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A CHARACTERISATION OF SOLUBLE *PST*-GROUPS

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Abstract

Let G be a finite group. A subgroup A of G is said to be *S-permutable* in G if A permutes with every Sylow subgroup P of G, that is, $AP = PA$. Let A_{sG} be the subgroup of A generated by all S-permutable subgroups of G contained in A and A^{sG} be the intersection of all S-permutable subgroups of G containing A. We prove that if G is a soluble group, then S-permutability is a transitive relation in G if and only if the nilpotent residual $G^{\mathfrak{N}}$ of G avoids the pair (A^{sG}, A_{sG}), that is, $G^{\mathfrak{N}} \cap A^{sG} = G^{\mathfrak{N}} \cap A_{sG}$ for every subnormal subgroup A of G.

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1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group.

Let $K \leq H$ and A be subgroups of G. Then we say that A *avoids* the pair (H, K) if $A \cap H = A \cap K$.

A subgroup H of G is said to be *Sylow permutable* or *S-permutable* [2, 3] in G if H permutes with every Sylow subgroup P of G, that is, $HP = PH$.

The S-permutable subgroups possess a series of interesting properties and they are closely related to subnormal subgroups. For instance, if H is an S-permutable subgroup of G, then H is subnormal in G (Kegel [10]), the normaliser $N_G(H)$ of H is also S-permutable in G (Schmid [12]) and the quotient H/H_G is nilpotent (Deskins [6]).

Note also that the S-permutable subgroups of G form a sublattice of the lattice of all subnormal subgroups of G (Kegel [10]) and this important result allows us to associate with each subgroup A of G two S-permutable subgroups of G: the *S-core* A_{sG} *of* A in G [13], that is, the subgroup of A generated by all S-permutable subgroups of G

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contained in A and the *S-permutable closure A^{sG} of A* in G [8], that is, the intersection of all S-permutable subgroups of G containing A.

The subgroups A_{sG} and A^{sG} have found numerous applications in the study of the structure of nonsimple groups (see, in particular, [8, 11, 13, 14]), and in this paper, we consider the use of such subgroups in the theory of PST-groups.

Recall that G is a *PST-group* [2, 3] if S-permutability is a transitive relation in G, that is, if K is an S-permutable subgroup of H and H is an S-permutable subgroup of G, then K is S-permutable in G. The description of soluble PST-groups was first obtained by Agrawal [1].

THEOREM 1.1 (Agrawal [1]). *Let $D = G^{\mathfrak{N}}$ be the nilpotent residual of a soluble group G, that is, the intersection of all normal subgroups N of G with nilpotent G/N. Then G is a PST-group if and only if D is an abelian Hall subgroup of G of odd order and every element of G induces a power automorphism in D.*

There are many other interesting characterisations of soluble PST-groups (see, for example, [3, Ch. 2]). In particular, a soluble group G is a PST-group if and only if every chief factor of G between A^G and A_G is central in G for every subgroup A of G such that A^G/A_G is nilpotent [5], and a soluble group G is a PST-group if and only if for every maximal subgroup V of every Sylow subgroup of G, there is a PST-subgroup T of G such that $G = VT$ [7].

In this paper, we prove the following result.

THEOREM 1.2. *Let $D = G^{\mathfrak{N}}$ be the nilpotent residual of a soluble group G. Then G is a PST-group if and only if D avoids the pair (A^{sG}, A_{sG}) for every subnormal subgroup A of G.*

2. Preliminaries

LEMMA 2.1. *If D avoids the pair (A^{sG}, A_{sG}) and for a minimal normal subgroup R of G we have either $R \leq D$ or $R \leq A$, then DR/R avoids the pair $((AR/R)^{s(G/R)}, (AR/R)_{s(G/R)})$.*

PROOF. First assume that $R \leq D$. Then

$$\begin{aligned}(DR/R) \cap (AR/R)^{s(G/R)} &= (D/R) \cap (A^{sG}R/R) = (D \cap A^{sG}R)/R \\ &= R(D \cap A^{sG})/R \leq R(D \cap A_{sG})/R.\end{aligned}$$

However,

$$R(D \cap A_{sG})/R \leq (D \cap (AR)_{sG})/R = (D/R) \cap (AR)_{sG}/R = (DR/R) \cap (AR/R)_{s(G/R)}.$$

Therefore, $(DR/R) \cap (AR/R)^{s(G/R)} \leq (DR/R) \cap (AR/R)_{s(G/R)}$ and hence

$$(DR/R) \cap (AR/R)^{s(G/R)} = (DR/R) \cap (AR/R)_{s(G/R)},$$

so DR/R avoids the pair $((AR/R)^{s(G/R)}, (AR/R)_{s(G/R)})$.

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Now assume that $R \le A$. Then

$$\begin{aligned}(DR/R) \cap (AR/R)^{s(G/R)} &= (DR/R) \cap (A^{sG}/R) = (DR \cap A^{sG})/R = R(D \cap A^{sG})/R \\ &\le R(D \cap A_{sG})/R \\ &\le (DR/R) \cap (A_{sG}/R) = (DR/R) \cap (A/R)_{s(G/R)}.\end{aligned}$$

Hence, DR/R avoids $((AR/R)^{s(G/R)}, (AR/R)_{s(G/R)})$. □

The following lemma is a corollary of [8, Lemmas 2.4 and 2.5].

LEMMA 2.2. *If $A \le E \le G$, then $A_{sG} \le A_{sE} \le A \le A^{sE} \le A^{sG}$.*

The following useful fact is obtained from [4, Proposition 2.2.8].

LEMMA 2.3. *Let N and E be subgroups of G, where N is normal in G. Then:*

(1) $(G/N)^{\mathfrak{N}} = G^{\mathfrak{N}}N/N$;
(2) $E^{\mathfrak{N}} \le G^{\mathfrak{N}}$; *and*
(3) *if $G = NE$, then $E^{\mathfrak{N}}N = G^{\mathfrak{N}}N$.*

LEMMA 2.4. *If the nilpotent residual $D = G^{\mathfrak{N}}$ of G avoids the pair (A^{sG}, A_{sG}) and $A \le E \le G$, then $E^{\mathfrak{N}}$ avoids the pair (A^{sE}, A_{sE}).*

PROOF. We have $A_{sG} \le A_{sE} \le A \le A^{sE} \le A^{G}$ by Lemma 2.2, and so from $A^{sG} \cap D = A_{sG} \cap D$ and Lemma 2.3(2), it follows that $E^{\mathfrak{N}} \cap A^{sG} \le E^{\mathfrak{N}} \cap A_{sG}$, where $E^{\mathfrak{N}} \cap A^{sE} \le E^{\mathfrak{N}} \cap A^{sG}$ and $E^{\mathfrak{N}} \cap A_{sG} \le E^{\mathfrak{N}_\sigma} \cap A_{sE}$.

Consequently, $E^{\mathfrak{N}} \cap A^{sE} \le E^{\mathfrak{N}} \cap A_{sE} \le E^{\mathfrak{N}} \cap A^{sE}$ and $E^{\mathfrak{N}} \cap A^{sE} = E^{\mathfrak{N}} \cap A_{sE}$. Hence, $E^{\mathfrak{N}}$ avoids the pair (A^{sEG}, A_{sE}). The lemma is proved. □

A group G is called *π-closed* if G has a normal Hall π-subgroup.

LEMMA 2.5. *Let $K \le H$ be normal subgroups of G, where H/K is π-closed. If either $K \le \Phi(G)$ or $K \le Z_\infty(H)$, then H is π-closed.*

PROOF. Let V/K be the normal Hall π-subgroup of H/K. Let D be a Hall π'-subgroup of K. Then D is a normal Hall π'-subgroup of V since K is nilpotent, so V has a Hall π-subgroup, E say, by the Schur–Zassenhaus theorem. It is clear that V is π'-soluble, so any two Hall π-subgroups of V are conjugated in V by the Hall–Chunikhin theorem on π-soluble groups.

Assume that $K \le \Phi(G)$. By a generalised Frattini argument, $G = VN_G(E) = DEN_G(E) = DN_G(E) = N_G(E)$ since $D \le K \le \Phi(G)$. Thus, E is normal in H, that is, H is π-closed since E is a Hall π-subgroup of H.

Finally, assume that $K \le Z_\infty(H)$ and then $D \le Z_\infty(V)$, so $V = D \rtimes E = D \times E$. Hence, E is characteristic in V and so normal in H. Thus, H is π-closed. The lemma is proved. □

LEMMA 2.6. *Let $D = G^{\mathfrak{N}}$ be the nilpotent residual of G and p a prime such that $(p-1, |G|) = 1$. If D is nilpotent and every subgroup of D is normal in G, then $(p, |D|) = 1$. Hence, the smallest prime in $\pi(G)$ belongs to $\pi(|G : D|)$. In particular, $|D|$ is odd and so D is abelian.*

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PROOF. Assume that p divides $|D|$. Then D has a maximal subgroup M such that $|D : M| = p$ and M is normal in G. It follows that $C_G(D/M) = G$, that is, $D/M \leq Z(G/M)$ since $(p-1, |G|) = 1$. However, G/D is nilpotent. Therefore, G/M is nilpotent by Lemma 2.5 and hence $D \leq M < D$, which is a contradiction. Therefore, the smallest prime in $\pi(G)$ belongs to $\pi(|G : D|)$. In particular, $|D|$ is odd and so D is abelian since D is a Dedekind group by hypothesis. The lemma is proved. □

DEFINITION 2.7. A subgroup D of G is a *special subgroup* of G if D is a normal Hall subgroup of G and every element of G induces a power automorphism in D.

LEMMA 2.8. *If D is a special subgroup of G and $N \trianglelefteq G$, then DN/N is a special subgroup of G/N.*

PROOF. It is clear that DN/N is a normal Hall subgroup of G/N and if $A/N \leq DN/N$, then $A = N(A \cap D)$, where $A \cap D$ is normal in G, so A/N is normal in G/N, that is, every element of G/N induces a power automorphism in DN/N. The lemma is proved. □

LEMMA 2.9 [3, Theorem 1.2.17]. *If A is a nilpotent S-permutable subgroup of G and V is a Sylow subgroup of A, then V is S-permutable in G.*

LEMMA 2.10. *If the nilpotent residual $D = G^{\mathfrak{N}}$ of G is a special subgroup of G and A is an S-permutable subgroup of G, then D avoids the pair (A^{sG}, A_{sG}).*

PROOF. Since $A_G \leq A_{sG} \leq A \leq A^{sG} \leq A^G$ by Lemma 2.2, it is enough to show that D avoids the pair (A^G, A_G). Assume this is false and let G be a counterexample of minimal order.

First we prove that $A \cap D = 1$. Indeed, assume that $N := A \cap D \neq 1$. Then $N \leq A_G$ and $D/N = (G/N)^{\mathfrak{N}}$ is a special subgroup of G/N by Lemma 2.8, and A/N is an S-permutable subgroup of G/N by [3, Lemma 1.2.7]. Therefore, $(G/N)^{\mathfrak{N}}$ avoids the pair $((A/N)^{G/N}, (A/N)_{G/N})$ by the choice of G, that is,

$$(G/N)^{\mathfrak{N}} \cap (A/N)^{G/N} = (G/N)^{\mathfrak{N}} \cap (A/N)_{G/N}.$$

However, $(A/N)^{G/N} = A^G/N$ and $(A/N)_{G/N} = A_G/N$, so

$$(G/N)^{\mathfrak{N}} \cap (A/N)^{G/N} = (D/N) \cap (A^G/N) = (D \cap A^G)/N$$

and

$$(G/N)^{\mathfrak{N}} \cap (A/N)_{G/N} = (D/N) \cap (A_G/N) = (D \cap A_G)/N.$$

Consequently, $D \cap A^G = D \cap A_G$. Hence, D avoids the pair (A^G, A_G), which is a contradiction.

Therefore, $A \cap D = 1$, so $AD/D \simeq A = P_1 \times \cdots \times P_t$, where P_i is the Sylow p_i-subgroup of A for all i. Then P_i is S-permutable in G by Lemma 2.9 and so $D \leq N_G(P_i)$ for all i by [3, Lemma 1.2.16]. Therefore, $D \leq N_G(A)$.

Let $\pi = \pi(D)$. Then G is π-soluble since every subgroup of D is normal in G by hypothesis. Moreover, D has a complement M in G since D is a Hall π-subgroup of G

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and for some $x \in G$, we have $A \leq M^x$ by the Chunikhin–Hall theorem [9, VI, Hauptsatz 1.7]. Finally, $D \leq N_G(A)$ and hence $A^G = A^{DM^x} = A^{M^x} \leq M_G \leq M$, so $A^G \cap D = 1$. Therefore, D avoids $(A^G, A_G) = (A^G, 1)$, contrary to the choice of G. The lemma is proved. □

3. Proof of Theorem 1.2

First suppose that D avoids the pair (A^{sG}, A_{sG}) for every subnormal subgroup A of G. We show that, in this case, G is a PST-group. Assume this is false and let G be a counterexample of minimal order. Then $D \neq 1$ since G/D is nilpotent and so G/D is a PST-group.

Claim 1. If R is a minimal normal subgroup of G, then G/R is a PST-group.

In view of the choice of G, it is enough to show that the hypothesis holds for G/R. First note that $DR/R = (G/R)^{\mathfrak{N}}$ by Lemma 2.3 and if A/R is a subnormal subgroup of G/R, then A is subnormal in G, so D avoids the pair (A^{sG}, A_{sG}) by hypothesis. Therefore, DR/R avoids the pair $((A/R)^{s(G/R)}, (A/R)_{s(G/R)})$ by Lemma 2.1. This proves Claim 1.

Claim 2. If E is a proper subnormal subgroup of G, then E is a PST-group.

Every subnormal subgroup A of E is subnormal in G, so D avoids the pair (A^{sG}, A_{sG}) by hypothesis. However, then $E^{\mathfrak{N}}$ avoids the pair (A^{sE}, A_{sE}) by Lemma 2.4. Hence, the hypothesis holds for E, so Claim 2 holds by the choice of G.

Claim 3. D is nilpotent and every subgroup of D is S-permutable in G. Hence, every chief factor of G below D is cyclic.

First we show that if $L \leq D$, where L is a minimal normal subgroup of G, then L is cyclic. Since G is soluble, $L \leq G_p$ for some Sylow subgroup G_p of G and then some maximal subgroup V of L is normal in G_p and V is subnormal in G. Assume that V is not S-permutable in G. Then $V \neq 1$ and $V^{sG} = L$, so $V^{sG} \cap D = L = V_{sG} \cap D < V < L$, which is a contradiction. Hence, V is S-permutable in G, so $G = G_p O^p(G) \leq N_G(V)$ by [3, Lemma 1.2.16]. Therefore, $V = 1$, so $|L| = p$.

Now we show that D is nilpotent. Assume that this is false and let R be a minimal normal subgroup of G. Then G/R is a PST-group by Claim 1.

Note also that $(G/R)^{\mathfrak{N}} = RD/R \simeq D/(D \cap R)$ by Lemma 2.3, where $(G/R)^{\mathfrak{N}}$ is abelian by Theorem 1.1, so $R \leq D$ and if N is a minimal normal subgroup of G, then $N = R$ since otherwise $D \simeq D/1 = D/(N \cap R)$ is abelian. Moreover, $|R| = p$ for some prime p and $R \nleq \Phi(G)$ by Lemma 2.5, so for some maximal subgroup M of G, we have $G = R \rtimes M$ and $C_G(R) \cap M$ is a normal subgroup of G, so $C_G(R) \cap M = 1$. Therefore, $C_G(R) = R(C_G(R) \cap M) = R$ and then $G/R = G/C_G(R)$ is cyclic. Hence, $R = D$ is nilpotent. This contradiction shows that D is nilpotent. So, for every subgroup A of D,

$$A^{sG} = D \cap A^{sG} = D \cap A_{sG} = A_{sG}.$$

Therefore, every subgroup of D is S-permutable in G.

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By Theorem 1.1 and Claim 1, every chief factor of G between R and D is cyclic, so every chief factor of G below D is cyclic by the Jordan–Hölder theorem for the chief series. Hence, Claim 3 holds.

Claim 4. D is a Hall subgroup of G.

Suppose that this is false and let P be a Sylow p-subgroup of D such that $1 < P < G_p$, where $G_p \in \mathrm{Syl}_p(G)$.

(a) $D = P$ is a minimal normal subgroup of G and $|D| = p$. Hence, $D \leq Z(G_p)$ and G_p is normal in G.

Let R be a minimal normal subgroup of G contained in D. Then R is a q-group for some prime q and $D/R = (G/R)^{\mathfrak{N}}$ is a Hall subgroup of G/R by Claim 1 and Theorem 1.1.

Suppose that $PR/R \neq 1$. Then $PR/R \in \mathrm{Syl}_p(G/R)$. If $q \neq p$, then $P \in \mathrm{Syl}_p(G)$. This contradicts the fact that $P < G_p$. Hence, $q = p$, so $R \leq P$ and therefore, $P/R \in \mathrm{Syl}_p(G/R)$ and again $P \in \mathrm{Syl}_p(G)$. This contradiction shows that $PR/R = 1$, which implies that $R = P$ is the unique minimal normal subgroup of G contained in D. Since D is nilpotent, a p'-complement E of D is characteristic in D and so it is normal in G. Hence, $E = 1$, which implies that $R = D = P$. Claim 3 implies that $|D| = p$, so $D \leq Z(G_p)$. Finally, since G/D is nilpotent and $D \leq G_p$, G_p is normal in G.

(b) $D \nleq \Phi(G)$. Hence, $G = D \rtimes M$ for some maximal subgroup M of G and $C_G(D) = D \times (C_G(D) \cap M)$.

This follows from part (a) since G is not nilpotent.

(c) If G has a minimal normal subgroup $L \neq D$, then $G_p = D \times L$. Hence, $O_{p'}(G) = 1$.

Indeed, $DL/L \simeq D$ is a Hall subgroup of G/L by Theorem 1.1 and Claim 1. Hence, $G_pL/L = DL/L$, so $G_p = D \times (L \cap G_p) = D \times L$ since G_p is normal in G by part (a). Thus, $O_{p'}(G) = 1$.

(d) $G_p \cap M \neq 1$ is normal in G.

Observe that $V := G_p \cap M$ is normal in M by part (a). Also from $G_p = G_p \cap D \rtimes M = D(G_p \cap M)$, where $D \leq Z(G_p)$ by part (a), it follows that V is normal G_p. Therefore, V is normal in G and $V \neq 1$ since $D < G_p$.

(e) If $L \leq G_p \cap M$, where L is a minimal normal subgroup of G, then $L = G_p \cap M$ and so $G_p = D \times L$ is abelian.

This follows from parts (c) and (d).

(f) Every normal subgroup Z of G contained in G_p with $1 \neq Z \neq G_p$ is G-isomorphic to either L or D. In particular, Z is a minimal normal subgroup of G and either $Z \in \{D, L\}$ or $D \simeq_G Z \simeq_G L$, and so $C_G(D) = C_G(Z) = C_G(L)$.

Assume that $D \neq Z \neq L$. If $Z \cap L \neq 1$, then $L \leq Z$ and so $Z = L(Z \cap D) = L$ since $1 \neq Z \neq G_p = LD$, which is a contradiction. Hence, $Z \cap L = 1$ and $Z \cap D = 1$. Therefore, $G_p = D \times Z = D \times L$ and so the G-isomorphisms $L \simeq LD/D = G_p/D = DZ/D \simeq Z$ and $D \simeq DL/L = G_p/D = LZ/L \simeq Z$ yield $D \simeq_G Z \simeq_G L$. In particular, Z is a minimal normal subgroup of G and $C_G(D) = C_G(Z) = C_G(L)$.

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(g) If $N = \langle ab \rangle$, where $D = \langle a \rangle$ and b is an element of order p in L, then $|N| = p$ and $N \cap D = N \cap L = 1$.

Since $G_p = D \times L$ is abelian by part (e) and $|D| = p$ by part (a), $|ab| = |N| = p$. Hence, $N \cap D = N \cap L = 1$ since $a \notin L$ and $b \notin D$.

(h) N is a minimal normal subgroup of G.

First we show that N is normal in G. In view of [3, Lemma 1.2.16] and part (e), it is enough to show that $N = N^{sG}$ is S-permutable in G. Assume that $N < N^{sG}$. Then $|N^{sG}| > p$. Since $G_p = DL$ by part (f) and $|D| = p$ by part (a),

$$|G_p : L| = p = |N^{sG}L/L| = |N^{sG}/(N^{sG} \cap L)|,$$

so $N^{sG} \cap L \neq 1$. However, $N^{sG} \cap L$ is S-permutable in G by [3, Theorem 1.2.19] and so $N^{sG} \cap L$ is normal in G by [3, Lemma 1.2.16] and part (e). Hence, $L \leq N^{sG}$ by the minimality of L. Then $N^{sG} = N^{sG} \cap G_p = L(N^{sG} \cap D)$. However, N is subnormal in G and so $N^{sG} \cap D = N_{sG} \cap D = 1$. Hence, $N^{sG} = L$ and then $N \cap L \neq 1$, in contrast to part (g). Hence, $N = N^{sG}$ and so N is normal in G. Therefore, N is a minimal normal subgroup of G since $|N| = p$. This proves part (h).

(i) The final contradiction to prove Claim 4.

In view of parts (f), (g) and (h), $C_G(D) = C_G(N) = C_G(L)$. However, $C_G(L) = G$ by part (e) since $G/D \simeq M$ is nilpotent and $L \leq M$. Therefore, $D \leq Z(G)$ and so G is nilpotent. This contradiction proves Claim 4.

Claim 5. Every subgroup A of D is normal in G. Hence, every element of G induces a power automorphism in D.

Since D is nilpotent by Claim 3, it is enough to consider the case when A is a p-group for some prime p. Moreover, A is S-permutable in G by Claim 3 and the Sylow p-subgroup D_p of D is a Sylow p-subgroup of G by Claim 4. Therefore, $G = D_pO^p(G) = DO^p(G) \leq N_G(A)$ by [3, Lemma 1.2.16]. This proves Claim 5.

Claim 6. D is an abelian group of odd order.

This follows from Lemma 2.6 and Claim 5.

Claim 7. The final contradiction.

From Claims 3–6, it follows that G is a PST-group by Theorem 1.1, in contrast to the choice of G. Hence, there is no minimal counterexample and G is a PST-group.

Finally, given that G is a PST-group, we show that D avoids the pair (A^{sG}, A_{sG}) for every subnormal subgroup A of G. There is a series $A = A_0 \trianglelefteq A_1 \trianglelefteq \cdots \trianglelefteq A_n = G$, so A is S-permutable in G since G is a PST-group. However, $D = G^{\mathfrak{N}}$ is a special subgroup of G by Theorem 1.1 and so D avoids the pair (A^{sG}, A_{sG}) by Lemma 2.10.

The theorem is proved.

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