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INVARIANT SUBSPACES IN THE BIDISC AND WANDERING SUBSPACES

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Abstract

Let M be a forward-shift-invariant subspace and N a backward-shift-invariant subspace in the Hardy space H^2 on the bidisc. We assume that $H^2 = N \oplus M$. Using the wandering subspace of M and N, we study the relations between M and N. Moreover we study M and N using several natural operators defined by shift operators on H^2.

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1. Introduction

Let Γ^2 be the torus, that is, the Cartesian product of two unit circles Γ in $\mathbb{C}$. Let $p = 2$ or $p = \infty$. The usual Lebesgue spaces, with respect to the Haar measure m on Γ^2, are denoted by $L^p = L^p(\Gamma^2)$, and $H^p = H^p(\Gamma^2)$ is the space of all f in L^p whose Fourier coefficients

$$\hat{f}(j, \ell) = \int_{\Gamma^2} f(z, w)\bar{z}^j \bar{w}^\ell \, dm(z, w)$$

vanish if at least one of j and ℓ is negative. Then H^p is called the *Hardy space*. As $\Gamma^2 = \Gamma_z \times \Gamma_w$, $H^p(\Gamma_z)$ and $H^p(\Gamma_w)$ denote the one-variable Hardy spaces.

Let P_{H^2} be the orthogonal projection from L^2 onto H^2. For ϕ in L^∞, the *Toeplitz operator* T_ϕ is defined by

$$T_\phi f = P_{H^2}(\phi f) \quad \text{for all } f \in H^2.$$

A closed subspace M of H^2 is said to be *forward-shift-invariant* if $T_z M \subset M$ and $T_w M \subset M$, and a closed subspace N of H^2 is said to be *backward-shift-invariant* if

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$T_z^* N \subset N$ and $T_w^* N \subset N$. Let P_M and P_N be the orthogonal projections from H^2 onto M and N, respectively. In this paper, we assume that $M \oplus N = H^2$, that is, $P_M + P_N = I$ where I is the identity operator on H^2. Let

$$A = P_M T_z P_N \quad \text{and} \quad B = P_N T_w^* P_M.$$

For ϕ in H^∞,

$$V_\phi f = P_M(\phi f) \quad \text{for all } f \in M \quad \text{and} \quad S_\phi f = P_N(\phi f) \quad \text{for all } f \in N.$$

Suppose that

$$\mathcal{V} = V_z V_w^* - V_w^* V_z \quad \text{and} \quad \mathcal{S} = S_z S_w^* - S_w^* S_z.$$

It is known [4] that $AB|_M = \mathcal{V}$ and $BA|_N = \mathcal{S}$. Guo and Yang [3] showed that AB is Hilbert–Schmidt under some mild conditions. In this paper, we study M or N when A, B, AB or BA is of finite rank. Izuchi and Nakazi [4] described an invariant subspace M or N with $A = 0$ or $B = 0$. Mandrekar [6], Ghatage and Mandrekar [2] and Nakazi [7, 8] described an invariant subspace M with $AB = 0$. Izuchi and Nakazi [4] and Izuchi *et al.* [5] described an invariant subspace N with $BA = 0$.

For a forward-shift-invariant subspace M, put

$$M_1 = \ker V_z^*, \quad M_2 = \ker V_w^* \quad \text{and} \quad M_0 = M_1 \cap M_2.$$

These are called *wandering subspaces* for M. In this paper, $[\cdot]$ denotes the closed span. For a backward-shift-invariant subspace N, with $M = H^2 \ominus N$, put

$$N_1 = [T_z^* M_1], \quad N_2 = [T_w^* M_2] \quad \text{and} \quad N_0 = N_1 \cap N_2.$$

These are called wandering subspaces for N.

In Section 2 we decompose and study M and N using the wandering subspaces M_1, M_2, N_1 and N_2. In Section 3 we study M and N when A or B is of finite rank. For an operator K, $r(K)$ denotes the rank of K. In Section 4 we show that $r(AB) = \dim N_1 \cap N_2$ in general, and $r(BA) = \dim M_1 \cap M_2$ under some mild conditions.

In this paper, for a bounded linear operator X on H^2, $\operatorname{ran} X = XH^2$ and $\ker X = \{f \in H^2 \mid Xf = 0\}$.

2. Wandering subspaces

Let M be a forward-shift-invariant subspace, and N be a backward-shift-invariant subspace with $H^2 = M \oplus N$. Put

$$M_z^\infty = \bigcap_{n=1}^{\infty} \{f \in M \mid \bar{z}^n f \in M\} \quad \text{and} \quad M_w^\infty = \bigcap_{n=1}^{\infty} \{f \in M \mid \bar{w}^n f \in M\},$$

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and

$$N_z^\infty = \bigcap_{n=1}^{\infty} \{f \in N \mid z^n f \in N\} \quad \text{and} \quad N_w^\infty = \bigcap_{n=1}^{\infty} \{f \in N \mid w^n f \in N\}.$$

In the case of one variable, $M_z^\infty = N_z^\infty = [0]$. In the case of two variables, M_z^∞ is also always $[0]$ but N_z^∞ may not be $[0]$. In fact, if $N \supset q_1 H^2(\Gamma_z)$ then $N_z^\infty \supset q_1 H^2(\Gamma_z)$ where $q_1 = q_1(z)$ is an inner function of one variable.

THEOREM 1. *Let N be a backward-shift-invariant subspace and $M = H^2 \ominus N$.*

(1) $M_z^\infty = M_w^\infty = [0]$ *and* $M = \sum_{n=0}^\infty \oplus T_z^n M_1 = \sum_{n=0}^\infty \oplus T_w^n M_2$.
(2) $N = [\bigcup_{n=0}^\infty T_z^{*n} N_1] \oplus N_z^\infty = [\bigcup_{n=0}^\infty T_w^{*n} N_2] \oplus N_w^\infty$.

PROOF. (1) is well known. To prove (2): if $f \in N_z^\infty$, then by definition $z^n f \in N$ for any $n \geq 1$, and hence f is orthogonal to $[\bigcup_{n=0}^\infty T_z^{*n} N_1]$. Conversely, suppose that f is orthogonal to $\bigcup_{n=0}^\infty T_z^{*n} N_1$. Since $f \perp N_1$, zf is orthogonal to $M_1 + zM$ because $N_1 = T_z^* M_1$ and $f \in N$. Hence $zf \in N$. Since $f \perp T_z^* N_1$, $z^2 f$ is orthogonal to $M_1 + zM$ because $T_z^* N_1 = T_z^{*2} M_1$ and $zf \in N$. Hence $z^2 f \in N$. By repeating the same argument, we can show that $z^n f$ belongs to N for any $n \geq 1$. This implies (2). □

COROLLARY 2. *Let N be a backward-shift-invariant subspace.*

(1) $N = N_z^\infty$ *if and only if* $N = H^2(\Gamma_z) \otimes (H^2(\Gamma_w) \ominus q_2 H^2(\Gamma_w))$ *where* $q_2 = q_2(w)$ *is an inner function of one variable.*
(2) $N = [\bigcup_{n=0}^\infty T_z^{*n} N_1]$ *if and only if for each nonzero f in N there exists $n \geq 1$ such that $z^n f \notin N$.*

PROOF. (1) If $N = N_z^\infty$ then $N_1 = 0$ and so $T_z^* M_1 = 0$. Hence $M_1 \subset H^2(\Gamma_w)$, so $M_1 = q_2 H^2(\Gamma_w)$ by a well-known theorem of Beurling [1]. Thus $M = q_2 H^2$, and so

$$N = H^2(\Gamma_z) \otimes (H^2(\Gamma_w) \ominus q_2 H^2(\Gamma_w)).$$

Conversely, if $M = q_2 H^2$ then $M_1 = q_2 H^2(\Gamma_w)$, and so $N_1 = T_z^* M_1 = 0$. Part (2) is clear by (2) of Theorem 1. □

By (1) of Theorem 1, both M_1 and M_2 are cyclic subspaces for T_z and T_w: that is,

$$\left[\bigcup_{(n,m) \geq (0,0)} T_z^n T_w^m M_j \right] = M \quad \text{for } j = 1, 2.$$

It may happen that

$$\left[\bigcup_{(n,m) \geq (0,0)} T_z^n T_w^m M_0 \right] = M,$$

where $M_0 = M_1 \cap M_2$. By (2) of Theorem 1, if $N_z^\infty = [0]$ or $N_w^\infty = [0]$ then N_1 or N_2 is a cyclic subspace for T_z^* and T_w^*: that is,

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$$\left[\bigcup_{(n,m)\geq(0,0)} T_z^{*n} T_w^{*m} N_j\right] = N \quad \text{for } j = 1, 2.$$

In general, N_0 may not be a cyclic subspace because $N_0 = [0]$ may happen. We can ask whether $T_z^* M_0$ or $T_w^* M_0$ is a cyclic subspace for T_z^* and T_w^* because $N_1 \supset T_z^* M_0$ and $N_2 \supset T_w^* M_0$. However, this is not true. If $M = zH^2$ then $N = H^2(\Gamma_w)$ and $M_0 = [z]$. Then $T_w^* M_0 = [0]$ and $T_z^* M_0 = [1]$.

EXAMPLE 1. Let $N = H^2(\Gamma_z) + H^2(\Gamma_w)$. Then the following hold.

(1) $N_1 = wH^2(\Gamma_w)$, $N_2 = zH^2(\Gamma_z)$ and $N_0 = [0]$.
(2) $[\bigcup_{n\geq 0} T_z^{*n} N_1] = wH^2(\Gamma_w)$, $[\bigcup_{n\geq 0} T_w^{*n} N_2] = zH^2(\Gamma_z)$ and $[\bigcup_{(n,m)\geq 0} T_z^{*n} T_w^{*m} N_0] = [0]$.
(3) $N_z^\infty = H^2(\Gamma_z)$ and $N_w^\infty = H^2(\Gamma_w)$.

EXAMPLE 2. Let $N = \mathbb{C}$ and $M = zH^2 + wH^2$. Then the following hold.

(1) $N_1 = N_2 = N_0 = \mathbb{C}$.
(2) $[\bigcup_{n\geq 0} T_z^{*n} N_1] = [\bigcup_{n\geq 0} T_w^{*n} N_2] = [\bigcup_{(n,m)\geq(0,0)} T_z^{*n} T_w^{*m} N_0] = N$.
(3) $N_z^\infty = N_w^\infty = [0]$.

EXAMPLE 3. Let

$$N = (H^2(\Gamma_z) \ominus q_1 H^2(\Gamma_z)) \otimes (H^2(\Gamma_w) \ominus q_2 H^2(\Gamma_w))$$

and $M = q_1 H^2 + q_2 H^2$, where $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are inner functions of one variable.

(1) $M_1 = q_1(H^2(\Gamma_w) \ominus q_2 H^2(\Gamma_w)) \oplus q_2 H^2(\Gamma_w)$ and $M_2 = q_2(H^2(\Gamma_z) \ominus q_1 H^2(\Gamma_z)) \oplus q_1 H^2(\Gamma_z)$.
(2) $N_1 = (T_z^* q_1)(H^2(\Gamma_w) \ominus q_2 H^2(\Gamma_w))$, $N_2 = (T_w^* q_2)(H^2(\Gamma_z) \ominus q_1 H^2(\Gamma_z))$ and $N_0 = \langle (T_z^* q_1)(T_w^* q_2)\rangle$.
(3) $[\bigcup_{n\geq 0} T_z^{*n} N_1] = [\bigcup_{n\geq 0} T_w^{*n} N_2] = [\bigcup_{(n,m)\geq(0,0)} T_z^{*n} T_w^{*m} N_0] = N$.

PROOF. (2) and (3) follow from (1). It is known [4] that

$$\begin{aligned} M = q_2 H^2 \oplus q_1(H^2 \ominus q_2 H^2) &= (H^2(\Gamma_z) \otimes q_2 H^2(\Gamma_w)) \\ &\oplus \{q_1 H^2(\Gamma_z) \otimes (H^2(\Gamma_w) \ominus q_2 H^2(\Gamma_w))\}. \end{aligned}$$

Hence (1) follows. □

3. $r(A) < \infty$ or $r(B) < \infty$

Recall that $A = P_M T_z P_N$ and $B = P_N T_w^* P_M$ (see Introduction). In this section, we are interested in the case when A or B is of finite rank. We know a characterization

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of $A = 0$ or $B = 0$ (see [3]). In fact, $A = 0$ if and only if $N = H^2$ or $N = H^2 \ominus qH^2$ where $q = q(w)$ is an inner function of one variable, and $B = 0$ if and only if $M = [0]$ or $M = qH^2$ where $q = q(z)$ is an inner function of one variable. In one-variable Hardy space, either A is of rank one for any N or B is of rank one for any M.

LEMMA 3. *Let M be a forward-shift-invariant subspace of H^2 and $N = H^2 \ominus M$.*

(1) $[\text{ran}A] \subseteq M_1$ *and* $\ker A = \{f \in N \mid T_z f \in N\} \oplus M$.
(2) $[\text{ran}A^*] = N_1$ *and* $\ker A^* = \{f \in M \mid T_z^* f \in M\} \oplus N$.
(3) $M_1 = [\text{ran}A] \oplus \{\ker A^* \ominus (T_z M \oplus N)\}$.
(4) $M = [\text{ran}A] \oplus (\ker A^* \ominus N)$ *and* $N = [\text{ran}A^*] \oplus (\ker A \ominus M)$.

PROOF. (1) By the definitions, $[\text{ran}A] = [P_M T_z N] \subseteq M_1$ because $T_z N$ is orthogonal to $T_z M$ and

$$\ker A = \{f \in N \mid T_z f \in N\} \oplus M.$$

(2) Since $T_z^* M = T_z^* M_1 \oplus M$, $[\text{ran}A^*] = [T_z^* M_1] = N_1$. By definition,

$$\ker A^* = \{f \in M \mid T_z^* f \in M\} \oplus N.$$

(3) is clear from (1) and the fact that $H^2 = [\text{ran}A] \oplus \ker A^*$. (4) is clear from (1), (2) and the fact that $H^2 = [\text{ran}A^*] \oplus \ker A$. $\square$

LEMMA 4. *Let M be a forward-shift-invariant subspace of H^2 and $N = H^2 \ominus M$.*

(1) $[\text{ran}A] = M_1 \ominus (M_1 \cap \ker T_z^*)$.
(2) $\ker A^* = (M_1 \cap \ker T_z^*) \oplus T_z M \oplus N$.

PROOF. (1) Since $T_z N \perp \ker T_z^*$, $T_z N \perp M_1 \cap \ker T_z^*$, so $P_M T_z N \perp M_1 \cap \ker T_z^*$. Hence, by (1) of Lemma 3, $[\text{ran}A] \subseteq M_1 \ominus (M_1 \cap \ker T_z^*)$. If $f \in M_1$ and $f \perp \text{ran}A$, then $f \perp T_z N$ and so $T_z^* f \perp N$. Hence $T_z^* f \in N \cap M$, because $T_z^* M_1 \perp M$. Hence $T_z^* f = 0$. (2) follows from (1), by (2) of Lemma 3. $\square$

LEMMA 5. *Let M be a forward-shift-invariant subspace of H^2. If $[\text{ran}A] \neq M_1$, then $M_1 = [\text{ran}A] \oplus q_2 H^2(\Gamma_w)$.*

PROOF. By Lemma 4, $M_1 \ominus [\text{ran}A] = M_1 \cap \ker T_z^*$ and $M_1 \cap \ker T_z^* \subset H^2(\Gamma_w)$ because $\ker T_z^* = H^2(\Gamma_w)$. Hence $w(M_1 \cap \ker T_z^*) \perp zM$, and so

$$w(M_1 \cap \ker T_z^*) \subseteq M_1 \cap \ker T_z^*.$$

By a theorem of Beurling [1], $M_1 \ominus [\text{ran}A] = q_2 H^2(\Gamma_w)$ for some one-variable inner function $q_2 = q_2(w)$. $\square$

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THEOREM 6. *Let M be a nonzero forward-shift-invariant subspace.*

(1) *If $r(A) < \infty$, then there is a one-variable inner function $q_2 = q_2(w)$ such that*

$$M_1 = \mathrm{ran} A \oplus q_2 H^2(\Gamma_w) \quad \textit{and} \quad M = q_2 H^2 \oplus \left\{ \sum_{j=0}^{\infty} \oplus (\mathrm{ran} A) z^j \right\}$$

where $q_2 = q_2(w)$ is a one-variable inner function.

(2) *If $r(B) < \infty$, then there is a one-variable inner function $q_1 = q_1(z)$ such that*

$$M_2 = \mathrm{ran} B^* \oplus q_1 H^2(\Gamma_z) \quad \textit{and} \quad M = q_1 H^2 \oplus \left\{ \sum_{j=0}^{\infty} \oplus (\mathrm{ran} B^*) w^j \right\}$$

where $q_1 = q_1(z)$ is a one-variable inner function.

(3) *If $r(A) < \infty$ and $r(B) < \infty$ then there exist two inner functions $q_1 = q_1(z)$ and $q_2 = q_2(w)$ such that $q_1 H^2 + q_2 H^2$ is a closed forward-shift-invariant subspace, $M \supseteq q_1 H^2 + q_2 H^2$, and*

$$\dim\{M_1 + M_2\}/\{q_1 H^2(\Gamma_z) + q_2 H^2(\Gamma_w)\} \le r(A) + r(B).$$

PROOF. Since $\dim M_1 = \infty$ by [7, Theorem 3], if $r(A) < \infty$ then $[\mathrm{ran} A] \ne M_1$, and so by Lemma 3, $M_1 = [\mathrm{ran} A] \oplus q_2 H^2(\Gamma_w)$ for some one-variable inner function $q_2 = q_2(w)$. This implies (1). If $r(B) < \infty$, then $r(B^*) < \infty$. Since $B^* = P_M T_w P_N$, (1) implies (2). If $r(A) < \infty$ and $r(B) < \infty$, (1) and (2) imply (3) because it is known [4] that $q_1 H^2 + q_2 H^2$ is closed. □

COROLLARY 7.

(1) *If $A = 0$, then $M = [0]$ or $M = q_2 H^2$ for some one-variable inner function $q_2 = q_2(w)$.*

(2) *If $B = 0$, then $M = [0]$ or $M = q_1 H^2$ for some one-variable inner function $q_1 = q_1(z)$.*

COROLLARY 8.

(1) *If $0 \le n \le \infty$ and $0 \le m \le \infty$, then there exist invariant subspaces M and N such that $r(A) = n$ and $r(B) = m$.*

(2) *If $r(B) = 0$, then $r(A) = 0$ or $r(A) = \infty$. If $r(A) = 0$, then $r(B)$ is 0 or ∞.*

PROOF. (1) Let $1 \le n < \infty$ and $1 \le m < \infty$. Suppose that $M = z^m H^2 + w^n H^2$. Then

$$\begin{aligned} M_1 &= w^n H^2(\Gamma_w) + [1, w, \ldots, w^{n-1}] z^m \quad \text{and} \\ M_2 &= z^m H^2(\Gamma_z) + [1, z, \ldots, z^{m-1}] w^n. \end{aligned}$$

By (1) and (2) of Theorem 6, $r(A) = n$ and $r(B) = m$.

(2) If $r(B) = 0$, then by (2) of Corollary 7, $M = [0]$ or $M = q_1 H^2$ where $q_1 = q_1(z)$ is a one-variable inner function. If $M = [0]$ then $r(A) = 0$ by definition. If $M = q_1 H^2$

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then $M_1 = q_1 H^2(\Gamma_w)$, and so if $r(A) < \infty$ then by (1) of Theorem 6 $M_1 \supset q_2 H^2(\Gamma_w)$ for some one-variable inner function $q_2 = q_2(w)$. This implies that q_1 is constant. Hence $M = H^2$, and so $A = 0$. □

For a finite Blaschke product q, $\deg q$ denotes the number of zeros of q counting multiplicity.

COROLLARY 9. *If $M = q_1 H^2 + q_2 H^2$ where $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are one-variable inner functions, then*

$$[\operatorname{ran} A] = q_1(H^2(\Gamma_w) \ominus q_2 H^2(\Gamma_w)) \quad \text{and} \quad [\operatorname{ran} B^*] = q_2(H^2(\Gamma_z) \ominus q_1 H^2(\Gamma_z)).$$

If $r(A) < \infty$ and $r(B) < \infty$, then $r(A) = \deg q_2$ and $r(B) = \deg q_1$.

COROLLARY 10. *Let M be a forward-shift-invariant subspace. If M is of finite codimension n, then $r(A) \le n$, $r(B) \le n$, and $M \supseteq q_1 H^2 + q_2 H^2$ where $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are one-variable finite Blaschke products.*

PROOF. By the definitions of A and B, it is clear that $r(A) \le n$ and $r(B) \le n$. The second statement follows from (3) of Theorem 6. □

PROPOSITION 11. *Let M be a forward-shift-invariant subspace. Then $M \supseteq q_1 H^2 + q_2 H^2$ for some one-variable inner functions $q_1 = q_1(z)$ and $q_2 = q_2(w)$ if and only if $[\operatorname{ran} A] \ne M_1$ and $[\operatorname{ran} B^*] \ne M_2$.*

PROOF. The 'if' part is clear by Lemma 5. If $M \supseteq q_1 H^2$, then $q_1 H^2(\Gamma_z)$ is orthogonal to wM, and so $q_1 H^2(\Gamma_z) \subseteq M_2$. Hence Lemma 4 implies that $[\operatorname{ran} B^*] \ne M_2$. Similarly we can prove that if $M \supseteq q_2 H^2$ then $[\operatorname{ran} A] \ne M_1$. □

PROPOSITION 12. *$N_1 = [\operatorname{ran} A^*]$ and $N_2 = [\operatorname{ran} B]$. Hence $\dim N_1 = r(A)$ and $\dim N_2 = r(B)$.*

PROOF. This follows from (2) of Lemma 3. □

4. $r(AB) < \infty$ or $r(BA) < \infty$

Let M be a forward-shift-invariant subspace and $N = H^2 \ominus M$. Recall the definitions of $\mathcal{V}$ and $\mathcal{S}$ in the Introduction. It is known [4] that $AB|_M = \mathcal{V}$ and $BA|_N = \mathcal{S}$. Then $AB = 0$ if and only if $\mathcal{V} = 0$, and $BA = 0$ if and only if $\mathcal{S} = 0$. We know the characterization of an invariant subspace such that $AB = 0$ or $BA = 0$. In fact, it is known (see [6–8]) that $AB = 0$ if and only if $M = qH^2$ for some inner function q. Recently it was proved (see [4, 5]) that $BA = 0$ if and only if

$$\begin{aligned} N &= (H^2(\Gamma_z) \ominus q_1 H^2(\Gamma_z)) \otimes (H^2(\Gamma_w) \ominus q_2 H^2(\Gamma_w)), \\ N &= (H^2(\Gamma_z) \ominus q_1 H^2(\Gamma_z)) \otimes H^2(\Gamma_w), \quad \text{or} \\ N &= H^2(\Gamma_z) \otimes (H^2(\Gamma_w) \ominus q_2 H^2(\Gamma_w)), \end{aligned}$$

where $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are one-variable inner functions. In this section, we study invariant subspaces such that $r(AB) < \infty$ or $r(BA) < \infty$.

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LEMMA 13. *Let M be a forward-shift-invariant subspace and $N = H^2 \ominus M$.*

(1) $r(BA) = \dim([P_M T_z N] \cap [P_M T_w N])$.

(2) $r(AB) = \dim([P_N T_z^* M] \cap [P_N T_w^* M])$.

PROOF. (1) Since $[BAH^2] = [B[\mathrm{ran} A]]$, $r(BA) = \dim((\ker B)^\perp \cap [\mathrm{ran} A])$. This implies (1) because $(\ker B)^\perp = [\mathrm{ran} B^*] = [P_M T_w N]$ and $[\mathrm{ran} A] = [P_M T_z N]_2$. (2) can be proved similarly. □

THEOREM 14. *Let M be a forward-shift-invariant subspace of H^2 and $N = H^2 \ominus M$.*

(1) *If $M_1 \cap \ker T_z^* = [0]$ and $M_2 \cap \ker T_w^* = [0]$, then $r(BA) = \dim M_1 \cap M_2$.*

(2) $r(AB) = \dim N_1 \cap N_2$.

PROOF. (1) By (1) and (2) of Lemma 3,

$$[\mathrm{ran} A] = [P_M T_z N] \subseteq M_1 \quad \text{and} \quad [\mathrm{ran} B^*] = [P_M T_w N] \subseteq M_2.$$

By Lemma 4, if $M_1 \cap \ker T_z^* = [0]$ then $[P_M T_z N] = M_1$, and if $M_2 \cap \ker T_w^* = [0]$ then $[P_M T_w N] = M_2$. Hence, $r(BA) = \dim M_1 \cap M_2$ by Lemma 13.

(2) Since $[P_N T_z^* M] = [P_N T_z^* M_1] = N_1$ and $[P_N T_w^* M] = [P_N T_w^* M_2] = N_2$, by Lemma 13 $r(AB) = \dim N_1 \cap N_2$. □

In (1) of Theorem 14, we need the condition $M_1 \cap \ker T_z^* = M_2 \cap \ker T_w^* = [0]$. In fact, $M_1 \cap M_2$ is always nontrivial but BA may be zero.

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