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Enriques surfaces and an Apollonian packing in eight dimensions

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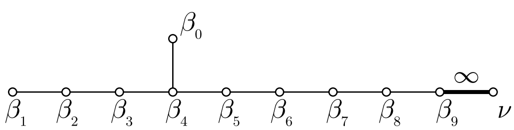
Abstract

We call a packing of hyperspheres in n dimensions an Apollonian sphere packing if the spheres intersect tangentially or not at all; they fill the n-dimensional Euclidean space; and every sphere in the packing is a member of a cluster of $n+2$ mutually tangent spheres (and a few more properties described herein). In this paper, we describe an Apollonian packing in eight dimensions that naturally arises from the study of generic nodal Enriques surfaces. The E_7, E_8 and Reye lattices play roles. We use the packing to generate an Apollonian packing in nine dimensions, and a cross section in seven dimensions that is weakly Apollonian. Maxwell described all three packings but seemed unaware that they are Apollonian. The packings in seven and eight dimensions are different than those found in an earlier paper. In passing, we give a sufficient condition for a Coxeter graph to generate mutually tangent spheres and use this to identify an Apollonian sphere packing in three dimensions that is not the Soddy sphere packing.

1. Introduction

That there is a connection between the Apollonian packing and rational curves on algebraic surfaces has only recently been explored (see [2, 10]). The connection so far has seemed a bit distant. In this paper, we show a rather spectacular connection. Let X be an Enriques surface that contains a smooth rational curve (a *nodal* curve), but is otherwise generic. Let Λ be its Picard group, modulo torsion. Then Λ is isomorphic to the Enriques lattice E_{10}, which is an even unimodular lattice of signature $(1, 9)$. Since $\Lambda \otimes \mathbb{R}$ is a Lorentz space $\mathbb{R}^{1,9}$, it has a 9-dimensional copy of hyperbolic space $\mathbb{H}^9$ naturally imbedded in it. Any nodal curve on X has self intersection -2, so represents a plane in $\mathbb{H}^9$, which in the Poincaré upper-half hyperspace model is represented by an 8-dimensional hemi-sphere. The boundary $\partial\mathbb{H}^9$ of $\mathbb{H}^9$, not including the point at infinity, is isomorphic to $\mathbb{R}^8$, and its intersection with a hyperbolic plane in $\mathbb{H}^9$ is a 7-dimensional hypersphere. In this way, the set of nodal curves on X gives us a configuration of hyperspheres in $\mathbb{R}^8$. In this paper, we show that this configuration is an Apollonian packing, by which we mean the hyperspheres intersect tangentially or not at all, they fill $\mathbb{R}^8$, the packing is crystallographic, and it satisfies the fundamental property that makes it Apollonian, which is that every sphere in the configuration is a member of a cluster of ten hyperspheres that are mutually tangent. (See Section 2.4 for precise definitions.)

Though the connection to Enriques surfaces is fascinating, this paper is really a study of sphere packings. The underlying algebraic geometry is explained in [1, 11]. Our starting point is the following Coxeter graph (also known as a Dynkin diagram):



(1.1)

Some may recognize that it generates the Reye Lattice. The graph also appears in Maxwell's paper [16, Table II, $N = 10$], in which it is described as generating a sphere packing, meaning the spheres intersect tangentially or not at all. It is one of many examples of what are sometimes called Boyd-Maxwell packings (see also [7]). Up until recently (see [4, 5]), it was believed that no Boyd-Maxwell packing in dimension $n \geq 4$ is Apollonian (see the *Mathematical Review* for [6], [15, p. 356], and [10, p. 33]). In [5], we describe a different Apollonian packing (though not a Boyd-Maxwell packing) in eight dimensions. The packing in this paper is more efficient, in the sense that its residual set is a subset of the residual set of the packing in [5]. This also shows that there can be different Apollonian packings in the same dimension.

In Section 2.4, we give a precise definition of higher dimensional Apollonian packings. This is the author's attempt to define what was likely meant or understood by Boyd, Maxwell, and their contemporaries. Note that it is different from the definition given in [4] (see Remark 2), and that our definition may still evolve as we better understand the subject.

A cross section of a sphere packing gives a sphere packing in a lower dimension. By taking a co-dimension one cross section perpendicular to nine of the ten mutually tangent spheres of our new packing in eight dimensions, we get a *weakly* Apollonian packing in seven dimensions, meaning there is a cluster of nine mutually tangent spheres, but not all spheres are a member of such a cluster. It too is described by Maxwell [16, Table II, $N = 9$, last line, first graph]. The resulting packing is different from and more efficient than the example in [5], which is Apollonian.

For Euclidean lattices, eight dimensions are special as it admits the even unimodular lattice E_8. The uniradial sphere packing (i.e., all spheres have the same radius) with spheres of radius $1/\sqrt{2}$ and centered at vertices of the lattice gives the densest uniradial sphere packing in eight dimensions [23]. With an appropriate choice of point at infinity, the Apollonian packing of this paper includes this uniradial sphere packing.

In [4], we generated Apollonian packings in dimension n from uniradial sphere packings in dimension $n - 1$. Using a similar procedure and the E_8 lattice, we generate an Apollonian packing in nine dimensions. This too appears in Maxwell's paper [16, Table II, $N = 11$, second graph].

This paper was directly inspired by a paper by Daniel Allcock [1], who reproved a result (see Theorem 2.1 below) that appears in [9]. Dolgachev attributes the original proof to Looijenga, based on an incomplete proof by Coble [8] from 1919.

The descriptions of the Apollonian packings of this paper do not require any background in algebraic geometry. We will therefore keep the algebraic geometry to a minimum, restricting it as much as possible to Subsection 2.5 and to remarks.

With respect to the organization of this paper, Section 2 on background includes a definition of higher dimensional analogs of the Apollonian circle packings, and the main result concerning Enriques surfaces from which the packing is derived; Section 3 is devoted to producing the Apollonian packing in eight dimensions; in Section 5, we look at a cross section; and in Section 6, we build an example in nine-dimensions. In Section 4, we describe a sufficient condition for a packing to have the (weak) Apollonian property and use it to find examples in Maxwell's list of packings, including a sphere packing in three dimensions that has the Apollonian property but is not the Soddy sphere packing.

2. Background

2.1. The pseudosphere in Lorentz space

Hyperspheres in $\mathbb{R}^n$ can be represented by $(n+2)$-dimensional vectors. Boyd calls such coordinates *polyspherical* coordinates and attributes them to Clifford and Darboux from the late 19th century [6]. The more modern interpretation is that they represent planes in $\mathbb{H}^{n+1}$ imbedded in an $(n+2)$-dimensional Lorentz space, which in turn represent hyperspheres on the boundary $\partial\mathbb{H}^{n+1}$. We refer the reader to [20] for more details.

Let us set $N = n + 2$.

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Given a symmetric matrix J with signature $(1, N-1)$, we define the Lorentz space, $\mathbb{R}^{1,N-1}$ to be the set of N-tuples over $\mathbb{R}$ equipped with the negative Lorentz product

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T J \mathbf{v}.$$

The surface $\mathbf{x} \cdot \mathbf{x} = 1$ is a hyperboloid of two sheets. Let us distinguish a vector D with $D \cdot D > 0$ and select the sheet $\mathcal{H}$ by

$$\mathcal{H}: \qquad \mathbf{x} \cdot \mathbf{x} = 1, \qquad \mathbf{x} \cdot D > 0.$$

We define a distance on $\mathcal{H}$ by

$$\cosh(|AB|) = A \cdot B.$$

Then $\mathcal{H}$ equipped with this metric is a model of $\mathbb{H}^{N-1}$, sometimes known as the *vector model*. Equivalently, one can define

$$V^+ = \{\mathbf{x} \in \mathbb{R}^{1,N-1} : \mathbf{x} \cdot \mathbf{x} > 0, \mathbf{x} \cdot D > 0\}$$

and $\mathcal{H} = V^+/\mathbb{R}^+$, together with the metric defined by

$$\cosh(|AB|) = \frac{A \cdot B}{|A||B|},$$

where $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ for $\mathbf{x} \in V$. For $\mathbf{x} \cdot \mathbf{x} < 0$, we define $|\mathbf{x}| = i\sqrt{-\mathbf{x} \cdot \mathbf{x}}$.

Hyperplanes in $\mathcal{H}$ are the intersection of $\mathcal{H}$ with hyperplanes $\mathbf{n} \cdot \mathbf{x} = 0$ in $\mathbb{R}^{1,N-1}$. Such a plane intersects $\mathcal{H}$ if and only if $\mathbf{n} \cdot \mathbf{n} < 0$. Let $H_\mathbf{n}$ represent both the plane $\mathbf{n} \cdot \mathbf{x} = 0$ in $\mathbb{R}^{1,N-1}$ and its intersection with $\mathcal{H}$. The direction of $\mathbf{n}$ distinguishes a half space

$$H_\mathbf{n}^+ = \{\mathbf{x} : \mathbf{n} \cdot \mathbf{x} > 0\},$$

in either $\mathbb{R}^{1,N-1}$ or $\mathcal{H}$.

The angle θ between two intersecting planes $H_\mathbf{n}$ and $H_\mathbf{m}$ in $\mathcal{H}$ is given by

$$|\mathbf{n}||\mathbf{m}| \cos\theta = \mathbf{n} \cdot \mathbf{m}, \tag{2.1}$$

where θ is the angle in $H_\mathbf{n}^+ \cap H_\mathbf{m}^+$. If $|\mathbf{n} \cdot \mathbf{m}| = |\,|\mathbf{n}||\mathbf{m}|\,|$, then the planes are parallel (i.e. tangent at infinity). If $|\mathbf{n} \cdot \mathbf{m}| > |\,|\mathbf{n}||\mathbf{m}|\,|$, then the planes are ultraparallel, and the quantity ψ in $|\mathbf{n}||\mathbf{m}| \cosh\psi = |\mathbf{n} \cdot \mathbf{m}|$ is the shortest hyperbolic distance between the two planes.

The group of isometries of $\mathcal{H}$ is given by

$$\mathcal{O}^+(\mathbb{R}) = \{T \in M_{N\times N} : T\mathbf{u} \cdot T\mathbf{v} = \mathbf{u} \cdot \mathbf{v} \text{ for all } \mathbf{u}, \mathbf{v} \in \mathbb{R}^{1,N-1}, \text{ and } T\mathcal{H} = \mathcal{H}.\}.$$

Reflection in the plane $H_\mathbf{n}$ is given by

$$R_\mathbf{n}(\mathbf{x}) = \mathbf{x} - 2\,\mathrm{proj}_\mathbf{n}(\mathbf{x}) = \mathbf{x} - 2\frac{\mathbf{n} \cdot \mathbf{x}}{\mathbf{n} \cdot \mathbf{n}}\mathbf{n}.$$

The group of isometries is generated by the reflections.

Let $\partial\mathcal{H}$ represent the boundary of $\mathcal{H}$, which is an $(N-2)$-sphere. It is represented by $\mathcal{L}^+/\mathbb{R}^+$ where

$$\mathcal{L}^+ = \{\mathbf{x} \in \mathbb{R}^{1,N-1} : \mathbf{x} \cdot \mathbf{x} = 0, \mathbf{x} \cdot D > 0\} = \partial V^+.$$

Given an $E \in \mathcal{L}^+$, let $\partial\mathcal{H}_E = \partial\mathcal{H} \setminus E\mathbb{R}^+$. Then $\partial\mathcal{H}_E$ equipped with the metric $|\cdot|_E$ defined by

$$|AB|_E^2 = \frac{2A \cdot B}{(A \cdot E)(B \cdot E)}$$

is the $(N-2)$-dimensional Euclidean space that is the boundary of the Poincaré upper half hyperspace model of $\mathcal{H}$ with E the point at infinity. In $\partial\mathcal{H}_E$, the plane $H_\mathbf{n}$ is represented by an $(N-3)$-sphere, which we denote with $H_{\mathbf{n},E}$ (or just $H_\mathbf{n}$ if E is understood, or sometimes just $\mathbf{n}$).

The curvature (the inverse of the radius, together with a sign) of $H_{\mathbf{n},E}$ is given by the formula

$$\frac{\mathbf{n} \cdot E}{||\mathbf{n}||}$$

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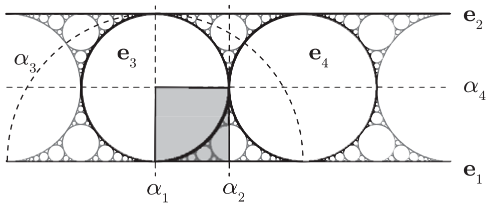

Figure 1. *The strip version of the Apollonian packing. The dotted curves α_i represent symmetries of the packing. The fundamental domain $\mathcal{F}_4$ for the group $\mathcal{O}^+_{\Lambda_4}$ is the region above the shaded square and above the plane H_{α_3} (represented by the dotted circle).*

using the metric $|\cdot|_E$ [4]. Here, $||\mathbf{n}|| = -i|\mathbf{n}| = \sqrt{-\mathbf{n}\cdot\mathbf{n}}$. By choosing a suitable orientation for $\mathbf{n}$, we get the appropriate sign for the curvature.

2.2. *Coxeter graphs*

Coxeter graphs can represent lattices in $\mathbb{R}^n$ or $\mathbb{R}^{1,n-1}$, or groups of isometries of $\mathbb{S}^{n-1}$ or $\mathbb{H}^{n-1}$. Each node in a Coxeter graph represents a plane, which can be represented by its normal vector. Two nodes are not connected if their corresponding planes are perpendicular. A regular edge between two nodes indicates that those planes intersect at an angle of $\pi/3$, so their normal vectors are at an angle of $2\pi/3$ (some ambiguity here). If the angle between two planes is $\pi/4$, then we indicate that with an edge subscripted (or superscripted) with a 4, or sometimes a double edge. In general, an edge with a superscript of m (or of multiplicity $m-2$) means the order of the composition of reflections in the two planes represented by the two nodes is m. A thick line, like the one in (1.1), means the two planes are parallel, and this is sometimes indicated with a superscript of ∞. A dotted edge means the two planes are ultraparallel.

The Coxeter graph represents the lattice $\mathbf{v}_1\mathbb{Z}\oplus\cdots\oplus\mathbf{v}_n\mathbb{Z}$, where the $\mathbf{v}_i$ are the nodes of the graph. Note that not all graphs give lattices, as the vectors may be linearly dependent. When the nodes are linearly independent (and sometimes when they are not), we get the bilinear form defined by the *incidence matrix* $[\mathbf{v}_i\cdot\mathbf{v}_j]$. We can normalize the vectors $\mathbf{v}_i$ (say) so that $\mathbf{v}_i\cdot\mathbf{v}_i=-1$. Then the angles noted by the edges define $\mathbf{v}_i\cdot\mathbf{v}_j$, where we take the angle between the normal vectors of the planes to be obtuse (so $\mathbf{v}_i\cdot\mathbf{v}_j\geq 0$).

The Coxeter graph can also represent a group, the *Weyl* group $\langle R_{\mathbf{v}_1},\ldots,R_{\mathbf{v}_n}\rangle$. When representing a group, it is often necessary for the vectors to be linearly dependent.

We will explain the notion of *weights* in the next subsection, which will give us an example to investigate.

2.3. *The Apollonian circle packing*

The Apollonian circle packing is a well- known object, and we assume the reader is already familiar with it. The goal of this section is to think of it in a way that more naturally generalizes to higher dimensions.

Let us consider the strip version of the Apollonian circle packing shown in Figure 1, and think of the picture as lying on the boundary $\partial\mathbb{H}^3$ of hyperbolic space. Then each circle (and the two lines) represents a plane in $\mathbb{H}^3$, which in turn are represented by their normal vectors. Let us represent four of the hyperbolic planes with the vectors $\mathbf{e}_i$ for $i=1,\ldots,4$, as shown in Figure 1. We orient $\mathbf{e}_i$ so that $H^+_{\mathbf{e}_i}$ contains $H_{\mathbf{e}_j}$ for $i\neq j$, and assign them norms of -2, meaning $\mathbf{e}_i\cdot\mathbf{e}_i=-2$. Then, by the tangency conditions, $\mathbf{e}_i\cdot\mathbf{e}_j=2$ for $i\neq j$ (see equation (2.1)). We define $J_4=[\mathbf{e}_i\cdot\mathbf{e}_j]$, which has signature (1, 3). (It is easy to see that 4 and -4 are eigenvalues, the latter with a 3-dimensional eigenspace.) Thus, J_4 defines a Lorentz product in $\mathbb{R}^{1,3}$.

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Note that $\mathbf{e}_i \cdot (\mathbf{e}_i + \mathbf{e}_j) = 0$, so $\mathbf{e}_i + \mathbf{e}_j$ is the point of tangency between the circles represented by $\mathbf{e}_i$ and $\mathbf{e}_j$. Thus, our point at infinity is $E = \mathbf{e}_1 + \mathbf{e}_2$, and the circle represented by $\mathbf{e}_i$ is $H_{\mathbf{e}_i,E}$.

There are obvious symmetries of the Apollonian packing, as shown in Figure 1. In the plane, the symmetries are reflection in the lines α_1, α_2, and α_4 and inversion in the circle α_3. The Apollonian packing is the orbit of $\mathbf{e}_1$ under the action of the group generated by these symmetries.

Thought of as actions in $\mathbb{H}^3$, the symmetries are reflection in the planes H_{α_i}, so the packing is the Γ_4-orbit of $H_{\mathbf{e}_1}$, where

$$\Gamma_4 = \langle R_{\alpha_1}, R_{\alpha_2}, R_{\alpha_3}, R_{\alpha_4} \rangle.$$

It is fairly easy to solve for α_i in the basis $\mathbf{e} = \{\mathbf{e}_1, ..., \mathbf{e}_4\}$. For example, we note that $\alpha_1 \cdot \mathbf{e}_i = 0$ for $i \neq 4$, since it is perpendicular to those planes. Solving, we get $\alpha_1 = (1, 1, 1, -1)$, up to scalars. The others are $\alpha_2 = (0, 0, -1, 1)$, $\alpha_3 = (0, -1, 1, 0)$, and $\alpha_4 = (-1, 1, 0, 0)$.

We let $\Lambda_4 = \mathbf{e}_1\mathbb{Z} \oplus ... \oplus \mathbf{e}_4\mathbb{Z}$, which we call the *Apollonian lattice*. Let

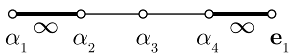
$$\mathcal{O}^+_{\Lambda_4} = \{T \in \mathcal{O}^+(\mathbb{R}) : T\Lambda_4 = \Lambda_4\}.$$

It is straight forward to verify that $R_{\mathbf{e}_i}$ and $R_{\alpha_i} \in \mathcal{O}^+_{\Lambda_4}$. Let $G_4 = \langle \Gamma_4, R_{\mathbf{e}_1} \rangle$. A fundamental domain $\mathcal{F}_4$ for this group is the region bounded by the four planes H_{α_i}, the pane $H_{\mathbf{e}_1}$, and with cusp E at the point at infinity:

$$\mathcal{F}_4 = H^+_{\alpha_1} \cap \cdots \cap H^+_{\alpha_4} \cap H^+_{\mathbf{e}_1}.$$

Since $\mathcal{F}_4$ has finite volume, we know G_4 has finite index in $\mathcal{O}^+_{\Lambda_4}$, and it is not hard to verify that the two are equal.

Let us also use this example to learn a little about Coxeter graphs. The Coxeter graph for the planes that bound $\mathcal{F}_4$ is the following:



(2.2)

We note, from our earlier calculations, that $\alpha_i \cdot \alpha_i = -8$. From the Coxeter graph, we therefore get $\alpha_1 \cdot \alpha_2 = 8$, $\alpha_2 \cdot \alpha_3 = 4$, $\alpha_3 \cdot \alpha_4 = 4$, and all other products $\alpha_i \cdot \alpha_j$ for $i < j$ are zero. This gives us the matrix

$$J_\alpha = [\alpha_i \cdot \alpha_j] = \begin{bmatrix} -8 & 8 & 0 & 0 \\ 8 & -8 & 4 & 0 \\ 0 & 4 & -8 & 4 \\ 0 & 0 & 4 & -8 \end{bmatrix}.$$

Our Lorentz product, in this basis, is $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^t J_\alpha \mathbf{y}$. Let $\Lambda_\alpha = \alpha_1\mathbb{Z} \oplus \cdots \oplus \alpha_4\mathbb{Z}$, so $\Lambda_\alpha \subset \Lambda_4$. Since $\det(J_\alpha) = 4\det(J_4)$, the index is two.

The fundamental domain $\mathcal{F}_4$ has five vertices: The point E at infinity, and the four vertices on H_{α_3}, the hemisphere above the dotted circle (see Figure 1). Let w_i be the intersection of the three α_j with $j \neq i$, so w_i is a vertex of $\mathcal{F}_4$ for $i = 1$ and 2, and w_3 is the point at infinity. The w_i's satisfy $w_i \cdot \alpha_j = 0$ for $i \neq j$, so in the basis $\alpha = \{\alpha_1, ..., \alpha_4\}$, are the rows of J_α^{-1}, up to scalars. To express these points in the basis $\mathbf{e}$, we multiply on the left by the change of basis matrix $Q = [\alpha_1|\alpha_2|\alpha_3|\alpha_4]$ whose columns are α_i; the columns of the resulting matrix are the w_i's. The w_i's are called *weights*. The incidence matrix for the w_is is $[w_i \cdot w_j] = J_\alpha^{-1} J_\alpha (J_\alpha^{-1})^t = (J_\alpha^{-1})^t = J_\alpha^{-1}$. The first and second diagonal elements of J_α^{-1} are positive, indicating that w_1 and w_2 lie in $\mathcal{H}$. The third diagonal element is 0, indicating w_3 lies on $\partial\mathcal{H}$, so is a cusp (it is the point E at infinity). The last diagonal element is negative, so w_4 represents a plane in $\mathcal{H}$. That plane is perpendicular to H_{α_1}, H_{α_2}, and H_{α_3}, so is the plane $H_{\mathbf{e}_1}$.

The other two vertices of $\mathcal{F}_4$ can be found in a similar way by replacing α_4 with $\mathbf{e}_1$. This is plain to see from the picture (see Figure 1). Combinatorially, the intersection of any three of the five faces should give us a potential vertex, but since H_{α_4} and $H_{\mathbf{e}_1}$ are parallel, any combination that includes those two will give at most their point of tangency.

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Remark 1. *The symmetry of the Coxeter graph about the node α_3 suggests that there is a symmetry that sends $\mathbf{e}_1$ to α_1, etc. That symmetry is reflection in the diagonal of slope one of the shaded square in Figure 1. However, since the norms of $\mathbf{e}_1$ and α_1 are different, it is not a symmetry of the lattice Λ_4. There is a different lattice that we could have considered, namely the one generated by the Coxeter graph (2.2) but with $\alpha_i^2 = -2$. There are some advantages to looking at that graph. Reflection in the diagonal is a symmetry of that lattice.*

2.4. Sphere packings

A *sphere packing* in $\mathbb{R}^n$ is a configuration of oriented $(n-1)$-spheres that intersect tangentially or not at all. By oriented, we mean each sphere includes either the inside (a ball) or the outside. The trivial sphere packing is a sphere and its complement. We will not consider trivial sphere packings.

Maxwell calls $\mathcal{P} \subset \mathbb{R}^{1,N-1}$ a *packing* if for all $\mathbf{n}, \mathbf{n}' \in \mathcal{P}$, there exists a positive constant k such that $\mathbf{n} \cdot \mathbf{n} = -k$ and $\mathbf{n} \cdot \mathbf{n}' \geq k$ [16]. Given a point E for the point at infinity, the packing $\mathcal{P}$ defines a sphere packing

$$\mathcal{P}_E = \bigcup_{\mathbf{n} \in \mathcal{P}} H^-_{\mathbf{n},E} \subset \partial\mathcal{H}_E \cong \mathbb{R}^n.$$

We call $\mathcal{P}_E$ a *perspective* of $\mathcal{P}$.

For example, the Apollonian packing in $\mathbb{R}^{1,3}$ is $\mathcal{P}_4 = \Gamma_4(\mathbf{e}_1)$ (the Γ_4-orbit of $\mathbf{e}_1$) and the strip packing of Figure 1 is $\mathcal{P}_{4,\mathbf{e}_1+\mathbf{e}_2}$. An Apollonian packing derived from a different initial cluster of four mutually tangent circles is a different perspective of $\mathcal{P}_4$.

We think of $\mathcal{P}$ as defining a *cone*

$$\mathcal{K}_{\mathcal{P}} = \bigcap_{\mathbf{n} \in \mathcal{P}} H^+_{\mathbf{n}},$$

in $\mathbb{R}^{1,N-1}$, or a *polyhedron*

$$\mathcal{K}_{\mathcal{P}} \cap \mathcal{H},$$

in $\mathbb{H}^{N-1}$. The *residual set* of a sphere packing is the complement of the sphere packing in $\partial\mathcal{H}_E \cong \mathbb{R}^n$ and is the intersection of $\mathcal{K}_{\mathcal{P}}$ with $\partial\mathcal{H}_E$. A sphere packing is *maximal* or *dense* if there is no space in the residual set where one can place another sphere of positive radius. It is *complete* if the residual set is of measure zero. A packing that is maximal (respectively complete) in one perspective is maximal (complete) in any perspective.

We call a packing *of lattice type* if $\mathcal{P}\mathbb{Z}$ forms a lattice in $\mathbb{R}^{1,N-1}$ [16]. We call a packing *of general lattice type* if there exists a lattice $\Lambda \subset \mathbb{R}^{1,N-1}$ so that a scalar multiple of $\mathbf{n}$ is in Λ for all $\mathbf{n} \in \mathcal{P}$, and $\mathcal{P}$ spans $\mathbb{R}^{1,N-1}$.

A subgroup $\Gamma \leq \mathcal{O}^+_\Lambda$ is called *geometrically finite* if it has a convex fundamental domain with a finite number of faces. A packing is called *crystallographic* if there exists a geometrically finite group $\Gamma \leq \mathcal{O}^+_\Lambda$ and a finite set $S \subset \Lambda$ so that

$$\mathcal{P} = \{\gamma(\mathbf{n})/|\gamma(\mathbf{n})| : \gamma \in \Gamma, \mathbf{n} \in S\}.$$

As the normality condition is not necessary, we will write $\mathcal{P} = \Gamma(S)$, the Γ-orbit of S. The residual set for the packing is the limit set of Γ.

Crystallographic packings are not always of lattice type, as was noted by Maxwell.

A crystallographic packing that is maximal is known to be complete. Furthermore, the Hausdorff dimension of the residual set is strictly less than $N-2$ [22].

The packings that are known as Boyd-Maxwell packings are crystallographic packings with the restriction that Γ be a reflective group (i.e. is generated by a finite number of reflections). Kontorovich and Nakamura include this requirement in their definition of a crystallographic packing [13].

We say a packing has the *weak Apollonian property* if it contains a cluster of $N = n+2$ mutually tangent spheres. We say it has the *Apollonian property* if every sphere is a member of a cluster of N

mutually tangent spheres. We call a packing *Apollonian* if it is crystallographic, maximal, and has the Apollonian property.

Missing in this definition is a notion of efficiency. It is conjectured that the residual set for the Apollonian circle packing in two dimensions has minimal Hausdorff dimension among all nontrivial circle packings [18], so an alternative definition would be that a packing is Apollonian if it is optimally efficient in that dimension. In $\mathbb{R}^7$, there exists a weakly Apollonian packing that is more efficient than a known Apollonian packing (see Section 3.3), suggesting that there is still much to learn about the subject.

Remark 2. *In* [4], *we give a different definition for an Apollonian packing. We begin with a set* $\{\mathbf{e}_1, ..., \mathbf{e}_N\}$ *with* $\mathbf{e}_i \cdot \mathbf{e}_i = -1$ *and* $\mathbf{e}_i \cdot \mathbf{e}_j = 1$ *for* $i \neq j$ *(so a cluster of N mutually tangent spheres), and define the lattice*

$$\Lambda_N = \mathbf{e}_1\mathbb{Z} \oplus \cdots \oplus \mathbf{e}_N\mathbb{Z}.$$

We pick $D \in \Lambda_N$ *with* $D \cdot D > 0$ *and such that* $D \cdot \mathbf{n} \neq 0$ *for any* $\mathbf{n} \in \Lambda_N$ *with* $\mathbf{n} \cdot \mathbf{n} = -1$. *(Such a D exists.) We define*

$$\mathcal{E}_{-1} = \{\mathbf{n} \in \Lambda_N : \mathbf{n} \cdot \mathbf{n} = -1, \mathbf{n} \cdot D > 0\}.$$

We define the cone

$$\mathcal{K}_N = \bigcap_{\mathbf{n} \in \mathcal{E}_{-1}} H_{\mathbf{n}}^+,$$

and the set

$$\mathcal{E}_{-1}^* = \{\mathbf{n} \in \mathcal{E}_{-1} : H_{\mathbf{n}} \text{ is a face of } \mathcal{K}_N\}.$$

The set $\mathcal{E}_{-1}^*$ *is what is called an Apollonian packing in that paper. It gives the Apollonian circle packing when* $N = 4$, *the Soddy sphere packing when* $N = 5$, *and yields Apollonian packings (as defined in this paper) in dimensions* $n = 4$ *through* 8 [4, 5].

It is not clear that $\mathcal{E}_{-1}^*$ is a packing for all N, as it is *a priori* possible that spheres in the set properly intersect. There is a very nice modularity argument to show that this cannot happen. We first note that if $\mathbf{n}$, $\mathbf{n}' \in \mathcal{E}_{-1}^*$, then $\mathbf{n} \cdot \mathbf{n}' \in \mathbb{Z}$, so if the spheres intersect but not tangentially, then $\mathbf{n} \cdot \mathbf{n}' = 0$ (so they intersect perpendicularly). Consider the difference $\mathbf{m} = \mathbf{n} - \mathbf{n}'$. If $\mathbf{n} \cdot \mathbf{n}' = 0$, then $\mathbf{m} \cdot \mathbf{m} = -2$. Let $\mathbf{m} = (m_1, ..., m_N)$ so

$$\mathbf{m} \cdot \mathbf{m} = -\sum_{k=1}^{N} m_k^2 + 2\sum_{i \le j} m_i m_j.$$

We consider this modulo four, which means we need only worry about the parity of each m_k. Suppose there are K odd m_k. Then

$$\mathbf{m} \cdot \mathbf{m} \equiv -K + 2\left(\frac{K(K-1)}{2}\right) \equiv K(K-2) \pmod 4.$$

Thus, $\mathbf{m} \cdot \mathbf{m} \neq 2 \pmod 4$, so there are no spheres that intersect perpendicularly.

Remark 3. *For a K3 surface X, let* $\Lambda = \mathrm{Pic}\,(X)$ *and let* $\mathcal{E}_{-2}^*$ *be the set of irreducible* -2 *curves on X. Then* $\mathcal{K}_{\mathcal{E}_{-2}^*}$ *is the ample cone for X. This was the motivation for the definitions given in* [4].

Using a result of Morrison [19], there exist K3 surfaces with intersection matrix $[J_{ij}]$ with $J_{ij} = 2 - 4\delta_{ij}$ and $N \le 10$. The set $\mathcal{E}_{-2}^*$ gives the same packing as described in Remark 2. By [14], the ample cone has no circular part for $N \ge 4$, so these packings are maximal.

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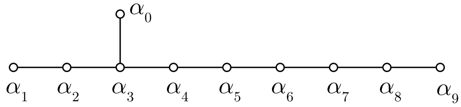
2.5. Enriques surfaces

For a nice introduction to Enriques surfaces, see [11]. Let X be an Enriques surface and let Λ be its Picard group modulo torsion. Then Λ is independent of the choice of X and is sometimes called the Enriques lattice E_{10}. It is an even unimodular lattice of signature $(1, 9)$ and can be decomposed as the orthogonal product of the E_8 lattice (with negative definite inner product) and the plane U equipped with the Lorentz product $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$: $\Lambda = E_8 \oplus U$.

A nice representation of E_{10} is given by the Coxeter graph T_{237} whose nodes have norm -2:



The subscript in T_{237} means the graph is a tree with three branches of length 2, 3, and 7, as above. The lattice is $\Lambda = \alpha_0\mathbb{Z} \oplus \cdots \oplus \alpha_9\mathbb{Z}$. Since $\alpha_i \cdot \alpha_i = -2$, we know $\alpha_i \cdot \alpha_j = 1$ if $\alpha_i\alpha_j$ is an edge of the graph, and 0 otherwise. Let $J = [\alpha_i \cdot \alpha_j]$, so for $\mathbf{x},\ \mathbf{y} \in \Lambda$ written in this basis, $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^t J \mathbf{y}$. Since J has integer entries and -2's along the diagonal, the lattice is even, meaning $\mathbf{x} \cdot \mathbf{x}$ is even for all $\mathbf{x} \in \Lambda$. One can verify $\det(J) = -1$, so the lattice is unimodular.

The reflections

$$R_{\alpha_i}(\mathbf{x}) = \mathbf{x} - 2\,\mathrm{proj}_{\alpha_i}(\mathbf{x}) = \mathbf{x} - 2\frac{\mathbf{x} \cdot \alpha_i}{\alpha_i \cdot \alpha_i}\alpha_i = \mathbf{x} + (\mathbf{x} \cdot \alpha_i)\alpha_i,$$

have integer entries, so are in $\mathcal{O}_\Lambda^+$. The inverse of J (which appears in [11, p.11]) has non-negative entries along the diagonal, so the polytope bounded by the faces H_{α_i} has finite volume. Thus, the Weyl group

$$W_{237} = \langle R_{\alpha_0}, \ldots, R_{\alpha_9} \rangle,$$

has finite index in $\mathcal{O}_\Lambda^+$.

A generic Enriques surface has no *nodal* curves, meaning it has no smooth rational curves. The moduli space of Enriques surfaces is ten dimensional. If X contains a nodal curve ν, then we call it a *nodal Enriques surface*. By the adjunction formula, $\nu \cdot \nu = -2$. Here we have abused notation by letting ν represent both the curve on X and its representation in Λ. Since distinct irreducible curves on X have non-negative intersection, an element of Λ represents at most one nodal curve on X. The moduli space of nodal Enriques surfaces is nine dimensional. The following is a rewording of a portion of [1, Theorem 1.2]:

Theorem 2.1 (Coble, Looijenga, Cossec, Dolgachev, Allcock). *Suppose X is a generic nodal Enriques surface with nodal curve ν. Let Λ be its Picard group modulo torsion. Then there exist $\beta_0, \ldots, \beta_9 \in \Lambda$ so that $\beta_i \cdot \beta_i = -2$ and $\beta_0, \ldots, \beta_9, \nu$ are the nodes of the Coxeter graph* (1.1). *Let*

$$\Gamma_\beta = \langle R_{\beta_0}, \ldots, R_{\beta_9} \rangle \cong W_{246}.$$

Then the image in Λ of all nodal curves on X is the Γ_β-orbit of ν.

As this version does not look much like Allcock's result, let us show how it follows. By (5) (of [1, Theorem 1.2]), $\mathrm{Aut}\,(X)$ acts transitively on the nodal curves on X. The image of $\mathrm{Aut}\,(X)$ in $\mathcal{O}_\Lambda^+$ is a subgroup of $\Gamma_\beta = W_{246}$ by (2), and by (3), contains a group identified as $\overline{W}_{246}(2)$ (see [1] for the definition of this group). By (4), $\overline{W}_{246}(2)$ acts transitively on the faces of the ample cone (the interior of $\mathrm{nef}\,(X)$), so the image of the nodal curves on X includes all of them. By (1), $\mathrm{nef}\,(X)$ is the union of the Γ_β-translates of the fundamental domain of the Coxeter group generated by the Coxeter graph (1.1) on page 2. The faces of the ample cone are therefore the Γ_β image of ν, as all other faces of the fundamental domain correspond to reflections in W_{246}. Hence, the nodal curves on X generate exactly the faces of the ample cone and is the Γ_β-orbit of ν.

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3. The Apollonian packing in eight dimensions

Theorem 3.1. *The packing* $\mathcal{P}_\beta = \Gamma_\beta(\nu)$ *is Apollonian.*

Proof. From Maxwell [16, Table II, $N = 10$], we know that this is a Boyd-Maxwell packing, so all we must verify is that it is maximal and that it contains a cluster of ten mutually tangent spheres. (Because Γ acts transitively on $\mathcal{P}_\beta$, the weak Apollonian property implies the Apollonian property.)

We define $J_\beta = [\beta_i \cdot \beta_j]$, which has -2 along the diagonal, 1 if $\beta_i\beta_j$ is an edge in Coxeter graph (1.1), and 0 otherwise. Taking the inverse,

$$J_\beta^{-1} = \frac{1}{2}\begin{bmatrix} 5 & 3 & 6 & 9 & 12 & 10 & 8 & 6 & 4 & 2 \\ 3 & 0 & 2 & 4 & 6 & 5 & 4 & 3 & 2 & 1 \\ 6 & 2 & 4 & 8 & 12 & 10 & 8 & 6 & 4 & 2 \\ 9 & 4 & 8 & 12 & 18 & 15 & 12 & 9 & 6 & 3 \\ 12 & 6 & 12 & 18 & 24 & 20 & 16 & 12 & 8 & 4 \\ 10 & 5 & 10 & 15 & 20 & 15 & 12 & 9 & 6 & 3 \\ 8 & 4 & 8 & 12 & 16 & 12 & 8 & 6 & 4 & 2 \\ 6 & 3 & 6 & 9 & 12 & 9 & 6 & 3 & 2 & 1 \\ 4 & 2 & 4 & 6 & 8 & 6 & 4 & 2 & 0 & 0 \\ 2 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 & -1 \end{bmatrix},$$

we find that the weights all have non-negative norm except w_9, which gives us $\nu = 2w_9 = [2, 1, 2, 3, 4, 3, 2, 1, 0, -1]$. (Recall that the weights are the rows of J_β^{-1}, and that the diagonal elements are their norms $w_i \cdot w_i$.) Since $w_9 \cdot \beta_i = \delta_{i9}$, we get $\nu \cdot \beta_9 = 2$ and $\nu \cdot \beta_i = 0$ for $i \neq 9$, as expected. This is our first sphere, which we label $s_0 = \nu = 2w_9$. We get s_1 by reflecting s_0 across H_{β_9}: $s_1 = R_{\beta_9}(s_0)$. We define s_i through $i = 9$ recursively by reflecting in subsequent planes: $s_{i+1} = R_{\beta_{9-i}}(s_i)$. It is straight forward to verify that $s_i \cdot s_j = 2$ if $i \neq j$, and of course, $s_i \cdot s_i = \nu \cdot \nu = -2$, so the set $\{s_0, ..., s_9\}$ is a cluster of ten mutually tangent spheres.

To show it is maximal, we can appeal to arguments like those in [4] and [5]. Let us instead appeal to Maxwell's Theorem 3.3 [16]. By this result, it is enough to show that all weights w_i are in the convex closure of $\Gamma(w_9)$. Following Maxwell's example, we note

$$w_{i-1} = w_i + R_{\beta_i} \cdots R_{\beta_9} w_9 = w_i + s_{10-i}/2 = (s_0 + ... + s_{10-i})/2,$$

for $5 \leq i \leq 9$, so these are all in the convex hull of $\Gamma(w_9)$. This leaves us with four more to check, of which w_1 is different, as it is a cusp and no spheres go through it.

We will use the map $R_{2w_1-\beta_9}$, which we verify is in Γ using a method of descent. Our method of descent is as follows: Given a vector $\mathbf{n}$, we descend to $R_{\beta_j}(\mathbf{n})$ if $\mathbf{n} \cdot \beta_j < 0$, and repeat if necessary. For $\mathbf{n} \cdot \mathbf{n} > 0$ (a point in $\mathcal{H}$), it is clear that at each step we are getting closer to the fundamental domain, and that descent stops when we reach it. What this method does for $\mathbf{n} \cdot \mathbf{n} < 0$ is less obvious, but is never-the-less useful.

With $\mathbf{n} = 2w_1 - \beta_9$, we eventually descend to β_8, so there exists $\gamma \in \Gamma$ so that $\mathbf{n} = \gamma\beta_8$. Thus, $R_\mathbf{n} = R_{\gamma\beta_8} = \gamma R_{\beta_8}\gamma^{-1} \in \Gamma$. The composition $R_{2w_1-\beta_9} \circ R_{\beta_9}$ is a parabolic translation with fixed point w_1 on $\partial\mathcal{H}$, so repeated applications converge to the line λw_1. More precisely, a messy but straightforward calculation shows

$$\lim_{k\to\infty} \frac{1}{2k^2}(R_{2w_1-\beta_9} \circ R_{\beta_9})^k(w_9) = w_1.$$

Thus w_1 is in the convex closure of $\Gamma(w_9)$.

Finally, we note that

$$\begin{aligned} w_2 &= (R_{\beta_0}(s_6) + R_{\beta_2}(s_8))/2 \\ w_0 &= w_1 + s_9/2 \\ w_3 &= R_{\beta_2}(w_2) + R_{\beta_1}(w_1). \end{aligned}$$

Thus, $\mathcal{P}_\beta$ is an Apollonian sphere packing. □

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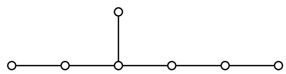
3.1. The strip version and the E_7 lattice

To better understand this packing, we describe it in a couple of ways. Let us begin with a strip version, which is an analog of Figure 1. We can think of Figure 1 as an infinite set of circles of constant diameter that are each centered at lattice points of a one-dimensional lattice, then sandwiched between two lines, and filled in using the symmetries of the lattice together with inversion in α_3 and reflection in α_4. We can describe $\mathcal{P}_\beta$ in a similar way:

Theorem 3.2. *Consider the E_7 lattice imbedded in a 7-dimensional subspace V of $\mathbb{R}^8$. Centered at each lattice point, place a hypersphere of radius $1/\sqrt{2}$, and bound these hyperspheres by two hyperplanes parallel to V and a distance $1/\sqrt{2}$ away from V, so that all the spheres are tangent to the two hyperplanes. Let one of the spheres be tangent to the two hyperplanes at A and B, and let σ be inversion in the hypersphere centered at A that goes through B. Consider the image of these spheres in the group generated by the symmetries of the E_7 lattice, the inversion σ, and reflection across the hyperplane V. The resulting configuration of hyperspheres is a perspective of $\mathcal{P}_\beta$.*

The E_7 lattice has the Coxeter diagram T_{234}:



It is common to assign each node the norm 2 so that the resulting bilinear form has integer entries, is even, and has determinant 2. Thus, the generating vectors have length $\sqrt{2}$, which is why the spheres above have radius $\sqrt{2}/2$. Adding a node to the appropriate branch gives us the group W_{244}, which acts transitively on the E_7 lattice [24, Table 2]. Alternatively, the E_7 lattice can be thought of as the W_{244} orbit of the point O, where O is the vertex of the canonical fundamental domain for W_{244} that lies at the intersection of the planes represented by the T_{234} subgraph. This is the representation we shall use.

Proof. Referring to J_β^{-1}, we note that w_8 has norm zero, so is a point on $\partial\mathcal{H}$. Let us consider the perspective $\mathcal{P}_{\beta,w_8}$, which lies in $\partial\mathcal{H}_{w_8} \cong \mathbb{R}^8$. Note that the spheres s_0 and s_1 are tangent at w_8, so are parallel Euclidean hyperplanes in $\partial\mathcal{H}_{w_8}$. These are our analogs of $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively, in Figure 1. Since $s_1 = R_{\beta_9}(s_0)$, the reflection R_{β_9} is the analog of the symmetry represented by α_4, and we take V to be $H_{\beta_9} \cap \partial\mathcal{H}_{w_8}$. Since $s_2 = R_{\beta_8}(s_1)$, the reflection R_{β_8} is σ, the analog of inversion in α_3. The sphere s_2 is the analog of $\mathbf{e}_3$, and the points A and B are the points of tangency $s_0 + s_2$ and $s_1 + s_2$, respectively. The rest of the generators are the analogs of α_1 and α_2. They form the group $\Gamma_7 = \langle R_{\beta_0}, ..., R_{\beta_7}\rangle \cong W_{244}$. It remains to understand what the orbit $\Gamma_7(s_2)$ looks like.

Note that the generators of Γ_7 fix w_8, so are Euclidean reflections in $\partial\mathcal{H}_{w_8} \cong \mathbb{R}^8$. The planes through which they reflect are all perpendicular to $V = H_{\beta_9} \cap \partial\mathcal{H}_{w_8} \cong \mathbb{R}^7$, so act as a group of reflections on V. Let O in V be the point of intersection of H_{β_i} for $i \le 6$. Since the planes H_{β_i} are perpendicular to s_2 for $i \le 6$ (since $\beta_i \cdot s_2 = 0$ for $i \le 6$), the point O must be the center of the sphere s_2. Thus, the Γ_7-orbit of s_2 is a set of congruent spheres that are centered on the lattice points of an E_7 lattice. Finally, since $s_3 = R_{\beta_7}(s_2)$, and s_3 is tangent to s_2, the spheres must be of maximal radius so that they intersect tangentially or not at all. That is, scaling so that the generating vectors of the E_7 lattice have length $\sqrt{2}$, the spheres all have radius $\sqrt{2}/2$. □

3.2. The E_8 structure

The E_8 lattice has the Coxeter diagram T_{235}, and the group W_{236} acts transitively on it [24, Table 2]. The lattice can also be thought of as the W_{236} orbit of the point O, where O is the vertex of the canonical fundamental domain for W_{236} that lies at the intersection of the planes represented by the T_{235} subgraph.

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Let us distinguish another vertex of the fundamental domain, the point P that lies at the intersection of the planes represented by the T_{226} subgraph.

Theorem 3.3. *Consider an arrangement of hyperspheres all of radius $1/\sqrt{2}$ and centered at the lattice points of the E_8 lattice in $\mathbb{R}^8$. (This is the densest uniradial sphere packing in $\mathbb{R}^8$.) Let σ be inversion in the sphere centered at P (described above) and with radius $1/\sqrt{2}$. Consider the image of this arrangement of spheres in the group generated by σ and the symmetries of the E_8 lattice. The resulting configuration is a perspective of $\mathcal{P}_\beta$.*

Proof. Referring to J_β^{-1}, we see that w_1 is also on $\partial\mathcal{H}$, as $w_1 \cdot w_1 = 0$. Let us consider the perspective $\mathcal{P}_{\beta,w_1}$. The planes H_{β_i} for $i \neq 1$ all go through w_1 (by the definition of w_1), so the corresponding reflections R_{β_i} are Euclidean symmetries of $\mathbb{R}^8 \cong \partial\mathcal{H}_{w_1}$. Let us first consider the group $\Gamma_8 = \langle R_{\beta_0}, R_{\beta_2}, ..., R_{\beta_9}\rangle \cong W_{236}$.

We let O be the point on $\partial\mathcal{H}_{w_1}$ where the planes $H_{\beta_0}, H_{\beta_2}, \ldots, H_{\beta_8}$ intersect, so $\Gamma_8(O)$ is an E_8 lattice in $\partial\mathcal{H}_{w_1}$. Since these planes intersect $s_0 = \nu$ perpendicularly, we know O is the center of s_0. The Γ_8-orbit of s_0 is therefore a set of spheres of constant radius centered on the vertices of an E_8 lattice. Since $R_{\beta_9}(s_0) = s_1$ is tangent to s_0, these spheres are of maximal radius such that any pair intersect tangentially or not at all. They give the densest uniradial sphere packing of $\mathbb{R}^8$ [23].

It remains to understand the action of R_{β_1}. The plane H_{β_1} represents a sphere in $\partial\mathcal{H}_{w_1}$. Since $\beta_1 \cdot w_1 = 1 = s_0 \cdot w_1$ and $\beta_1 \cdot \beta_1 = -2 = s_0 \cdot s_0$, the spheres have the same radii. Finally, $\beta_1 \cdot \beta_i = 0$ for $i = 0, 3, ..., 9$, so is centered at the point of intersection on V of the planes $H_{\beta_0}, H_{\beta_3}, \ldots, H_{\beta_9}$, which corresponds to the vertex P of the fundamental domain for W_{236}. Thus, the described configuration is congruent to the perspective $\mathcal{P}_{\beta,w_1}$. □

3.3. Comparison with the packing in [4]

Let

$$\Lambda_\beta = \beta_0\mathbb{Z} \oplus \cdots \oplus \beta_9\mathbb{Z},$$
$$\Lambda_{10} = s_0\mathbb{Z} \oplus \cdots \oplus s_9\mathbb{Z}.$$

and let $J_{10} = [s_i \cdot s_j]$. Then $\det(J_{10}) = -2^{22}$ while $\det(J_\beta) = -4$. It would appear that the packing $\mathcal{P}_{10} = \mathcal{E}^*_{-2}$ generated by Λ_{10} (see Remark 2) is very different than $\mathcal{P}_\beta$. However, we can modify the underlying lattice for $\mathcal{P}_\beta$ in the following way. Consider the sublattice

$$\Lambda'_\beta = 2\beta_0\mathbb{Z} \oplus \cdots \oplus 2\beta_8\mathbb{Z} \oplus \nu\mathbb{Z}.$$

and its incidence matrix J'_β. Then $\Lambda_{10} \subset \Lambda'_\beta \subset \Lambda_\beta$ and $\det(J'_\beta) = -2^{20}$. Thus, Λ_{10} is a sublattice of Λ'_β of index two. Though we have changed the underlying lattice for $\mathcal{P}_\beta$, we have not changed its geometry. Furthermore, in this basis, $\mathcal{P}_\beta = \mathcal{E}^*_{-2}$. Thus, $\mathcal{K}_{\mathcal{P}_{10}} \supset \mathcal{K}_{\mathcal{P}_\beta}$, so the residual set for the packing $\mathcal{P}_\beta$ is a subset of the residual set for $\mathcal{P}_{10}$. Thus, $\mathcal{P}_\beta$ is a more *efficient* packing than $\mathcal{P}_{10}$.

Remark 4. *There exists a K3 surface X with* Pic $(X) = \Lambda'_\beta$*, so we can think of $\mathcal{P}_\beta$ as the ample cone for some K3 surface.*

4. The Apollonian property and Maxwell's paper

In this section, we identify the relevant property of the Coxeter graph (1.1) that implies the existence of a maximal cluster of mutually tangent spheres and use this to identify the Apollonian sphere packings in Maxwell's paper [16].

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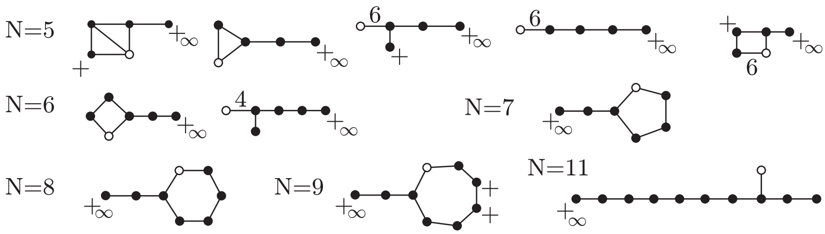
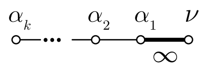


Figure 2. *The eleven other graphs in [7] that satisfy the conditions of Theorem 4.1 for* $k = N - 1$.

Theorem 4.1. *Let* $\mathcal{P} = \Gamma(S)$ *be a crystallographic sphere packing. Suppose there exists an element* ν *of* $\mathcal{P}$ *and a reflective subgroup of* Γ *that generates a Coxeter graph of the form* T_k *with* ν *attached to one of the ends by a bold edge. Then* $\mathcal{P}$ *includes a cluster of* $k+1$ *mutually tangent spheres.*

The notation T_k represents a tree with one branch of length k, which is consistent with our notation used earlier. When used to represent a Coxeter graph or lattice, it is sometimes denoted A_k, which is Vinberg's notation (see [24, Table I]).

Proof. Let us label the graph T_k and ν as follows:



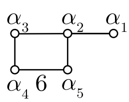
and set $\alpha_i \cdot \alpha_i = \nu \cdot \nu = -2$. Let $s_1 = \nu$ and $s_{i+1} = R_{\alpha_i}(s_i)$ for $i = 1, ..., k$. Then $s_i \cdot s_i = -2$ for all i. To show the set $\{s_1, ..., s_{k+1}\}$ is a cluster of $k+1$ mutually tangent spheres, we need to show $s_i \cdot s_j = 2$ for $i \neq j$. We prove this using induction on the following statements: (a) $s_i = s_{i-1} + 2\alpha_{i-1}$; (b) $\alpha_i \cdot s_i = 2$; (c) $\alpha_j \cdot s_i = 0$ for $j > i$; and (d) $s_i \cdot s_j = 2$ for $i < j$. We leave the details to the reader. □

While this result may not be surprising, and may even be obvious, it nevertheless seems to have escaped any serious notice. With this result in mind, we look at Maxwell's Table II [16] and Chen and Labbé's Appendix [7] in search of candidates for Apollonian packings. We find twelve candidates: The one in the introduction and the eleven listed in Figure 2. The hollow node indicates that removing it yields the desired subgraph. The $+$ is Maxwell's notation to indicate that the weight at that point is real, so represents a plane. We have modified his notation a bit, using $+_\infty$ to indicate that the weight is a plane that is parallel to its associated node, while $+$ indicates that the plane is ultraparallel to its associated node.

Maxwell notes that the first four graphs for $N = 5$ yield the same sphere packing [16, Table I], which is the Soddy sphere packing [21]. The two graphs in $N = 6$ also give the same packing. The packings for $N = 6$, 7, and 8 are the subject of [4], and their Coxeter graphs are given in [5]. The graphs for $N = 9$ and $N = 11$ are the subjects of the next two sections.

The last graph for $N = 5$ is a pleasant surprise, as it shows that there are different ways of filling in the voids of an initial configuration of five mutually tangent spheres in $\mathbb{R}^3$, yet still get a sphere packing where every sphere is a member of a cluster of five mutually tangent spheres. The packing is Apollonian, but not lattice like. Unlike the Soddy packing, there is no perspective where all the spheres have integer curvature.

Let us label the Coxeter graph as follows:



(4.1)

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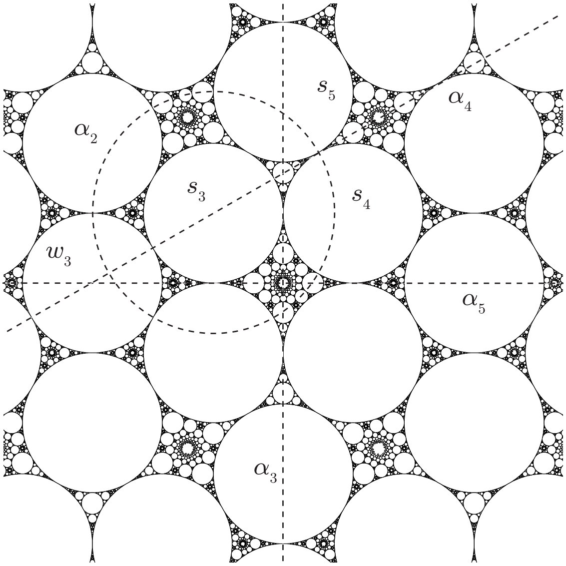

Figure 3. *The horizontal cross section of the strip version of the non-Soddy Apollonian sphere packing with Coxeter graph (4.1). The dotted lines represent the symmetries. Note that the dotted circle represents inversion in a sphere that intersects this plane at an angle of* $\pi/3$*. This picture and the one in Figure 4 were generated using McMullen's Kleinian groups program [17].*

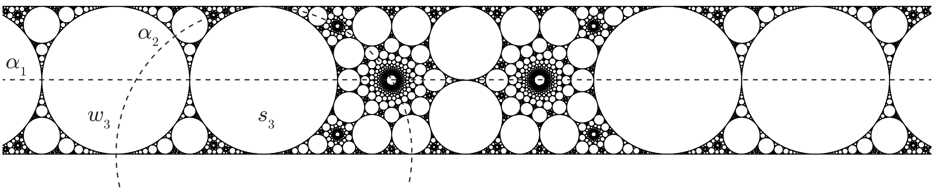

Figure 4. *The cross section on the plane* H_{α_4} *of the strip version of the non-Soddy Apollonian sphere packing. As a circle packing, this cross section is weakly Apollonian.*

so

$$J_\alpha = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & \sqrt{3} \\ 0 & 1 & 0 & \sqrt{3} & -2 \end{bmatrix}.$$

We let $s_1 = w_1, s_2 = R_{\alpha_1}(s_1), \ldots, s_5 = R_{\alpha_4}(s_4)$. We note $w_2 = s_1 + s_2$ and let it be the point at infinity, giving us a strip version of the packing. The cross section on the plane α_1 is shown in Figure 3, and the cross section on the plane α_4 is shown in Figure 4.

Remark 5. *If we invert the strip packing in the sphere* w_3 *and scale to get a sphere of curvature* -1*, then the planes* s_1 *and* s_2 *become spheres with curvature 2 tangent at the center of the sphere* w_3*. The six spheres surrounding* w_3 *become a hexlet of spheres with curvature 3, just like those in the sphere packing described by Soddy [21]. How the space between these spheres is filled in, though, is different from how Soddy does it. The spheres* s_3 *and* s_4*, in this perspective, have curvature* $7 + 4\sqrt{3}$.

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5. A cross section in $\mathbb{R}^7$

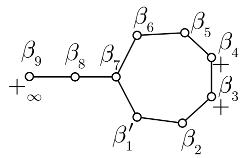
Given a sphere packing in dimension n, a codimension one cross section is a sphere packing in dimension $n-1$. If the sphere packing contains a cluster of $n+2$ mutually tangent spheres (i.e. has the Apollonian property), then by choosing a cross section perpendicular to $n+1$ of these spheres, we get a sphere packing in one lower dimension that has (at least) the weak Apollonian property.

Theorem 5.1. *A cross section perpendicular to nine spheres in a cluster of ten mutually tangent spheres in $\mathcal{P}_\beta$ yields the sphere packing in $\mathbb{R}^7$ with Coxeter graph labeled $N=9$ in Figure 2. This packing is of general lattice type.*

Proof. Let H be the plane perpendicular to s_i for $i=0,\ldots,8$. Then H has normal vector

$$h=[-1,3,2,1,0,0,0,0,0,0].$$

Note that $h\cdot\beta_i=0$ for $i=2,\ldots,9$. Knowing the expected Coxeter graph, we solve for β_1' so that we get:



Thus we want $\beta_1'\cdot\beta_i=0$ for $i\neq 0$, 2, or 7; $\beta_1'\cdot h=0$; $\beta_1'\cdot\beta_2=\beta_1'\cdot\beta_7=1$; and $\beta_1'\cdot\beta_1'=-2$. We get $\beta_1'=[2,1,1,2,3,2,1,0,0,0]$. Using descent, we show $R_{\beta_1'}\in\Gamma$. The set $\{\beta_1',\beta_2,\ldots,\beta_9\}$ forms a basis of the subspace H. Let J_7 be the incidence matrix for this basis of H, and let w_i' be the weights. Since $s_0\cdot h=0$, we get $s_0=2w_9'$ (so $w_9=w_9'$). Let $\Gamma_7=\langle R_{\beta_1'},R_{\beta_2},\ldots,R_{\beta_9}\rangle$. Then the spheres in $\Gamma_7(s_0)$ all intersect H perpendicularly. Most spheres in $\mathcal{P}_\beta$ miss H and some are tangent to H, but there are some that intersect H at an angle that is not right. In particular, let $s_{10}=R_{\beta_0}(s_6)$ and $s_{11}=R_{\beta_0}(s_9)$. Then, $s_{10}\cdot h=4$ and $s_{11}\cdot h=-4$, while $h\cdot h=-14$, so these two spheres intersect H but not perpendicularly nor tangentially. The difference in signs (the ± 4) indicates that the centers of the spheres are on opposite sides of H. The intersection of s_{10} with H is found by projecting the vector s_{10} onto H to get

$$n=s_{10}-\frac{s_{10}\cdot h}{h\cdot h}h.$$

We note that $n\cdot h=0$, $n\cdot\beta_1'=0$, and $n\cdot\beta_i=0$ for $i\neq 0$, 1, or 4, so n is a scalar multiple of w_3'. Similarly, the projection of s_{11} onto H is a scalar multiple of w_4'. Thus, the intersection of $\mathcal{P}_\beta$ with H contains the packing

$$\mathcal{P}_7=\Gamma_7(\{s_0,w_3',w_4'\}).$$

To show that they are equal, we show that $\mathcal{P}_7$ is maximal, so there is no room for the intersection to include anything more than $\mathcal{P}_7$. We use Maxwell's technique again (see the proof of Theorem 3.1). That is, we show that the nonreal weights w_i' are in the convex closure of the Γ_7-orbit of the real weights w_9', w_3, and w_4'. We note that

$$\begin{aligned}
w_8'&=w_8=(s_0+s_1)/2\\
w_7'&=w_7=(s_0+s_1+s_2)/2\\
w_1'&=(s_0+s_1+s_2+s_3+s_4+s_5+s_6+s_7+s_8)/14.
\end{aligned}$$

By symmetry, we get w_6' by going the other way around the loop in the Coxeter graph. That is,

$$w_6'=(s_0+s_1+s_2+s_3+s_4'+s_5'+s_6'+s_7'+s_8')/14,$$

where $s_4'=R_{\beta_1'}(s_3)$, $s_5'=R_{\beta_2}(s_4')$, $\ldots$, $s_8'=R_{\beta_5}(s_7')$. Finally, to get w_2', we first note that

$$w_2'=(w_9-R_{w_2'}(w_9'))/14.$$

The map $R_{\mathbf{n}}$ is a reflection for $\mathbf{n}\cdot\mathbf{n}<0$. The map $-R_{\mathbf{n}}$ for $\mathbf{n}\cdot\mathbf{n}>0$ is a *Cartan* involution, which is the -1 map on $\mathcal{H}$ through the point $\mathbf{n}\in\mathcal{H}$. There is no reason to expect $-R_{w_2'}(w_9)$ to be in $\Gamma_7(w_9)$ (similar quantities are not), but we prove it is by using our method of descent. By symmetry, w_5' is also in the convex closure of $\Gamma_7(w_9)$.

The spheres in $\Gamma_7(s_0)$ in the perspective with w_8 the point at infinity (or any other lattice point for the point at infinity) all have curvature an integer multiple of $\sqrt{2}$, while those in $\Gamma_7(w_3')$ and $\Gamma_7(w_4')$ all have curvature an integer multiple of $\sqrt{42}/3$. Thus, the packing is not of lattice type, though it is of general lattice type, since all spheres have integer coordinates in the basis $\{\beta_1', \beta_2, ..., \beta_9\}$. □

Remark 6. *Every sphere in $\Gamma_7(s_0)$ is a member of a cluster of 9 mutually tangent spheres. The same cannot be said of the spheres in the orbits of w_3' and w_4'.*

Remark 7. *A similar cross section of the Apollonian packings in* [4] *and* [5] *intersect all spheres perpendicularly, so give the related Apollonian packing in one dimension lower. A similar cross section of the example in Section 4 gives the packing in Figure 4, which is a weakly Apollonian packing.*

Remark 8. *The spheres s_{10} and s_{11} are tangent at the point $P_1=s_{10}+s_{11}$, which lies in the subspace H and is on $\partial\mathcal{H}$. Let $P_2=R_{\beta_3}(P_1)$.*

For two points A and B on $\partial\mathcal{H}$, define the *Bertini involution* to be

$$\phi_{A,B}(\mathbf{x})=2\frac{(A\cdot\mathbf{x})B+(B\cdot\mathbf{x})A}{A\cdot B}-\mathbf{x}.$$

This is the map that is -1 on $\partial\mathcal{H}_A$ through the point B.

The Bertini involutions ϕ_{P_1,w_8} and ϕ_{P_1,P_2} both preserve the lattice $\beta_1'\mathbb{Z}\oplus\beta_2\mathbb{Z}\oplus\cdots\oplus\beta_9\mathbb{Z}$. Note that $\phi_{P_1,P_2}(w_4')=-w_4'$, so ϕ_{P_1,P_2} sends everything on one side of the plane $H_{w_4'}$ to the other side. The map ϕ_{P_1,w_8} sends w_3' to w_4'. Let

$$\Gamma_7'=\langle\Gamma_7,\phi_{P_1,P_2},\phi_{P_1,w_8}\rangle.$$

Since $\Gamma_7(s_0)$ does not intersect $H_{w_3'}$ or $H_{w_4'}$, $\Gamma_7'(s_0)$ is a sphere packing. Because $\mathcal{P}_7$ is maximal, $\Gamma_7'(s_0)$ is also maximal. It is lattice like and Apollonian. Since $\Gamma_7\le\Gamma_7'$, the limit set of Γ_7 is a subset of the limit set of Γ_7'. Since these are the residual sets of the respective packings, the packing $\mathcal{P}_7$, which is only weakly Apollonian, is more efficient than the Apollonian packing $\Gamma_7'(s_0)$.

The packing $\Gamma_7'(s_0)$ is the packing described in [5]. The cone $\mathcal{K}_{\Gamma_7'(s_0)}$ is the ample cone for a class of K3 surfaces. The packing $\mathcal{P}_7$, though, cannot be the ample cone of any K3 surface.

6. A sphere packing in $\mathbb{R}^9$

The sphere packing generated by the Coxeter graph with $N=11$ in Figure 2 can be described as follows: Let H be an 8-dimensional subspace of $\mathbb{R}^9$ (with normal vector h) and let us place spheres of radius $1/\sqrt{2}$ at each vertex of a copy of the E_8 lattice imbedded in H. Let us place two hyperplanes parallel to H a distance of $1/\sqrt{2}$ on either side, so that they are tangent to all the spheres. Let us distinguish the sphere s_0 centered at the origin of the E_8 lattice and let its points of tangencies with the hyperplanes be A and B. Let σ be inversion in the sphere centered at A and through B. The packing is the image of these spheres under the group generated by σ, the symmetries of the E_8 lattice, and reflection R_h in H.

To see this, recall that the spheres centered on the vertices of the E_8 lattice are generated by the image of s_0 under the action of the Weyl group W_{236}, which is $\langle R_{\beta_0}, R_{\beta_2}, ..., R_{\beta_9}\rangle$ for the T_{236} subgraph of Graph (1.1). In this context, these reflections act on $\mathbb{R}^9$. The hyperplane H_{β_i} (the plane in $\mathbb{R}^9$ perpendicular to $\beta_i\in H\subset\mathbb{R}^9$) is perpendicular to s_0 for $i\neq 9$, and hence also perpendicular to σ. (We ignore $i=1$ in this discussion.) Note that H_{β_9} is tangent to s_0. Let $s_1=R_{\beta_9}(s_0)$, giving us the cross section shown in Figure 5. We note that σ and β_9 are at an angle of $2\pi/3$. We note that h is perpendicular to β_i for all i, and h and

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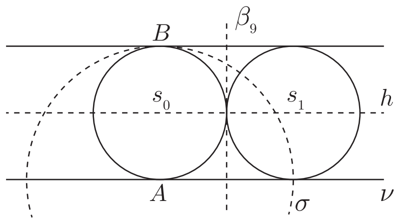
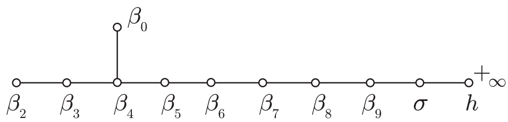


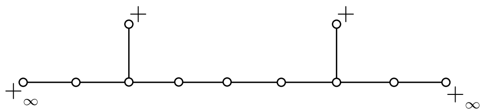
Figure 5. *A cross section perpendicular to H, s_0, and s_1.*

σ intersect at an angle of $2\pi/3$. Finally, let ν represent the plane tangent to s_0 at A. Then ν is parallel to h, perpendicular to σ, and perpendicular to β_i for all i. This gives us the Coxeter graph



as desired, where ν is the weight at the node h. Maxwell verifies that this packing is maximal (see the discussion after Theorem 3.3 in [16]).

Remark 9. *Maxwell also presents the packing with Coxeter graph*



which generates the same packing. Maxwell identifies an invariant of packings of lattice type and notes that these two packings have the same invariant, but presumably could not show equivalence.

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