Can. J. Math., Vol. XXXVI, No. 6, 1984, pp. 1031-1045

ON EXPLICIT DECOMPOSITION FOR POSITIVE POLYNOMIALS ON [−1, +1] WITH APPLICATIONS TO EXTREMAL PROBLEMS

R. PIERRE

1. Introduction. The following well known inequality was first proved by Bernstein [**2**].

THEOREM A. *If* $p_n(x)$ *is a polynomial of degree n, such that* $|p_n(x)| \leqq 1$ *for* $-1 \leqq x \leqq +1$, *then*

(1) $\quad |p_n'(x)| \leqq n(1 - x^2)^{-1/2}, \quad -1 < x < +1.$

The dominant $n(1 - x^2)^{-1/2}$ is best possible only at the zeros of the Tchebychev polynomial

$$T_n(x) = \cos(n \text{ arc cos } x),$$

but the bound is precise at every interior point as far as the exponent of n is concerned.

Theorem A was extended to the case of higher derivatives by Duffin and Schaeffer in [**4**]. In that paper they make extensive use of the oscillation property of the polynomial $T_n(x)$ and of the related function

$$S_n(x) = \sin(n \text{ arc cos } x).$$

The relationship between these two functions and the majorant $q(x) \equiv 1$ appearing in the hypothesis of Theorem A is best illustrated by the following equation

(2) $\quad 1 = (T_n(x))^2 + (S_n(x))^2 = (T_n(x))^2 + (1 - x^2)(T_n'(x)/n)^2.$

That such a decomposition plays an important role in Theorem A was recognized by Bernstein himself (see [**3**]). This observation lead him to the following generalisation of Theorem A.

THEOREM B. *If* $p_n(x)$ *is a polynomial of degree n, satisfying*

$$|p_n(x)| \leqq [M^2(x) + (1 - x^2)N^2(x)]^{1/2} \quad \textit{for } -1 \leqq x \leqq +1$$

where $M(x)$ *and* $N(x)$ *are real polynomials of degree* l *and* $l - 1$ *respectively* ($l \leqq n$) *such that* $M(x) > 0$ *and* $N(x) > 0$ *for* $x > 1$ *and their zeros in* $[-1, +1]$ *alternate, then, for* $x \in (-1, +1)$

Received September 7, 1982 and in revised form May 8, 1984. This work was supported by the NSERC under Grant A-3514 and by a grant of the Gouvernement du Québec.

$$(3) \quad |p_n'(x)|(1 - x^2)^{1/2} \leqq \{[(n - l)M(x) + xN(x) + (x^2 - 1)N'(x)]^2 + (1 - x^2)[(n - l)N(x) + M'(x)]^2\}^{1/2}.$$

In this paper, we would like to develop a method to obtain an explicit decomposition of the type (2) for a polynomial $q(x)$ positive on $[-1, +1]$. This will enable us not only to reformulate Theorem B for a majorant of the form $\sqrt{q(x)}$ but also to obtain informations on $\max_{[-1,+1]} |p_n'(x)|$ in various cases.

2. An explicit decomposition for positive polynomials. Let $q(x)$ be a polynomial of degree k such that $q(x) > 0$ for $x \in [-1, +1]$. For every n satisfying $2n \geqq k$, there exist polynomials $\tau_n(x)$ and $v_{n-1}(x)$ of degree n and $n - 1$ respectively for which

$$(4) \quad q(x) = (\tau_n(x))^2 + (1 - x^2)(v_{n-1}(x))^2.$$

Moreover their zeros are all in $[-1, +1]$ and interlace. Results of that type have been obtained in various forms by Luckas (see **[8]**), Karlin and Shapley **[5]** and others, but the proofs are not constructive. Our proof will give an explicit formula for $\tau_n(x)$ in a form which will closely relate it to $T_n(x)$.

Let us first suppose that $q(x)$ is a perfect square, i.e., $q(x) = (q_1(x))^2$ where $q_1(x)$ is a polynomial of degree $j = k/2 \leqq n$ having all its roots in $\mathbf{C}\setminus[-1, +1]$.

We begin with a heuristic remark. If the polynomials $\tau_n(x)$ and $q_1(x)$ are related by (4), there exist $(n + 1)$ points $-1 = x_0 < x_1 < \ldots < x_n = +1$ such that

$$(5) \quad (q_1(x))^2 - \tau_n^2(x) = c \prod_{i=1}^{n-1} (x - x_i)^2(1 - x^2)$$

while

$$(6) \quad \tau_n'(x)q_1(x) - \tau_n(x)q_1'(x) = \prod_{i=1}^{n-1} (x - x_i)r(x),$$

where $r(x)$ is of degree j if $j < n$ and of degree at most $(j - 1)$ if $j = n$. Using (5) and (6) putting $y = \tau_n/q_1$, we obtain

$$(7) \quad \frac{(y')^2}{1 - y^2} = \frac{h_1^2(x)}{q_1^2(x)} \frac{1}{(1 - x^2)}.$$

Here $h_1^2(x) = r^2(x)c^{-1}$. Obviously, if one was able to properly choose $h_1(x)$ in (7), one would obtain $\tau_n(x)$ through integration, namely

$$\tau_n(x) = \pm q_1(x)\cos\left(\int_1^x \frac{h_1(t)}{q_1(t)} \frac{dt}{\sqrt{1 - t^2}}\right).$$

From now on, for every $u \in \mathbf{C}\backslash[-1, +1]$, we will denote by $(u^2 - 1)^{1/2}$ the determination of $\sqrt{u^2 - 1}$ for which $|u + \sqrt{u^2 - 1}| > 1$ and denote by $\sqrt{x}$ the positive square root of a positive number.

Consideration of equation (5) for large x and of equation (6) at a zero of $q_1(x)$ will lead to the following choice of $h_1(x)$, at least up to the sign of the second term (which will be justified later).

$$(8) \quad h_1(x) = \sum_{s=1}^{l} m_s(z_s^2 - 1)^{1/2} \frac{q_1(x)}{x - z_s} - (n - j)q_1(x)$$

where $z_1, \ldots, z_l$ are the zeros of $q_1(x)$ of multiplicity $m_1, \ldots, m_l$ respectively.

To proceed more formally, we need the following.

LEMMA 1. *If*

$$q_1(x) = \prod_{s=1}^{l} (x - z_s)^{m_s}$$

is a polynomial of degree $j = \sum_{s=1}^{l} m_s$, *positive on* $[-1, +1]$, *if* $h_1(x)$ *is defined by* (8) *and* $H_1(x)$ *is given by*

$$(9) \quad H_1(x) = \int_1^x \frac{h_1(t)}{q_1(t)} \frac{dt}{\sqrt{1 - t^2}}$$

then, for $x \in (-1, +1)$

$$\text{(i)} \quad H_1(x) = \sum_{s=1}^{l} m_s\delta_s(x) + (n - j)\text{arc cos } x$$

where

$$\delta_s(x) = 2 \text{ Arc tan } \left[\frac{\omega_s + 1}{\omega_s - 1} \sqrt{\frac{1 - x}{1 + x}}\right], \quad \omega_s = z_s + (z_s^2 - 1)^{1/2}$$

and Arc tan(0) = 0.

$$\text{(ii)} \quad \cos\left(\sum_{s=1}^{l} m_s\delta_s(x)\right) = \frac{n_j(x)}{q_1(x)},$$

$$\sin\left(\sum_{s=1}^{l} m_s\delta_s(x)\right) = \frac{(1 - x^2)^{1/2}m_{j-1}(x)}{q_1(x)},$$

where $n_j(x)$ *and* $m_{j-1}(x)$ *are polynomials of degree* j *and* $j - 1$ *respectively.*

Proof. (i) From (8) and (9), one readily obtains

$$H_1(x) = \sum_{s=1}^{l} m_s\delta_s(x) + (n - j)\text{arc cos } x$$

where

$$(\dagger) \quad \delta_s(x) = (z_s^2 - 1)^{1/2} \int_1^x \frac{dt}{(t - z_s)\sqrt{1 - t^2}}.$$

Under the changes of variables $t = \cos\theta$ and $u = tg\frac{\theta}{2}$, $(\dagger)$ is transformed into

$$\delta_s(x) = -(z_s^2 - 1)^{1/2} \int_0^{\sqrt{\frac{1-x}{1+x}}} \frac{du}{(1 - z_s) - (1 + z_s)u^2}.$$

Introducing

$$z_s = \frac{1}{2}\left(\omega_s + \frac{1}{\omega_s}\right), \ |\omega_s| > 1 \quad \text{and} \quad w_s = \frac{\omega_s + 1}{\omega_s - 1},$$

we set

$$u = \frac{1}{w_s} v,$$

which, considering that Re $(w_s) > 0$ leads to

$$\delta_s(x) = \int_0^{w_s\sqrt{\frac{1-x}{1+x}}} \frac{dv}{1 + v^2}$$
$$= 2 \text{ Arc tan } \left[\frac{\omega_s + 1}{\omega_s - 1} \sqrt{\frac{1 - x}{1 + x}}\right],$$

with the proposed determination of arc tan z.

(ii) Let us first compute $\cos(\delta_1(x))$ and $\sin(\delta_1(x))$. We get

$$\cos(\delta_1(x)) = \frac{1 - tg^2\left(\frac{\delta_1(x)}{2}\right)}{1 + tg^2\left(\frac{\delta_1(x)}{2}\right)} = \frac{z_1 x - 1}{z_1 - x},$$

while

$$\sin(\delta_1(x)) = \frac{2tg\left(\frac{\delta_1(x)}{2}\right)}{1 + tg^2\left(\frac{\delta_1(x)}{2}\right)} = \frac{(z_1^2 - 1)^{1/2}\sqrt{1 - x^2}}{(z_1 - x)}.$$

Assertion (ii) now follows by induction on the number j of zeros of $q_1(x)$.

We are now in position to state and prove the main result of this section.

THEOREM 1. *Let $q_1(x)$ be a polynomial positive on $[-1, +1]$. If $h_1(x)$ and $H_1(x)$ are defined by* (8) *and* (9) *respectively, then the functions*

$$\tau_n(x) = q_1(x)\cos(H(x)) \quad and \quad v_{n-1}(x) = \frac{q_1(x)\sin(H(x))}{\sqrt{1 - x^2}}$$

are real polynomials of degree n and $(n - 1)$ respectively satisfying the following properties:

a) $(q_1(x))^2 = \tau_n^2(x) + (1 - x^2)v_{n-1}^2(x)$,

b) *there exist* $(2n + 1)$ *points* $-1 = x_0 < y_1 < x_1 < y_2 < \ldots < y_n < x_n = +1$ *such that*

$$\begin{cases} \tau_n(x_s) = (-1)^{n-s}\, q_1(x_s), & s = 0, \ldots, n, \\ v_{n-1}(x_s) = 0, & s = 1, \ldots, n - 1, \end{cases}$$

whereas

$$\tau_n(y_s) = 0,$$
$$v_{n-1}(y_s) = \frac{(-1)^{n-s}q_1(y_s)}{\sqrt{1 - y_s^2}} \quad s = 1, \ldots, n.$$

Proof. It follows from Lemma 1 that

$$\tau_n(x) = n_j(x)T_{n-j}(x) - (1 - x^2)m_{j-1}(x)(T'_{n-j}(x)/(n - j))$$

and that

$$v_{n-1}(x) = m_{j-1}(x)T_{n-j}(x) + n_j(x)(T'_{n-j}(x)/(n - j))$$

which implies that $\tau_n(x)$ and $v_{n-1}(x)$ are polynomials. Now, since $q_1(x)$ is a real polynomial, its zeros are either real or conjugate. If $z_l = \bar{z}_s$ we will have

$$\delta_l(x) = \overline{\delta_s(x)};$$

hence $H_1(x)$ is real for $x \in [-1, +1]$ and so are $\tau_n(x)$ and $v_{n-1}(x)$.

Since $\tau_n(x)$ and $v_{n-1}(x)$ obviously verify a), it is enough to complete the proof, to show that $H_1(x)$ is a strictly decreasing function on $[-1, +1]$, such that $H_1(-1) = n\pi$ and $H_1(1) = 0$. Studying the sign of

$$\sqrt{1 - x^2}\, H_1'(x) = \frac{h_1(x)}{q_1(x)} = \sum_{s=1}^{l} \frac{m_s(z_s^2 - 1)^{1/2}}{(x - z_s)} - (n - j)$$

we first observe that, for real z_s, our choice of the determination of $\sqrt{z_s^2 - 1}$ implies that

$$z_s(z_s^2 - 1)^{1/2} > 0$$

and thus, that

$$\frac{(z_s^2 - 1)^{1/2}}{x - z_s} < 0 \quad \text{for } x \in [-1, +1].$$

On the other hand, if z_s is a complex zero, so is $\bar{z}_s$; setting

$$z_s = \frac{1}{2}\left(\omega + \frac{1}{\omega}\right) \quad \text{where } \omega = \mathrm{r}e^{i\theta},\ r > 1,$$

we get

$$\frac{(z_s^2 - 1)^{1/2}}{x - z_s} + \frac{(\bar{z}_s^2 - 1)^{1/2}}{x - \bar{z}_s} = \frac{(r - r^{-1})[\,(\cos\theta)x - 2^{-1}(r + r^{-1})\,]}{|x - z|^2}$$

which is strictly negative for $x \in [-1, +1]$. This implies that $H_1'(x) < 0$ on $[-1, +1]$, hence that $H_1(x)$ decreases on that segment. That $H_1(1) = 0$ is obvious, whereas the fact $H_1(-1) = n\pi$ follows directly from Lemma 1 (i) since

$$\delta_s(-1) = 2 \lim_{x \to -1^+} \operatorname{Arc\,tan}\left[\frac{\omega_s + 1}{\omega_s - 1}\sqrt{\frac{1 - x}{1 + x}}\right] = \pi.$$

To obtain the decomposition (4) in the case where $q(x)$ is not a perfect square, it is enough to prove the following corollary.

COROLLARY 1. *Let*

$$q(x) = c \prod_{s=1}^{l} (z - z_s)^{m_s}$$

be a polynomial of degree $k = \sum_{s=1}^{l} m_s$ *positive on* $[-1, +1]$. *Let* $h(x)$ *and* $H(x)$ *be defined by*

(10) $$h(x) = \sum_{s=1}^{l} m_s(z_s^2 - 1)^{1/2} \frac{q(x)}{x - z_s} - (2n - k)q(x)$$

and

(11) $$H(x) = \int_1^x \frac{h(t)}{q(t)} \frac{dt}{\sqrt{1 - t^2}}.$$

Then, for each n, such that, $2n \geqq k$, the functions

(12) $$\tau_n(x) = \sqrt{q(x)} \cos(\tfrac{1}{2}H(x))$$

and

(13) $$v_{n-1}(x) = \sqrt{q(x)} \frac{\sin(\frac{1}{2}H(x))}{\sqrt{1 - x^2}}$$

are real polynomials of degree n and $(n - 1)$ respectively all of whose zeros lie in $[-1, +1]$ and separate one another.

Proof. Let

$$t_{2n}(x) = q(x)\cos(H(x));$$

since $H(-1) = 2n\pi$, the polynomial $2^{-1}(t_{2n}(x) + q(x))$ has only double zeros in $(-1, +1)$. Theorem 1 implies that there are n such zeros hence this polynomial is a perfect square. This implies that there exists a polynomial $r_n(x)$ such that

$$r_n^2(x) = 2^{-1}(t_{2n}(x) + q(x)) = \tau_n^2(x).$$

If we choose $r_n(x)$ such that $r_n(-1) = \tau_n(-1)$, we will have

$$r_n(x) \equiv \tau_n(x).$$

Considering $2^{-1}(q(x) - t_{2n}(x))$ we argue similarly for $v_{n-1}(x)$.

The last statement follows directly from the fact that as x increases from -1 to $+1$, $H(x)$ decreases from $2n\pi$ to 0.

It should be clear that the same device will give an explicit decomposition for positive polynomials on $[-1, +1]$ as a sum of non-negative polynomials of odd degree. For example, if $q(x)$ is a positive polynomial of degree $2n - 1$ then

$$q(x) = (1 + x)(t_{n-1}(x))^2 + (1 - x)(s_{n-1}(x))^2$$

where

$$t_{n-1}(x) = \sqrt{q(x)}\cos\left(\frac{1}{2}\sum_{1}^{2n-1} \delta_i(x)\right)(\sqrt{1 + x})^{-1} \quad \text{and}$$

https://doi.org/10.4153/CJM-1984-059-3 Published online by Cambridge University Press

$$s_{n-1}(x) = \sqrt{q(x)}\sin\left(\frac{1}{2}\sum_{1}^{2n-1} \delta_i(x)\right)(\sqrt{1-x})^{-1}.$$

We will content ourselves with this remark and use only the representation (4) in what follows.

3. Pointwise bound. Let $q(x)$, $\tau_n(x)$ be as in Corollary 1 and $s_n(x)$ be defined by

(14) $s_n(x) = \sqrt{q(x)}\sin(\tfrac{1}{2}H(x))$.

If t is an arbitrary real number, we wish to estimate $|p_n'(t)|$ where $p_n(x)$ is a polynomial of degree $n \geqq k/2$ satisfying

(15) $|p_n(x)| \leqq \sqrt{q(x)}$ for $x \in [-1, +1]$.

Using the notation of Theorem 1, set

$$\omega(x) = \prod_{s=0}^{n} (x - x_s) = c\sqrt{1-x^2}s_n(x)$$

and

$$\omega_l(x) = \frac{\omega(x)}{x - x_l} \quad \text{for } l = 0, \ldots, n.$$

If $\xi_1 \leqq \xi_2 \leqq \ldots \leqq \xi_{n-1}$ and $\eta_1 \leqq \eta_2 \leqq \ldots \leqq \eta_{n-1}$ denote the roots of

$$\omega_n'(x) = 0, \ \omega_0'(x) = 0$$

respectively, then, applying Theorem 1 of [7], we see that the inequality

(16) $|p_n'(t)| \leqq |\tau_n'(t)|$

is valid for every t lying outside the interval (ξ_1, η_{n-1}).

The case $t \in (\xi_1, \eta_{n-1})$ is covered by Theorem B. Our proof will depend on the following

LEMMA 2. *Let $q(x)$, $\tau_n(x)$ and $s_n(x)$ be as above, then, the trigonometric polynomial of order n*

$$\tau_n(\cos\theta) + is_n(\cos\theta) = \sqrt{q(\cos\theta)}\exp\left(i\left(\frac{H(\cos\theta)}{2}\right)\right)$$

has all its roots in $\text{Im}(\theta) \geqq 0$.

Proof. If $z_1, \ldots, z_k$ denote the k zeros of $q(x)$ and

$$\omega_s = z_s + (z_s^2 - 1)^{1/2}, \quad s = 1, \ldots, k,$$

then

$$e^{i\delta_s(\cos\theta)} = \frac{-e^{-i\theta}}{2\omega_s(\cos\theta - z_s)}[e^{i\theta}\omega_s - 1]^2$$

from which one readily deduces that

$$\tau_n(\cos\theta) + is_n(\cos\theta) = ce^{i(n-k)\theta}\prod_1^k (e^{i\theta}\omega_s - 1)$$

where c is constant. The lemma now follows from the fact that $|\omega_s| > 1$ for $s = 1, \ldots, k$.

We can now reformulate Theorem B.

THEOREM 2. *If $p_n(x)$ is a polynomial of degree $n \geqq k/2$ satisfying* (15), *then, for each $t \in (-1, +1)$,*

(17) $(p_n'(t))^2 \leqq (\tau_n'(t))^2 + (s_n'(t))^2$

where $\tau_n(x)$ and $s_n(x)$ are given by (12) *and* (14) *respectively.*

Proof. By definition of τ_n and s_n, setting $x = \cos\theta$, we can put (15) under the form

$$|p_n(\cos\theta)| \leqq |\tau_n(\cos\theta) + is_n(\cos\theta)|.$$

Using Lemma 2, we see that the hypotheses of Levin's theorem, (see [**1**], p. 226) are satisfied, so that, for every real θ, we have

$$|\sin\theta\, p_n'(\cos\theta)| \leqq |\sin\theta(\tau_n'(\cos\theta) + is_n'(\cos\theta))|,$$

which is the desired inequality.

We first remark that, in the case where $\sqrt{q(x)}$ is of the form

$$|\tau_l(x) + is_l(x)|, \quad l \leqq n,$$

the functions $\tau_n(\cos\theta)$, $s_n(\cos\theta)$, $\tau_l(\cos\theta)$ and $s_l(\cos\theta)$ are related by

$$\tau_n(\cos\theta) + is_n(\cos\theta) = e^{+i(n-l)\theta}(\tau_l(\cos\theta) + is_l(\cos\theta)).$$

Differentiating both sides with respect to θ and putting $x = \cos\theta$ we see that, in this case, the right hand side of (17) can be written

$$(-\sqrt{1-x^2}\, s_l'(x) + (n-l)\tau_l(x))^2 + \left(\frac{(n-l)s_l(x)}{\sqrt{1-x^2}} + \tau_l'(x)\right)^2.$$

A comparison with the right hand side of (3) will show that Theorem 2 includes Theorem B completely.

Secondly, it should be noted that, while inequality (16) is best possible at every point, inequality (17) is best possible only at the zeros of $s_n'(t)$. Exact estimates for $t \in (-1, +1)$ would require a much deeper study in

https://doi.org/10.4153/CJM-1984-059-3 Published online by Cambridge University Press

the spirit of [**6**]. We will not need to do that and conclude this section with the following corollary which combines the inequalities (16) and (17) in a form that is useful in the next one.

COROLLARY 2. *Let* $a_1 < a_2 < \ldots < a_n$ *denote the roots of* $s_n'(x)$. *If* $p_n(x)$ *is a polynomial of degree* $n \geqq k/2$ *satisfying* (15), *then*

$$(18)\quad (p_n'(t))^2 \leq \begin{cases} (\tau_n'(t))^2 & \text{for } t \notin [a_1, a_n] \\ \dfrac{1}{4q(t)}\left\{(q'(t))^2 + \dfrac{h^2(t)}{1-t^2}\right\} & \text{for } t \in [a_1, a_n]. \end{cases}$$

Proof. The second part of the inequality can be verified directly by computing the right-hand side of (17). The first one will be a consequence of (16) if we show that $a_1 \leqq \xi_1$ and $a_n \geqq \eta_{n-1}$. We verify the first inequality, the second one is obtained mutatis mutandis.

By definition of $\omega_n(x)$, we have

$$\omega_n'(x) = \left(\frac{1+x}{1-x}\right)^{-1/2}\left[\frac{1}{(1-x)^2}\, s_n(x) + \left(\frac{1+x}{1-x}\right) s_n'(x)\right],$$

and thus,

$$\operatorname{sign}(\omega_n'(a_1)) = \operatorname{sign}(s_n(a_1)).$$

On the other hand, using the fact that $H(x)$ decreases from $2n\pi$ as x increases from -1, we see that for small positive ϵ,

$$\operatorname{sign}(s_n(-1+\epsilon)) = \operatorname{sign}(s_n'(-1+\epsilon)) = \operatorname{sign}(\omega_n'(a_1)).$$

It follows from there that

$$\operatorname{sign}(\omega_n'(a_1)) = \operatorname{sign}(\omega_n'(-1+\epsilon)),$$

which implies $\xi_1 \geqq a_1$.

4. Global bounds. The use of inequality (1) to get an estimate for

$$M = \sup\left\{ \max_{[-1,+1]} |p_n'(x)| \;\middle|\; |p_n(x)| \leqq 1 \text{ for } x \in [-1, +1] \right\}$$

is classical (see [**4**]). It amounts essentially to proving that if $q(t) \equiv 1$, the right hand side of inequality (18) defines an increasing function on (0, 1), so that

$$M = |T_n'(1)| = n^2.$$

Setting

https://doi.org/10.4153/CJM-1984-059-3 Published online by Cambridge University Press

$$(19)\quad M_q(t) = \frac{1}{4q(t)}\left\{ (q'(t))^2 + \frac{h^2(t)}{1 - t^2} \right\}$$

we would like to study the behaviour of the function $M_q(t)$ for two different types of majorant $q(t)$. It will turn out that this behaviour is greatly influenced by the nearness of the zeros of $q(t)$ to the end points of $[-1, +1]$.

A. $q(t) = (\beta^2 - t^2)^k$.

Here β is a real number greater than 1 and k an integer. As pointed out by Videnskii in [**10**], this case is of interest in view of the work of Dzyadyk on the approximation of functions in the Lip_α class. Videnskii himself studied this question in the case $k = 1$. Let us suppose that $k \geqq 2$. According to (10) we can write

$$(20)\quad h(x) = -2(\beta^2 - x^2)^{k-1}[k\,\beta\sqrt{\beta^2 - 1} + (n - k)(\beta^2 - x^2)]$$

which upon substitution in (19) gives

$$(21)\quad M_q(t) = \frac{(\beta^2 - t^2)^{k-2}}{(1 - t^2)}\{B + Ct^2 + Dt^4\}.$$

Here

$$B = [k\,\beta\sqrt{\beta^2 - 1} + \beta^2(n - k)]^2 \geqq 0$$
$$C = -[2(n - k)^2\beta^2 + 2k\beta\sqrt{\beta^2 - 1}(n - k) - k^2] \leqq 0$$
$$D = n(n - 2k) \geqq 0,$$

where the last two inequalities are valid if $n \geqq 2k$ which we now suppose to be true. Differentiating (21), we get

$$(22)\quad M_q'(t) = \frac{(\beta^2 - t^2)^{k-3}}{(1 - t^2)^2}\, 2t\{H + Gt^2 + It^4 + Kt^6\}$$

where

$$H = (C + B)\beta^2 - B(k - 2)$$
$$G = 2D\beta^2 - C(k - 1) + B(k - 3)$$
$$I = -D\beta^2 + C(k - 2) - Dk$$
$$K = (k - 1)D.$$

Set

$$A(t) = H + Gt + It^2 + Kt^3.$$

If $k = 2$, $A(\beta^2) = 0$, hence the numerator in (22) reduces to

$$(\beta^2 - t^2)^{k-3}2tA(t^2) = 2t(-Dt^4 + 2Dt^2 + (B + C)).$$

The graph of the function $y = -Dt^2 + 2Dt + (B + C)$ being a concave parabola with vertex at $t = +1$ we see that, for $k = 2$, $M_q'(t)$ is increasing on (0, 1). For $k > 2$, we consider

$$A'(t) = A'(1) + A''(1)(t - 1) + \frac{A'''(1)}{2}(t - 1)^2.$$

Now

$$A'(1) = G + 2I + 3K = (k - 3)(B + C + D) \geqq 0,$$

while

$$\frac{A''(1)}{2} = I + 3K = -(\beta^2 - 1)[n(n - 2k) + 2(k - 2)(n - k)^2]$$
$$- 2\beta k\sqrt{\beta^2 - 1}(n - k)(k - 2) - k^2(k - 2)$$

is negative and

$$\frac{A'''(1)}{6} = K \geqq 0.$$

Thus $A'(t)$ is positive on (0, 1) which implies that $A(t)$ is increasing on that interval.

This leads us to the following generalization of Videnskii's result [**10**].

THEOREM 3. *Let $p_n(x)$ be a polynomial of degree $n \geqq 2k$ satisfying*

$$|p_n(x)| \leqq (\beta^2 - x^2)^{k/2}$$

where $\beta \geqq 1$ and $k \geqq 2$, then

$$(23) \quad \max_{[-1,+1]} |p_n'(x)| \leqq \max\left(\sqrt{M_q(0)}, |\tau_n'(1)|\right)$$

where $M_q(t)$ is given by (21) *and $\tau_n(x)$ by*

$$\tau_n(x) = (\beta^2 - x^2)^{k/2} \cos\left(-\int_1^x \frac{k\beta\sqrt{\beta^2 - 1}\, dt}{(\beta^2 - t^2)\sqrt{1 - t^2}} + (n - k) \arccos x\right).$$

Proof. Let us first suppose that $\beta > 1$. Since for $t \in (-1, +1)$

$$\tau_n(t)s_n'(t) - s_n(t)\tau_n'(t) = \frac{h(t)}{2\sqrt{1 - t^2}} < 0,$$

we can use Rolle's theorem to conclude that the zeros of $s_n'(t)$ and $\tau_n'(t)$ separate each other, hence that $\tau_n'(t)$ is increasing on the interval $(a_n, 1)$.

Using $A(1) = (B + C + D)(\beta^2 - 1) > 0$, we obtain

$$\lim_{t \to +1^-} M_q'(t) = +\infty.$$

In view of the preceding discussion, we see that $M_q(t)$ has at most one local extremum in (0, 1) which will then be a minimum. Inequality (23) now follows from Corollary 2.

In the case $\beta = 1$,

$$M_q(t) = (1 - t^2)^{k-2}\{(n - k)^2(1 - t^2) + k^2t^2\}$$

is a decreasing function. By continuity, the inequality

$$|p_n'(t)| \leqq \sqrt{M_q(t)},$$

which is valid for $t \in (-1, +1)$ is also valid at the end points, hence

$$\max_{[-1,+1]} |p_n'(t)| \leqq \max_{[-1,+1]} \sqrt{M_q(t)} = \sqrt{M_q(0)} = (n - k).$$

From (21) and the definition of $\tau_n(x)$ we obtain, using some elementary computation, that

$$\sqrt{M_q(0)} = \beta^{k-2}[k\beta\sqrt{\beta^2 - 1} + \beta^2(n - k)]$$

while

$$\tau_n'(1) = \Big\{-k(\beta^2 - 1)^{k/2-1} + (\beta^2 - 1)^{k/2}\left(\frac{\beta k\sqrt{\beta^2 - 1}}{1 - \beta^2} - (n - k)\right)^2\Big\}.$$

This implies that

$$\lim_{\beta \to \infty} \frac{|\tau_n'(1)|}{\sqrt{M_q(0)}} = n,$$

whence $\sqrt{M_q(0)} < |\tau_n'(1)|$ if β is big enough. Obviously, if this inequality is true (23) is best possible.

On the other hand, since for $\beta = 1$, $\tau_n'(1) = 0$, we see that, if β is small, the inequality $\sqrt{M_q(0)} \geqq |\tau_n'(1)|$ is valid. If so, (23) is again best possible when n is odd, but not when n is even. The study of this last case would require a more precise local estimate than the one provided by Corollary 2.

B. $q(t) = (q_1(t))^2$, where $q_1(t) \neq 0$ if $\operatorname{Re}(t^2) > 0$.

In this second case, we suppose that the majorant $\sqrt{q(t)}$ is an even real polynomial $q_1(t)$ of degree j which has no zero in

$$\operatorname{Arg}(t) \in \left(\frac{-\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right).$$

A subclass of this class of majorants has been studied by Videnskii in [**9**] where he supposed that all the zeros of $q(t)$ are purely imaginary.

Here again, we would like to show that the function

$$M_q(t) = (q_1'(t))^2 + \frac{h_1^2(t)}{(1 - t^2)}$$

is increasing on (0, 1). For this, we verify that $(q_1'(t))^2$ and $h_1^2(t)$ are polynomials with positive coefficients.

Let R_1 be the set of zeros of $q_1(t)$ lying in $\operatorname{Arg}(t) \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$ and R_2 be the set of zeros of $q_1(t)$ lying on the positive imaginary axis. In view of the conditions imposed on $q_1(t)$, we see that

$$(24) \quad q_1(t) = C \prod_{z \in R_1} |t^2 - z^2|^2 \prod_{z \in R_2} (t^2 - z^2).$$

Clearly, in both products, each factor has positive coefficients and so the same is true of $(q_1'(t))^2$.

From (8) and (24), we get

$$h_1(t) = q_1(t)\left\{2 \sum_{z \in R_1} \operatorname{Re}\left(\frac{z(z^2 - 1)^{1/2}}{t^2 - z^2}\right) + \sum_{z \in R_2} \frac{z(z^2 - 1)^{1/2}}{t^2 - z^2} - (n - j)\right\}.$$

If $z \in R_2$, there exists a positive α such that $z = i\alpha$, hence

$$z(z^2 - 1)^{1/2} = -\alpha\sqrt{\alpha^2 + 1} \leqq 0.$$

On the other hand if $z \in R_1$, set

$$z = \frac{1}{2}\left(\omega + \frac{1}{\omega}\right)$$

where $\omega = re^{i\theta}$, $r > 1$. Then

$$\operatorname{Re}\left(\frac{z(z^2 - 1)^{1/2}}{t^2 - z^2}\right) = \frac{(r^4 - 1)}{4r^2|t^2 - z^2|^2} \cdot \{(\cos 2\theta)t^2 - |z|^2\}.$$

In order to verify that $h_1^2(t)$ has positive coefficients it is enough to check that $\cos 2\theta \leqq 0$ which follows from the fact that $\operatorname{Re}(z^2) \leqq 0$.

In view of the above discussion, we can conclude that $M_q(t)$ is increasing. The next result now follows from Corollary 2.

THEOREM 4. *Let $q_1(x)$ be an even polynomial of degree j with real coefficients, which has no zero in* $\operatorname{Arg}(t) \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$.

Let $h_1(x)$ and $H_1(x)$ be defined by (8) *and* (9) *respectively. If $p_n(x)$ is a polynomial of degree $n \geqq j$ satisfying $|p_n(x)| \leqq |q_1(x)|$ for $x \in [-1, +1]$, then*

$$\max|p_n'(x)| \leqq |\tau_n'(1)| = q_1'(1) + \frac{h_1^2(1)}{q_1(1)}$$

where

$$\tau_n(x) = q_1(x)\cos(H_1(x)).$$

In conclusion we remark that, as suggested by cases A and B, it would be interesting to determine the class of polynomials $q(t)$ for which the function $M_q(t)$ is increasing. Would this, for example be true under the condition that $q(t)$ has positive coefficients? If the answer was yes, it could lead to interesting asymptotic results for majorants

$$\phi(t) = \sum_0^{\infty} a_n t^n \quad \text{where } a_n \geqq 0$$

and, in particular, for incomplete polynomials.

Acknowledgement. The author would like to thank the referee for many valuable suggestions and improvements on the original version.

References

1. R. P. Boas, Jr., *Entire functions* (Academic Press, 1954).
2. S. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné*, Mémoire de l'Académie Royale de Belgique *4* (1912), 1-104.
3. ——— *Estimates of derivatives of polynomials*, Collected papers, 1, paper n° *46* (1952), 497-499.
4. R. J. Duffin and A. C. Schaeffer, *On some inequalities of S. Bernstein and W. Markov for derivatives of polynomials*, Bull. Am. Math. Soc. *44* (1938), 289-297.
5. S. Karlin and L. S. Shapley, *Geometry of moment spaces*, Mem. Am. Math. *12* (1953).
6. W. Markov, *Über Polynome, die in einem gegebenen Intervalle möglichst wenig von Null abweichen*, Math. Ann. *77* (1916), 218-258.
7. R. Pierre and Q. I. Rahman, *On a problem of Turán about polynomials III*, Can. J. Math. *34*(1982), 888-889.
8. G. Szegö, *Orthogonal polynomials*, American Mathematical Society Colloquium Publications *23*. Published by the American Mathematical Society. (Providence, Rhode Island, Third edition, 1967).
9. V. S. Videnskii, *Generalization of a theorem of A. A. Markov on the estimation of the derivative of a polynomial*, Dokl. Akad. Nauk. SSSR *125* (1959), 15-18.
10. ——— *Generalization of the inequalities of V. A. Markov*, Dokl. Akad. Nauk. SSSR *120* (1958), 447-450.

Université Laval,
Québec, Québec