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REGULAR DIGRAPHS CONTAINING A GIVEN DIGRAPH

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ABSTRACT. Let the maximum degree d of a digraph D be the maximum of the set of all outdegrees and indegrees of the points of D. We prove that every digraph D of order P and maximum degree d has a d-regular superdigraph H with at most $d+1$ more points, and that this bound, which is independent of p, is best possible.

1. **Introduction.** In the first book on graph theory ever written, Dénes König proved that for every graph G, of order p and maximum degree d, there is a d-regular graph H containing G as an induced subgraph. Paul Erdös and Paul Kelly solved the extremal problem of determining the minimum number of points which must be added to a given graph G to obtain such a supergraph H. Lowell Beineke and Raymond Pippert extended this result to digraphs. A related problem was studied by Jin Akiyama, Hiroshi Era and Frank Harary when they considered G as a subgraph of H which is not necessarily induced and showed that at most $d+2$ new points are needed. We now settle the corresponding problem for digraphs.

A *digraph* D has a set V of $p \geq 1$ points and a set X of $q \leq p(p-1)$ arcs, each of which is an ordered pair (u, v) of distinct points. The *outdegree* $\text{od}(u)$ of point u in D is the number of arcs from u and its *indegree* $\text{id}(u)$ is the number of arcs to u. The *maximum degree* d of digraph D is the maximum of the set of all outdegrees and indegrees of the points.

The *digraph* DG *of a graph* G is obtained when each (undirected) line uv of G is replaced by the symmetric pair of arcs (u, v) and (v, u). In particular DK_p is the digraph of the complete graph K_p.

The previous results concerning regular supergraphs are now stated in chronological order. Throughout, p is the number of points and d the maximum degree. The *maximum deficiency* of graph G with minimum degree δ is $d-\delta$. If $(d_1, d_2, \ldots, d_p)$ is the degree sequence of G, then the *total deficiency* of G is $s = \sum (d-d_i) = pd-2q$.

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THEOREM A (König [6]). *For every graph G there is a d-regular graph H containing G as an induced subgraph.*

THEOREM B (Erdös and Kelly [3, 4]). *Let G be a graph with p points, maximum degree d, maximum deficiency e, and total deficiency s. The minimum number of new points in a d-regular supergraph H containing G as an induced subgraph is the smallest integer m satisfying* (1) $md \geq s$, (2) $m^2-(d+1)m+s \geq 0$, (3) $m \geq e$, (4) $(m+p)d$ *is even. Further, this bound is best possible.*

THEOREM C (Beineke and Pippert [2]). *Let D be an oriented graph (asymmetric digraph) with maximum degree d, sum of in-deficiencies s and maximum combined deficiency t. The minimum number of new points in a d-regular oriented supergraph of D is the smallest integer m satisfying* (1) $m \geq t$, (2) $md \geq s$, (3) $\binom{m}{2} \geq md - s$.

When D is a digraph having maximum in- or out-deficiency r, m is the least integer such that (1) $m \geq r$, (2) $md \geq s$, (3) $m(m-1) \geq md - s$.

THEOREM D (Akiyama, Era and Harary [1]). *For every graph G there is a d-regular supergraph H having at most $d+2$ new points and this bound, which is independent of p, is best possible.*

2. **The result.** The proof given below modifies that of [1] to the case of digraphs. In a *d-regular digraph*, each point has both indegree and outdegree d.

THEOREM 1. *Every digraph D, of order p and maximum degree d, has a d-regular superdigraph H with at most $d+1$ more points and this bound, which is independent of p, is best possible.*

Proof. We begin by filling D with additional arcs without exceeding d. If there are two points u, v in D such that $\mathrm{od}(u)$, $\mathrm{id}(v) < d$ and $\mathrm{arc}(u, v)$ is not in D, then add this arc to D. Continue this until no such pair of points remains and call D' the resulting superdigraph of D. At the end of this process, $V(D') = V$ is partitioned into four subsets A_i such that, with $\mathrm{od}(u)$ and $\mathrm{id}(u)$ now referring to D',

$$A_1 = \{u : \mathrm{od}(u) < d, \mathrm{id}(u) = d\},$$
$$A_2 = \{u : \mathrm{od}(u) < d, \mathrm{id}(u) < d\},$$
$$A_3 = \{u : \mathrm{od}(u) = d, \mathrm{id}(u) < d\},$$
$$A_4 = \{u : \mathrm{od}(u) = d = \mathrm{id}(u)\}.$$

Let $a_i = |A_i|$, $i = 1, 2, 3, 4$. For each point u in D', call $\mathrm{im}(u) = d - \mathrm{id}(u) =$ the *in-deficiency* of u (with the letter "m" standing for missing) and similarly $\mathrm{om}(u) = d - \mathrm{od}(u) =$ the *out-deficiency* of u.

We now show that $a_1+a_2<d$, from which it follows at once by directional duality that $a_2+a_3<d$. Each point $w\in A_3$ has positive in-deficiency while points $u\in A_1$ and $v\in A_2$ have positive out-deficiency. Now if u and v are not adjacent to w, then D' has not yet been completely constructed. Hence both u and v are adjacent to w, so $\mathrm{id}(w)\geq a_1+a_2$. But as $w\in A_3$, $\mathrm{id}(w)<d$ and we have $a_1+a_2<d$.

Obviously $\sum \mathrm{im}(u)=\sum \mathrm{om}(u)$ with the sum taken over all points u in D'.

LEMMA. *Let r, p be positive integers with $r<p$ and let s, t be nonnegative integers such that $2s+t=p$. Then there exist two digraphs D_1, D_2 with p points in both of which s points have degree pair $(r, r-1)$, another s have $(r-1, r)$, and the remaining t points have $(r-1, r-1)$ in D_1 and (r, r) in D_2.*

Proof. We first construct D_1, and begin by taking p even. It is well-known, König [6, p. 85], that K_p has a 1-factorization into $p-1$ 1-factors F_i. In $r-1$ of these, replace each edge by a symmetric pair of arcs. Then in F_r take any s remaining edges and orient them arbitrarily to make them arcs. The result is D_1 for p even.

When p is odd, we use the well-known decomposition of K_p into $(p-1)/2$ hamiltonian cycles [5, p. 89] and make $r-1$ of these cycles directed. Now orient the rth cycle C to become a directed cycle C', and retain any s arcs of C' no two of which are consecutive while discarding the remaining $p-s$ arcs. This completes D_1 when p is odd.

The construction of D_2 is the same for the first $r-1$ steps. But for the final step it must be modified. When p is even, in addition to orienting any s edges of F_r, we also take $t/2$ additional edges of F_r and convert them to symmetric pairs of arcs. And when p is odd, we take the directed cycle C' above and delete any s nonconsecutive arcs, retaining the remaining $p-s$ arcs. The resulting digraph D_2 has the specified degree pairs.

We can now continue the proof of the theorem.

We add a set W of $d+1$ new points $w_0, \ldots, w_d$ to D'. Let the points of $A_1\cup A_2$ be $u_1, \ldots, u_m$, and join u_1 to the first $\mathrm{om}(u_1)$ points $w_0, w_1, \ldots,$ then u_2 to the next $\mathrm{om}(u_2)$ points of W, and so forth in a cyclic manner. Similarly, let $v_1, v_2, \ldots, v_n$ be the points in $A_2\cup A_3$ and join to v_1 the last $\mathrm{im}(v_1)$ points $w_d, w_{d-1}, \ldots,$ to v_2 the preceding $\mathrm{im}(v_2)$ points, etc. In the resulting digraph E, all points of D have both out- and in-degree equal to d, while in E the degree pairs of the points of W are either all $(x, x-1)$, $(x-1, x)$ and $(x-1, x-1)$ or all $(x, x-1)$, $(x-1, x)$ and (x, x), where $0<x\leq d$ since $0<a_1+a_2<d$ and $0<a_2+a_3<d$. In both cases it follows from the lemma that digraph E can be extended to d-regularity, by embedding D_1 or D_2 as required into the deficient points of E.

To prove that the bound $d+1$ is best possible, we exhibit two digraphs D

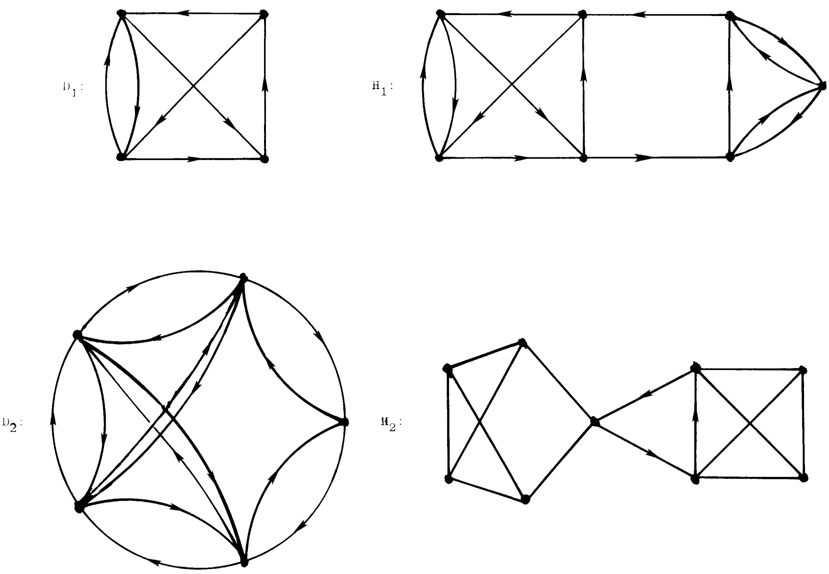

Figure 1. Two digraphs and their smallest regular superdigraphs.

having maximum degree d that require $d+1$ new points to obtain a desired d-regular superdigraph H. The first of these is a digraph D_1 with $p=4$ and $d=2$. In Fig. 1, it is verified that $d+1=3$ new points suffice, and the role of the point w mentioned as K_1 in the proof is shown in the construction of the 2-regular superdigraph H_1.

The second of these has d odd and is a symmetric digraph D_2 with $p=5$ points and $d=3$ so that $r=4$. In fact D_2 is DG_2 where G_2 is the graph used as an illustration in [1]. In accordance with Theorem D above, this graph G_2 requires five, i.e. $d+2$, new points in order to build a 3-regular supergraph. However, the digraph D_2 requires only four new points as shown in Fig. 1, in which each symmetric pair of arcs in H_2 is depicted for simplicity as an undirected edge.

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