb{P}^2, \mathfrak{A}_5)$  and  $\text{Am}(\mathbb{P}^1 \times \mathbb{P}^1, \mathfrak{A}_5)$  are not equal. Thus  $\mathbb{P}^2$  and  $\mathbb{P}^1 \times \mathbb{P}^1$  are not  $\mathfrak{A}_5$ -birationally equivalent. This also follows from [17, theorem 6.6.1].

### Appendix B. Conic bundles

Let  $X$  be a smooth projective variety, and let  $G$  be a finite subgroup in  $\text{Aut}(X)$ . Suppose that there exists a  $G$ -equivariant conic bundle  $\eta: X \rightarrow Y$  such that  $Y$  is smooth, and the morphism  $\eta$  is flat. These assumptions mean that  $\eta$  is regular conic bundle in the sense of [Sar82, definition 1.4]. Note that  $G$  naturally acts on  $Y$ . But this action is not necessarily faithful. In general, we have an exact sequence of groups

$$1 \longrightarrow G_\eta \longrightarrow G \longrightarrow G_Y \longrightarrow 1,$$

where  $G_Y$  is a subgroup in  $\text{Aut}(Y)$ , and the subgroup  $G_\eta$  acts trivially on  $Y$ . If

$$\text{Pic}(X)^G = \eta^*\text{Pic}(Y)^{G_Y} \oplus \mathbb{Z},$$

we say that the conic bundle  $\eta: X \rightarrow Y$  is a  $G$ -minimal or  $G$ -standard (cf. [Pro18, section 1] and [Sar82, definition 1.12]). In this case, the conic bundle  $\eta: X \rightarrow Y$  is a  $G$ -Mori fiber space.

**LEMMA B.1.** *Let  $G_1 \subset G_\eta$  be a non-trivial subgroup and let  $\text{Fix}(G_1)$  be its fixed point set. Then  $\text{Fix}(G_1)$  does not contain any component of a reduced fiber.*

*Proof.* Let  $C = \eta^{-1}(Q)$  be a reduced fiber and let  $C_1 \subset C$  be an irreducible component and let  $P \in C$  be a smooth point. Assume that  $\text{Fix}(G_1) \supset C_1$ . Consider the exact sequence

$$0 \longrightarrow T_{P,C} \longrightarrow T_{P,X} \longrightarrow \eta^*(T_{Q,Y}) \longrightarrow 0. \tag{B.1.1}$$

If  $G_1$  does not act faithfully on the curve  $C_1$ , then it does not act faithfully on the tangent space  $T_{P,C}$ , and it also acts trivially on the tangent space  $T_{P,Y}$ . Thus, in this case,  $G_1$  does not act faithfully on  $T_{P,X}$ , which is impossible by Lemma 2.3, since  $G_1$  acts faithfully on  $X$  by assumption.

Suppose, in addition, that  $G_\eta \simeq \mu_n$  for  $n \geq 2$ . Let  $B$  be the union of codimension one subvarieties in  $X$  that are pointwise fixed by the subgroup  $G_\eta$ . Then  $B$  is smooth (see Lemma 2.3). Since all smooth fibers of the conic bundle  $\eta$  are isomorphic to  $\mathbb{P}^1$ , we see that the subgroup  $G_\eta$  fixes exactly two points in each (see Lemma B.1). Thus, if  $C$  is a general fiber of  $\eta$ , then the intersection  $B \cap C$  consists of two distinct points. This implies that  $B \cdot C = 2$  for every fiber  $C$  of the conic bundle  $\eta$ . Hence, the morphism  $\eta$  induces a generically two-to-one morphism  $\phi: B \rightarrow Y$ .

LEMMA B.2. *Let  $C$  be a reduced fiber of the conic bundle  $\eta$ . If  $n = |G_\eta| \geq 3$ , then  $G_\eta$  leaves invariant every irreducible component of  $C$ .*

*Proof.* We may assume that  $C$  is reducible. Then  $C = C_1 \cup C_2$ , where  $C_1$  and  $C_2$  are smooth irreducible rational curves that intersect transversally at one point. Denote this point by  $O$ . Then  $O$  is fixed by  $G_\eta$ .

If  $n = |G_\eta|$  is odd, then both  $C_1$  and  $C_2$  are  $G_\eta$ -invariant. Thus, to complete the proof, we may assume that  $n$  is even and neither  $C_1$  nor  $C_2$  is  $G_\eta$ -invariant. Let us seek for a contradiction.

Let  $z$  be a generator of the group  $G_\eta$ . Then  $z$  swaps the curves  $C_1$  and  $C_2$ . In particular, this shows that

$$B \cap C = B \cap C_1 = B \cap C_2 = C_1 \cap C_2 = O.$$

On the other hand, the element  $z^2$  leaves both curves  $C_1$  and  $C_2$  invariant. By Lemma B.1, we see that  $z^2$  acts faithfully on both these curves. Since  $C_1 \simeq \mathbb{P}^1$ , the element  $z^2$  fixes a point  $P \in C_1$  such that  $P \neq O$ . Moreover, using (B.1.1), we see that there exists a two-dimensional subspace in the tangent space  $T_{P,X}$  consisting of zero eigenvalues of the element  $z^2$ . By [BB73, theorem 2.1] this shows that  $z^2$  pointwise fixes a surface  $B_1$  in  $X$  such that  $P \in B_1$ .

Since  $P \notin B$ , we see that  $B_1$  is not an irreducible component of the surface  $B$ . On the other hand, the surface  $B$  is also pointwise fixed by  $z^2$ , because it is pointwise fixed by  $z$ . Let  $C'$  be a general fiber of the conic bundle  $\eta$ . Then  $G_\eta$  acts faithfully on  $C'$  by Lemma B.1, so that  $z^2$  fixes exactly two points in  $C'$ . On the other hand, it also fixes all points of the intersection  $B_1 \cap C'$  and all points of the intersection  $B \cap C'$ . This shows that  $z^2$  fixes at least three points in  $C'$ , which is absurd. The obtained contradiction shows that  $z$  does not swap the curves  $C_1$  and  $C_2$ , which completes the proof of the lemma.

Let  $\Delta \subset Y$  be the discriminant locus of  $\eta$ , i.e. the locus consisting of points  $P \in Y$  such that the scheme fiber  $\eta^{-1}(P)$  is not isomorphic to  $\mathbb{P}^1$ . Then  $\Delta$  is a (possibly reducible) reduced  $G_Y$ -invariant divisor that has at most normal crossing singularities in codimension 2. If  $P$  is a smooth point of  $\Delta$ , then the fiber of  $\eta$  over  $P$  is isomorphic to a reducible reduced conic in  $\mathbb{P}^2$ . If  $P$  is a singular point of  $\Delta$ , then  $F$  is isomorphic to a non-reduced conic in  $\mathbb{P}^2$ .

LEMMA B.3. *Suppose that  $G_\eta$  leaves invariant every irreducible component of each reduced fiber of the conic bundle  $\eta$ . Then  $\phi: B \rightarrow Y$  is an étale double cover over  $Y \setminus \text{Sing}(\Delta)$ .*

*Proof.* Fix a point  $Q \in Y$  such that  $Q \notin \text{Sing}(\Delta)$ . Let  $C$  be the fiber of the conic bundle of  $\eta$  over the point  $Q$ . By Lemma B.2, the center  $G_\eta$  acts faithfully on every irreducible component of the fiber  $C$ . In particular, we see that  $C \not\subset B$ . Since  $B \cdot C = 2$ , we see that

either  $|B \cap C|$  consists of two distinct points or  $|B \cap C|$  consists of a single point. In the former case, the morphism  $\phi$  is étale over  $Q$ . Thus, we may assume that the intersection  $B \cap C$  consists of a single point. Denote this point by  $P$ .

Suppose first that  $C$  is smooth. Then  $C$  is tangent to  $B$  at the point  $P$ . Then  $T_{P,C} \subset T_{P,B}$ , so that  $G_\eta$  acts trivially on  $T_{P,C}$ . This is impossible by Lemma 2.3, since  $G_\eta$  acts faithfully on  $C$ .

We see that  $C = C_1 \cup C_2$ , where  $C_1$  and  $C_2$  are smooth irreducible rational curves that intersect transversally at one point. Denote this point by  $O$ . If  $P \neq O$ , then  $T_{P,C_1} \subset T_{P,B}$ . As above, this leads to a contradiction, since  $G_\eta$  acts faithfully on the curve  $C_1$ . Thus, we have  $O = P$ .

Recall that  $G_\eta$  is cyclic by assumption. Denote by  $z$  its generator. Then  $z$  must fix a point  $P' \in C_1$  that is different from  $P$ . Using the exact sequence

$$0 \longrightarrow T_{P',C_1} \longrightarrow T_{P',X} \longrightarrow \eta^*(T_{Q,Y}) \longrightarrow 0,$$

we see that there exists a two-dimensional subspace in the tangent space  $T_{P',X}$  consisting of zero eigenvalues of the element  $z$ . By [BB73, theorem 2.1] this shows that  $z$  pointwise fixes a surface  $B'$  in  $X$  such that  $P' \in B'$ .

Since  $P = B \cap C$ , we see that  $P' \notin B$ , so that  $B'$  is not an irreducible component of the surface  $B$ . On the other hand, the surface  $B'$  is also pointwise fixed by  $z$ . This contradicts the definition of the surface  $B$ .

**COROLLARY B.4.** *If  $|G_\eta| \geq 3$ , then  $\phi: B \rightarrow Y$  is an étale double cover over  $Y \setminus \text{Sing}(\Delta)$ .*

**COROLLARY B.5.** *Suppose that  $|G_\eta| \geq 3$  or  $G_\eta$  leaves invariant every irreducible component of each reduced fiber of the conic bundle  $\eta$ . Suppose also that  $\eta: X \rightarrow Y$  is  $G$ -minimal, and  $Y$  is simply connected. Then  $\Delta = 0$ , and there exists a central extension  $\tilde{G}_Y$  of the group  $G_Y$  such that*

$$X \simeq \mathbb{P}(\mathcal{L}_1 \oplus \mathcal{L}_2)$$

for some  $\tilde{G}_Y$ -linearisable line bundles  $\mathcal{L}_1$  and  $\mathcal{L}_2$  on  $Y$ , where the splitting  $\mathcal{L}_1 \oplus \mathcal{L}_2$  is also  $\tilde{G}_Y$ -invariant, and  $\eta: X \rightarrow Y$  is a natural projection.

*Proof.* By Lemma B.3 and Corollary B.4, the morphism  $\phi: B \rightarrow Y$  is an étale double cover over  $Y \setminus \text{Sing}(\Delta)$ . Since  $Y$  is simply connected, we see that

$$B = B_1 \cup B_2,$$

where  $B_1$  and  $B_2$  are rational sections of the conic bundle  $\eta$ . Now  $G$ -minimality implies that  $\Delta = 0$ , so that  $\eta$  is a  $\mathbb{P}^1$ -bundle. Since  $\phi: B \rightarrow Y$  is an étale double cover, we see that  $B$  is a disjoint union of the divisors  $B_1$  and  $B_2$ , and both  $B_1$  and  $B_2$  are sections of the  $\mathbb{P}^1$ -bundle  $\eta$ . This implies the remaining assertions of the corollary.

Now we are ready to prove:

**THEOREM B.6.** *Let  $X$  be a rationally connected threefold that is faithfully acted upon by the group  $6.A_6$ . Then there exists no  $6.A_6$ -equivariant dominant rational map  $\pi: X \dashrightarrow S$  such that its general fibers are irreducible rational curves or irreducible rational surfaces.*

*Proof.* Using  $6.\mathfrak{A}_6$ -equivariant resolution of singularities and indeterminacies, we may assume that both  $X$  and  $S$  are smooth, and  $\pi$  is a morphism. If  $S$  is a curve, then we immediately obtain a contradiction, since  $\mathfrak{A}_6$  cannot faithfully act on a rational curve, and  $6.\mathfrak{A}_6$  cannot faithfully act on a rational surface. Thus, we may assume that  $S$  is a surface, and the general fiber of  $\pi$  is  $\mathbb{P}^1$ . Using [Avi14, theorem 1], we may assume that  $\pi : X \rightarrow S$  is  $6.\mathfrak{A}_6$ -minimal standard conic bundle.

Since  $X$  is rationally connected, we see that  $S$  is rational. Thus, none of the groups  $6.\mathfrak{A}_6$ ,  $3.\mathfrak{A}_6$  and  $2.\mathfrak{A}_6$  can faithfully act on  $S$ . This shows that there exists an exact sequence of groups

$$1 \longrightarrow G_\pi \longrightarrow 6.\mathfrak{A}_6 \longrightarrow G_S \longrightarrow 1,$$

where  $G_S$  is a subgroup in  $\text{Aut}(S)$  that is isomorphic to  $\mathfrak{A}_6$ , and  $G_\pi$  is the center of the group  $6.\mathfrak{A}_6$  that acts trivially on  $S$ . Applying Corollary B.5, we see that there exists a central extension  $\tilde{G}_S$  of the group  $G_S \simeq \mathfrak{A}_6$  such that

$$X \simeq \mathbb{P}(\mathcal{L}_1 \oplus \mathcal{L}_2)$$

for some  $\tilde{G}_S$ -linearisable line bundles  $\mathcal{L}_1$  and  $\mathcal{L}_2$  on  $S$ , where the splitting  $\mathcal{L}_1 \oplus \mathcal{L}_2$  is also  $\tilde{G}_S$ -invariant.

A priori, we know that  $\tilde{G}_S$  is one of the following groups  $\mathfrak{A}_6$ ,  $2.\mathfrak{A}_6$ ,  $3.\mathfrak{A}_6$  or  $6.\mathfrak{A}_6$ . On the other hand, the induced action of  $\tilde{G}_S$  on  $X$  gives our action of  $6.\mathfrak{A}_6$  on the threefold  $X$ . This shows that  $\text{Am}(S, \mathfrak{A}_6) \simeq \mu_6$ . On the other hand, it follows from [DI09] that  $S$  is  $\mathfrak{A}_6$ -birational to  $\mathbb{P}^2$ , where  $\text{Am}(\mathbb{P}^2, \mathfrak{A}_6) \simeq \mu_3$ , contradicting Lemma A.1.

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