UNIFORM BOUNDEDNESS FOR GROUPS

IRVING GLICKSBERG

1. Let G and H be locally compact abelian groups with character groups G^*, H^*, and let $< \cdot , \cdot >$ denote the pairing between a group and its dual.

In 1952 Kaplansky proved the following result, using the structure of locally compact abelian groups and category arguments.[1]

THEOREM 1.1. *Let $\tau: G \to H$ be an algebraic homomorphism for which there is a dual $\tau^*: H^* \to G^*$ (so that $< \tau g, h^* > = < g, \tau^* h^* >$ for all g in G, h^* in H^*). Then τ is continuous.*

The result bears a striking similarity to a well-known fact about Banach spaces which is a consequence of uniform boundedness;[2] the present note is devoted to an analogous "uniform boundedness" for groups, which yields a non-structural proof of Kaplansky's theorem.

THEOREM 1.2. *If K is a subset of G which is (conditionally[3]) compact in the topology of pointwise convergence on G^*, then K is (conditionally) compact in G.*

Regarding "boundedness" in the group situation as meaning "conditional compactness," 1.2 is the exact analogue of uniform boundedness. Indeed, 1.2 states that if K is bounded in the topology of pointwise convergence on the dual (that is, weakly bounded), then K is bounded in the topology of uniform convergence on bounded subsets of the dual. We can carry the analogy a bit further; Theorem 3.1 below gives an analogue of the Banach-Steinhaus theorem which, necessarily, requires pointwise boundedness not only of the original set of maps, but also of their duals.

Another interpretation of 1.2 can be made in terms of the almost periodic compactification G^a of G; 1.2 says the only subsets of G which are compact when "imbedded" in G^a are the compact subsets themselves.

Theorem 1.1 has the rather interesting consequence that the topology of G is determined by the *set* of continuous characters it produces (Corollary 2.4), and also that any measurable homomorphism of G into H is continuous (Corollary 2.3). From this last fact one obtains an analogue of the open mapping theorem (Theorem 4.1).

Received May 31, 1961. Presented to the American Mathematical Society, Nov. 28, 1959. Work supported in part by the United States Air Force Office of Scientific Research.

[1]The special case in which τ is $(1 - 1)$ and onto was proved by Helson in 1953 without the use of structure. Both Kaplansky's and Helson's proofs are unpublished.

[2]We shall use "uniform boundedness" for Banach spaces to refer only to the fact that a weak (or weak*) bounded subset is strongly bounded.

[3]"Conditionally compact" means "having compact closure."

https://doi.org/10.4153/CJM-1962-017-3 Published online by Cambridge University Press

Our results proceed, of course, from compactness rather than category arguments. Theorem 1.2 itself is a simple corollary of a result in which compactness in a given topology implies compactness in a stronger one: the result of Grothendieck (**1**, Théorème 5) that, for bounded subsets of [4] $C(X)$, X compact, compactness in the topology of pointwise convergence implies weak compactness.

The notation used below is standard; for various facts concerning topological groups the reader is referred to (**2**; **4**; **5**). The letters G, H will always denote locally compact abelian groups, X, Y Banach spaces. Where convenient, for $g \in G$, we shall denote the corresponding function $g^* \to < g, g^* >$ on G^* simply by g.

2. Proof of Theorem 1.2. Suppose K is compact in the topology of pointwise convergence on G^*. Clearly K is closed in G, and thus it is sufficient to show K is contained in some compact subset of G.

Suppose not. Then there is a net $\{g_\delta\}$ of elements of K which tend to infinity. If K^* is any compact subset of G^*, $\{g|K^*: g \in K\}$ is a subset of $C(K^*)$ which is compact in the topology of pointwise convergence on K^*, and thus, by the result of Grothendieck (**1**, Théorème 5) cited above, is weakly compact in $C(K^*)$. Consequently there is a g in K for which $g|K^*$ is a weak cluster point of the net $\{g_\delta|K^*\}$.

Thus for each f in $L_1(G^*)$ supported by K^* we have

$$f^\wedge(g) = \int f(g^*) < g, g^* > dg^*$$

a cluster point of the net of numbers $f^\wedge(g_\delta)$. Since these converge to 0 by the Riemann-Lebesgue lemma, $f^\wedge(g) = 0$ for each such f. Taking K^* with interior we can choose such f so as to approximate a point mass, and thus obtain $f^\wedge(g) \neq 0$, our contradiction.

Proof of Theorem 1.1. Let V be a compact neighbourhood of the identity g_0 of G. The fact that $< \tau g, h^* > = < g, \tau^* h^* >$ tells us that $g \to < \tau g, h^* >$ is continuous for each h^* in H^*, and thus that τ is a continuous map from G into H taken in the topology of pointwise convergence on H^*. Consequently τV is compact in the topology of pointwise convergence, hence compact by 1.2.

Now if $\{g_\delta\}$ is a net of points of V converging to the identity g_0 then $< \tau g_\delta, h^* > \to < \tau g_0, h^* >$ for all h^* so that $\{\tau g_\delta\}$ can have at most the cluster point τg_0 in the compact set τV. Consequently $\tau g_\delta \to \tau g_0$ and τ is continuous, completing our proof.

The existence of a dual τ^* of τ is clearly equivalent to the continuity of $g \to < \tau g, h^* >$ for each h^* in H^*. Thus we have

COROLLARY 2.1. *Let $\tau: G \to H$ be an algebraic homomorphism which is continuous when H is taken in the topology of pointwise convergence on H^* (that is, for which $g \to < \tau g, h^* >$ is continuous, all h^*). Then τ is continuous.*

[4] $C(X)$ will denote as usual the space of all bounded continuous complex valued functions on X, $C_0(X)$ those vanishing at ∞.

More generally

COROLLARY 2.2. *Let K be a locally compact Hausdorff space and $f: K \to H$ any function which is continuous when H is taken in the topology of pointwise convergence on H^*. Then f is continuous.*

For if V is a compact neighbourhood of $k \in K$ then $f(V)$ is compact by 1.2. Thus if $\{k_\delta\}$ is a net of points of V converging to k we conclude as in the proof of 1.1 that $\{f(k_\delta)\}$ has at most the cluster point $f(k)$ in $f(V)$, whence $f(k_\delta) \to f(k)$ as before.

The following consequence of 1.2 was suggested by David Lowdenslager.

COROLLARY 2.3. *If $\tau: G \to H$ is an algebraic homomorphism which is measurable in the sense that $\tau^{-1} F$ is locally Borel for each closed subset F of H, then τ is continuous.*

By Corollary 2.1 we need only show the measurable function $\chi: g \to < \tau g, h^* >$ is continuous for each h^* in H^*, and thus the proposition is reduced to the special case in which H is the circle group T^1. This special case seems to be well known, but we include the following proof, probably typical, since doing so has the advantage of showing what difficulties[5] have to be met in the direct extension of its argument to the general situation.

Since $\chi: G \to T^1$ is measurable, as is well known, there is a character g^* in G^* which coincides locally almost everywhere with χ. Thus $\chi_0 : g \to < g, g^* >^{-1}\chi(g)$ is again a homomorphism of G into T^1 which is measurable, and takes the value 1 locally almost everywhere. $G_0 = \{g : \chi_0(g) = 1\}$ is clearly an algebraic subgroup of G, and contains some set of positive measure. But χ_0 is constant on cosets mod G_0, and thus assumes a value $\neq 1$ only if it does so on a set of positive measure, which is impossible.

COROLLARY 2.4. *Let $\mathfrak{g}$ be an abelian group which becomes a locally compact topological group under two topologies $\mathfrak{T}_1$, $\mathfrak{T}_2$, and let $\mathfrak{g}_i{}^*$ be the set of homomorphisms of $\mathfrak{g}$ into the circle group T^1 which are $\mathfrak{T}_i$-continuous, $i = 1, 2$. Then if $\mathfrak{g}_1{}^* = \mathfrak{g}_2{}^*$, the topologies coincide.* (In particular, any non-discrete locally compact abelian group has "discontinuous characters.")

For if G_1 and G_2 are the two locally compact groups produced, the injection map of G_1 onto G_2 has the identity map of characters as its dual, and thus is continuous by 1.1.

As was pointed out in the Introduction, 1.2 says any subset K of G which is compact in G^a is compact in G; thus compactness in any topology finer than that of pointwise convergence on G^* implies compactness in the latter, hence in G.

COROLLARY 2.5. *If $K \subset G$ is compact in any topology finer than that of pointwise convergence on G^*, K is compact in G.*

[5]For example, the fact (used below) that a product of complex-valued measurable functions is measurable requires some countable properties of the range.

Finally we note

COROLLARY 2.6. *Let* $[\cdot, \cdot]$ *be any separately continuous pairing between* G *and* H, *that is, a map of* $G \times H$ *into the circle group for which*

$$g \to [g, h] \text{ and } h \to [g, h]$$

are continuous homomorphisms. Then $[\cdot, \cdot]$ *is jointly continuous.*

This is precisely the analogue of a well-known fact about bilinear functionals on $X \times Y$, and follows from 1.1 exactly as in the Banach space situation: $[\cdot, \cdot]$ provides us with a pair of dual maps $\tau: G \to H^*$, $\tau^*: H \to G^*$, satisfying $[g, h] = < h, \tau g > = < g, \tau^* h >$. Thus τ is continuous, and joint continuity follows from the joint continuity of $< \cdot, \cdot >$.

An application of 2.2 can be made to show 2.6 holds with T^1 replaced by any locally compact abelian group K. (Consider $< [g, h], k^* >$.)

3. For Banach spaces, some consequences of uniform boundedness are results of the Banach-Steinhaus type, for example, the fact that a pointwise bounded set of continuous linear maps of X into Y is uniformly bounded. The direct analogue for groups would state that if T is a set of continuous homomorphisms of G into H with $\{\tau g : \tau \in T\}$ conditionally compact for each g in G, then T would be conditionally compact in the space $\text{Hom}(G, H)$ of all continuous homomorphisms topologized by uniform convergence on compact subsets of G; and since $T = \text{Hom}(G, T^1) = G^*$ satisfies these requirements no such result can hold. However, imposing pointwise boundedness on the set T^* of dual maps as well yields a valid result.

THEOREM 3.1. *Let* $\text{Hom}(G, H)$ *be the space of all continuous homomorphisms from* G *into* H, *in the compact-open topology, and let* $T \subset \text{Hom}(G, H)$. *Then* T *is conditionally compact if (and, of course, only if)*

$$Tg = \{\tau g : \tau \in T\} \text{ and } T^*h^* = \{\tau^* h^* : \tau \in T\}$$

are conditionally compact subsets of H *and* G^*, *respectively, for each* g *in* G, h^* *in* H^*.

Proof. Clearly $\text{Hom}(G, H)$ is complete in the natural uniform structure provided by the base of entourages of the form

$$\{(\sigma, \tau) : \sigma g - \tau g \in U, \text{ all } g \text{ in } K\}$$

where K is a compact subset of G and U is a neighbourhood of the identity of H. Similarly $\text{Hom}(G, H^a)$, where H^a is the almost periodic compactification of H, is complete in the corresponding uniform structure, which has a *subbase* of entourages of the form

$$(3.1) \qquad \{(\sigma, \tau) : | < \sigma g, h^* > - < \tau g, h^* > | < \epsilon, \text{ all } g \text{ in } K\}.$$

Considering T as a subset of $\text{Hom}(G, H^a)$ we have T precompact in the

https://doi.org/10.4153/CJM-1962-017-3 Published online by Cambridge University Press

corresponding uniform structure; indeed, given a compact subset K of G, $\epsilon > 0$ and $h^* \in H^*$, the set of functions $\{\tau^*h^*|K: \tau \in T\}$ is by hypothesis conditionally compact in $C(K)$ so that there are τ_i, $i = 1,2, \ldots, n$, for which, given τ in T, there is an i satisfying $| < g, \tau^*h^* > - < g, \tau_i^*h^* > | < \epsilon$ for all g in K, whence (τ, τ_i) lies in (3.1). Consequently the closure S of T in $\mathrm{Hom}(G, H^a)$ is compact.

Moreover, since $Tg = \{\tau g: \tau \in T\}$ has compact closure F_g in H, and F_g remains compact in the topology of pointwise convergence on H^*, each τ in S has τg in F_g; thus each τ in S maps G into H, and is continuous by 2.1, so $S \subset \mathrm{Hom}(G, H)$.

Now let $K \subset G$ be compact, and let $\tau_\delta \to \tau$ in the compact subset S of $\mathrm{Hom}(G, H^a)$. For any h^* in H^*, $< \tau_\delta g, h^* > \to < \tau g, h^* >$ uniformly on compact subsets of G so that $\tau_\delta^* h^* \to \tau^* h^*$ in G^*; thus if $k_\delta \to k$ in K we have

$$< \tau_\delta k_\delta, h^* > = < k_\delta, \tau_\delta^* h^* > \to < k, \tau^* h^* > = < \tau k, h^* >$$

since $< \cdot , \cdot >$ is jointly continuous. Hence the map $(\tau, k) \to \tau k$ from $S \times K$ into H in the topology of pointwise convergence on H^* is continuous. And since $S \times K$ is compact, Corollary 2.2 implies $(\tau, k) \to \tau k$ is continuous from $S \times K$ into H.

Thus, given any neighbourhood U of the identity of H, a standard compactness argument yields a finite set $\{\tau_1, \ldots, \tau_n\} \subset S$ for which, given any τ in S, there is an i satisfying

$$\tau k - \tau_i k \in U, \text{ all } k \text{ in } K,$$

so that S (and *a fortiori* T) is precompact in $\mathrm{Hom}(G, H)$; since $\mathrm{Hom}(G, H)$ is complete this is all we had to prove.

As an immediate consequence of the proof we have

COROLLARY 3.2. *A subset T of* $\mathrm{Hom}(G, H)$ *is compact if and only if T is compact in* $\mathrm{Hom}(G, H^a)$, *and then both topologies, being comparable, coincide on T.*

Let G^d denote G in the discrete topology. If T is a compact subset of $\mathrm{Hom}(G, H)$, so that it is compact in $\mathrm{Hom}(G, H^a)$, then it is clearly compact in the less fine topology of $\mathrm{Hom}(G^d, H^a)$. Compactness in this last topology amounts to compactness of the set of induced pairings

$$(g, h^*) \to < \tau g, h^* > , \tau \in T,$$

in the topology of pointwise convergence on $G \times H^*$. On the other hand, if the set of induced pairings is compact (or T is compact in $\mathrm{Hom}(G^d, H^a)$) then Tg is compact in H^a, thus by 1.2 compact in H, while T^*h^* is compact in $G^{d*} = G^{*a}$, and thus compact in G^*. Consequently T is conditionally compact in $\mathrm{Hom}(G, H)$ by 3.1; trivially T must be closed, and thus

COROLLARY 3.3. *A subset T of* $\mathrm{Hom}(G, H)$ *is compact if and only if the set of induced pairings $(g, h^*) \to < \tau g, h^* >$, $\tau \in T$, is compact in the topology of pointwise convergence on $G \times H^*$.*

https://doi.org/10.4153/CJM-1962-017-3 Published online by Cambridge University Press

4. Proofs of the open mapping theorem for Banach spaces utilize the same category argument which yields uniform boundedness. As is well known, the analogue for groups fails except in special cases (for a detailed investigation of such questions, see **(3)**); in this section one of these cases will be obtained as a trivial consequence of Corollary 2.3.

For $\tau \in \mathrm{Hom}(G, H)$ let G_τ denote its kernel, and $\bar{\tau}$ the induced isomorphism of G/G_τ into H. Clearly τ and $\bar{\tau}$ are onto, or open, simultaneously.

THEOREM 4.1. *Let τ be a continuous homomorphism of G onto H. Then τ is open if (and, of course, only if) $\bar{\tau}F$ is locally Borel for each closed F in G/G_τ.*

For the proof we need only note that $\bar{\tau}^{-1}$ is measurable in the sense of Corollary 2.3, and thus continuous. A well-known consequence is

COROLLARY 4.2. *Let τ be a continuous homomorphism of G onto H. If G, or even G/G_τ, is σ-compact then τ is open.*

For each closed subset F of G/G_τ is σ-compact, and thus has a σ-compact image $\bar{\tau}F$ in H.

COROLLARY 4.3. *Let $\tau: G \to H$ be an algebraic homomorphism which has a closed graph. If the projection of the graph onto G maps closed sets onto locally Borel sets, then τ is continuous. In particular if G and H are σ-compact, closure of the graph of τ implies continuity.*

The continuous projection is open by 4.1 and thus has a continuous inverse $g \to (g, \tau g)$, so τ is continuous. The particular case (which includes the case of G and H connected) follows as does 4.2 since $G \times H$, and thus the closed graph, must be σ-compact. Indeed in this particular case 4.3 yields a strengthened form of Corollary 2.1.

COROLLARY 4.4. *Let G and H be σ-compact, and E a subset of H^* which generates H^* topologically. Then if $g \to < \tau g, h^* >$ is continuous for all h^* in E, τ is continuous.*

The proof is trivial; one need only verify that the graph is closed.

We should note that there is no satisfactory analogue of the bornological aspect of Banach spaces (that bounded maps are continuous). This is quite clear when "bounded" is taken as "conditionally compact," but one might insist that bounded sets be closed, and then the question becomes: if $\tau: G \to H$ maps compact sets onto compact sets, is τ continuous? That the answer is no can be seen from the fact (Corollary 2.4) that the compact product group $\Pi_1^\infty \{0,1\}$, where $\{0,1\}$ is the two-element group, has a "discontinuous character" τ, which can only assume two values.

5. Viewing the topology of G as that induced by the w^* topology of $L_1(G^*)^*$ (or of $C_0(G^*)^{**}$) suggests the following extension of 1.2. (Recall that for S

locally compact, $C_0(S)^*$ consists of all finite regular Borel measures on S, while in a natural way $C(S)$ yields a subspace of $C_0(S)^{**}$.)

THEOREM 5.1. *Let S be a locally compact Hausdorff space, E a bounded subset of $C(S)$. Then E is w^* compact in $C_0(S)^{**}$ if (and, of course, only if) E is compact in the topology of pointwise convergence on S.*

When S is compact[6] this is precisely Grothendieck's Théorème 5 **(1)**; otherwise it is a simple corollary. Indeed suppose $\mathfrak{F}$ is an ultrafilter on E (which must converge to some f_0 in E in the topology of pointwise convergence); we need only show $\mathfrak{F}$ converges in the w^* topology.

On the bounded set E the w^* topology is produced by the dense set of measures μ with compact carriers K_μ, and thus it suffices to show

$$\int f_0 d\mu = \lim_{\mathfrak{F}} \int f d\mu$$

for such μ. But in the weak topology of $C(K_\mu)$, $E|K_\mu$ is compact by Grothendieck's result so that on it the weak topology coincides with that of pointwise convergence; hence we have

$$\int f_0 d\mu = \lim_{\mathfrak{F}} \int f d\mu$$

as desired.

The same proof yields more general versions. For example, we can take a vector subspace X of $C_0(S)^*$ equipped with some locally convex topology, E a set of (possibly unbounded!) continuous functions on S producing elements of X^*, and, under suitable conditions, show compactness of E under pointwise convergence implies w^* compactness in X^*. We need only know that E is bounded on compact sets and that the measures in X with compact carriers produce the w^* topology on E (as is the case if, say, they are dense in X and E is an equicontinuous subset of X^*).

Theorem 1.2 is, of course, an immediate consequence[7] of 5.1; indeed 5.1 says K is w^* compact in $C_0(G^*)^{**}$, hence, *a fortiori* in $L_1(G^*)^*$, that is, in G. Finally, we have a simple extension of Kaplansky's Theorem 1.1.

COROLLARY[8] 5.2. *Let S and T be locally compact Hausdorff spaces and f a bounded complex valued function on $S \times T$ which is separately continuous. Then $s \to \int f(s,t)\mu(dt)$ is continuous for each μ in $C_0(T)^*$.*

[6]At the other extreme, when S is discrete 5.1 says nothing since the topologies coincide.

[7]Actually 1.2 is far more elementary than 5.1, since it does not really depend on Eberlein's Theorem (of **(1)**) on which Grothendieck's Théorème 5 rests. For in our proof of 1.2 we need only take a sequence tending to ∞, rather than a net; applying Grothendieck's construction **(1**, Theorem 5**)** of a pointwise convergent subsequence and Lebesgue's dominated convergence theorem thus avoids all questions of weak compactness.

[8]If S is metric the result is, of course, trivial.

We need only note that $s \to f(s, \cdot)$ is a continuous map into $C(T)$ in the topology of pointwise convergence, so that a compact neighbourhood V of x_0 has an image $\{f(x, \cdot) : x \in V\}$ which is w^* compact in $C_0(T)^{**}$; continuity now follows as in the proof of 1.1.

References

1. A. Grothendieck, *Critères de compacité dans les espaces fonctionelles généraux*, Amer. J. Math., *74* (1952), 168–186.
2. L. H. Loomis, *An introduction to abstract harmonic analysis* (New York, 1953).
3. B. J. Pettis, *On continuity and openness of homomorphisms in topological groups*, Ann. Math. (2), *52* (1950), 293–308.
4. L. Pontrjagin, *Topological groups* (Princeton, 1939).
5. A. Weil, *L'intégration dans les groupes topologiques et ses applications* (Paris, 1940).

University of Notre Dame