

GROUPS WITH A (B, N) -PAIR AND LOCALLY TRANSITIVE GRAPHS

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1. Introduction.

Let Γ be an undirected graph and G a subgroup of $\text{aut}(\Gamma)$. We denote by $\partial(x, y)$ the distance between two vertices x and y , by $E(\Gamma)$ the edge set of Γ , by $V(\Gamma)$ the vertex set of Γ , by $\Gamma(x)$ the set of neighbors of the vertex x and by $G(x)^{\Gamma(x)}$ the permutation group induced by the stabilizer $G(x)$ on $\Gamma(x)$. For each $i \in \mathbb{N}$, let $G_i(x) = \{a \mid a \in G(y) \text{ for every } y \text{ with } \partial(x, y) \leq i\}$. An s -path is an ordered sequence (x_0, \dots, x_s) of $s + 1$ vertices x_i with $x_i \in \Gamma(x_{i-1})$ for $1 \leq i \leq s$ and $x_i \neq x_{i-2}$ for $2 \leq i \leq s$. For each vertex x , let $W_s(x)$ be the set of s -paths (x_0, \dots, x_s) with $x = x_0$. We say that the graph Γ is locally (G, s) -transitive if for every vertex x , $G(x)$ acts transitively on $W_s(x)$ but not on $W_{s+1}(x)$ (compare [1], [11]). If, in addition, G acts transitively on $V(\Gamma)$, then Γ is called (G, s) -transitive; otherwise Γ is bipartite with vertex blocks V_0 and V_1 and G acts transitively on both V_0 and V_1 , assuming that Γ is connected and $s \geq 1$.

Now let G be a finite group with a (B, N) -pair whose Weyl group is a dihedral group D_{2n} of order $2n$ ($n \geq 2$) and Γ be the incidence graph of the associated coset geometry as defined in [3, p. 129] (or [2, (15. 5. 1)]). The graph Γ has the following properties:

(A) $V(\Gamma) = V_0 \cup V_1$ with $V_0 \cap V_1 = \emptyset$ and $\Gamma(x) \subseteq V_{1-i}$ for every vertex $x \in V_i$ ($i = 0$ and 1). For $i = 0$ and 1 there exists a $d_i \in \mathbb{N}$ such that $|\Gamma(x)| = d_i + 1$ for every vertex $x \in V_i$. The diameter of Γ is n and the girth $2n$.

(B) Γ is locally $(G, n + 1)$ -transitive.

A generalized n -gon of order (d_0, d_1) is, by definition, an incidence structure whose incidence graph has the properties listed in (A).

W. Feit and G. Higman have shown in [3] that finite generalized n -gons of order (d_0, d_1) with $d_0 d_1 > 1$ exist only for $n = 2, 3, 4, 6, 8$ and 12 , that

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$n = 8$ is possible only when the squarefree part of $d_0 d_1$ is equal to two and that $n = 12$ is possible only when d_0 or $d_1 = 1$. The only known finite groups with a (B, N) -pair whose Weyl group is isomorphic to D_{2n} with $n = 3$ (resp. $n = 4$, $n = 6$) and whose generalized n -gon is of order (d_0, d_1) with $d_0 = d_1$ are (essentially) the Chevalley groups $A_2(q)$ (resp. $B_2(q)$, $G_2(q)$) (with $q = d_0$). Let $\Gamma_{n,q}$ denote the corresponding graph.

We prove here the following theorems:

(1.1) *Let p be a prime, r and $s \in \mathbb{N}$ with $r \geq 1$ and $s \geq 2$ and $q = p^r$. Let Γ be a finite undirected connected graph regular of valency $q + 1$ and G a subgroup of $\text{aut}(\Gamma)$ such that Γ is locally (G, s) -transitive and $\text{PSL}(2, q) \cong G(x)^{\Gamma(x)} \cong \text{P}\Gamma\text{L}(2, q)$ for every vertex x . Then $s \leq 5$ or $s = 7$. Let (x_1, \dots, x_s) be an arbitrary $(s - 1)$ -path. Then $G_1(x_1) = 1$ if $s = 2$ and $G_1(x_1) \cap G_1(x_2) \cap G(x_3) \cap \dots \cap G(x_s) = 1$ otherwise.*

(1.2) *Let Γ, G , etc. be as in (1.1) with $q \geq 3$ and $s \in \{4, 5, 7\}$. In addition, suppose that $s \neq 5$ if $q = 3$. Let $H_{3,q} = A_2(q)$, $H_{4,q} = B_2(q)$, $H_{6,q} = G_2(q)$ and $G_{n,q} = \text{aut}(\Gamma_{n,q}) \cong \text{aut}(H_{n,q})$ for $n = 3, 4, 6$; $H_{n,q}$ is to be considered as a subgroup of $G_{n,q}$. Let $k = \{x, y\}$ be an edge of Γ , $\Delta_i = \{w \in V(\Gamma) \mid \partial(i, w) \leq s - 2\}$ for $i = x$ and y and $\Delta = \Delta_x \cup \Delta_y$. Then there exists a bijective map $\varphi: \Delta \rightarrow V(\Gamma_{s-1,q})$ mapping edges onto edges such that:*

(a) *For $i = x$ and y and for each $g \in G(i)$ (resp. $g \in G(k)$, where $G(k)$ is the stabilizer in G of the unordered pair $\{x, y\}$), there exists a unique element $h \in G_{s-1,q}((i)\varphi)$ (resp. $h \in G_{s-1,q}((k)\varphi)$) such that $(w)h = (w)\varphi^{-1}g\varphi$ for every $w \in (\Delta_i)\varphi$ (resp. $w \in (\Delta)\varphi = V(\Gamma_{s-1,q})$).*

(b) *For $i = x$ and y and for each $h \in H_{s-1,q}((i)\varphi)$ (resp. $h \in H_{s-1,q}((k)\varphi)$), there exists a unique element $g \in G(i)$ (resp. $g \in G(k)$) such that $(w)h = (w)\varphi^{-1}g\varphi$ for each $w \in (\Delta_i)\varphi$ (resp. $w \in (\Delta)\varphi$).*

In particular, $H_{s-1,q}((i)\varphi) \cong G(i) \cong G_{s-1,q}((i)\varphi)$ for $i = x$ and y and $H_{s-1,q}((k)\varphi) \cong G(k) \cong G_{s-1,q}((k)\varphi)$.

In the following theorem, $\hat{G}_{4,2}$ denotes the unique subgroup of $\text{aut}(\Gamma_{4,2}) \cong \text{P}\Gamma\text{L}(2, 9)$ isomorphic to $\text{PGL}(2, 9)$. The reader can check that $\Gamma_{4,2}$ is $(\hat{G}_{4,2}, 4)$ -transitive.

(1.3) *Let Γ, G , etc. be as in (1.1) with $q = 2$ and $s \in \{4, 5, 7\}$. Let (X, Y) be an arbitrary 1-path of $\Gamma_{s-1,2}$. Then there exists a map $\varphi: k = \{x, y\} \rightarrow \{X, Y\}$ such that $H_{s-1,2}((i)\varphi) \cong G(i)$ for $i = x$ and y . Either $H_{s-1,2}((k)\varphi) \cong G(k) \cong G_{s-1,2}((k)\varphi)$ or $s = 4$ and $G(k) \cong \hat{G}_{4,2}(K)$ where K is any edge of*

$\Gamma_{4,2}$.

In the first part of the proof of (1.1) we show that $s \leq 5$ or $s \in \{7, 9, 13\}$. Note the remarkable coincidence with the numbers $n = 2, 3, 4, 6, 8$ and 12 obtained in [3] as the solution to a completely different sort of problem. To exclude $s = 9$ when $p = 2$ and $q \geq 4$ we construct a generalized 8-gon of order (q, q) , thus obtaining a contradiction from [3]. To proceed in the case $q \equiv 3 \pmod{4}$, we require [6, (8.2.11)] in order to prove that $PGL(2, q) \cong G(x)^{\Gamma(x)}$ for some vertex x . In the proof of (1.2) we use the characterizations of the graphs $\Gamma_{n,q}$ given in [5, Theorem 1.8], [7, Theorem 2] and [12, (4.4)]. Otherwise, the arguments contained in this paper are elementary and self-contained.

When proving (1.2), we include the case that $q = 3$ and $s = 5$, making the additional assumption that $G(x)^{\Gamma(x)} \cong PGL(2, 3)$ for every vertex x . The conclusion reached is that $G_{4,3}(X)$ induces $PGL(2, 3)$ on $\Gamma_{4,3}(X)$ for every vertex X of $\Gamma_{4,3}$. Since this is not so, it follows that $G(x)^{\Gamma(x)} \cong PGL(2, 3)$ does not hold for all vertices x of Γ when $q = 3$ and $s = 5$; in particular, G cannot act transitively on $V(\Gamma)$.

Theorems (1.2) and (1.3) imply that $G_{s-1,q}((k)\varphi)$ contains an element exchanging $(x)\varphi$ and $(y)\varphi$ if $G(k)$ contains an element exchanging x and y . Thus $\Gamma_{s-1,q}$ is $(G_{s-1,q}, s)$ -transitive if Γ is (G, s) -transitive. For $n = 4$ and 6 , $\Gamma_{n,q}$ is $(G_{n,q}, n + 1)$ -transitive if and only if $p = n/2$ (see [2]). Hence we have the following corollary:

(1.4) *Let Γ, G , etc. be as in (1.1). If G acts transitively on $V(\Gamma)$ (i.e., if Γ is (G, s) -transitive), then $p = 2$ if $s = 5$ and $p = 3$ if $s = 7$.*

For other relevant results consult [4] and [9] where, however, completely different methods are used from those developed here.

2. Proof of (1.1): $s \in \{2, 3, 4, 5, 7, 9, 13\}$

Let Γ and G satisfy the hypotheses of (1.1). If $W = (x_0, \dots, x_t)$ is any t -path (t arbitrary), we set $G(W) = G(x_0, \dots, x_t) = G(x_0) \cap \dots \cap G(x_t)$ and $G_i(W) = G_i(x_0, \dots, x_t) = G_i(x_0) \cap \dots \cap G_i(x_t)$ for each $i \in N$. If $b \in G(x)$, x a vertex, we denote by $|b|_x$ the order of the permutation that b induces on $\Gamma(x)$. We will often use integers to denote vertices of Γ .

For each vertex x , let $\bar{G}(x)$ be the largest subgroup of $G(x)$ such that $\bar{G}(x)^{\Gamma(x)} \cong PGL(2, q)$ and $f_x = [\bar{G}(x) \cap G(y, x, z): G_1(x)]$ where y and z are

any two neighbors of x . A t -path $(0, \dots, t)$ will be called good if $[G(W) \cap \bar{G}(i): G(W) \cap G_i(i)] = f_i$ for each i with $1 \leq i \leq t - 1$.

(2.1) *If $W = (0, \dots, t)$ is a good t -path, then there exists a vertex $t + 1$ such that $(0, \dots, t, t + 1)$ is a good $(t + 1)$ -path.*

Proof. Clearly all 1- and 2-paths are good, so that we can assume $t \geq 2$. Let $W_1 = (1, \dots, t)$. By induction, there exists an element $b_t \in G(W_1) \cap \bar{G}(t)$ with $|b_t|_t = f_t$ and $(p, |b_t|) = 1$. For $1 \leq i \leq t - 1$ there exists an element $b_i \in G(W) \cap \bar{G}(i)$ with $|b_i|_i = f_i$ and $(p, |b_i|) = 1$. The subgroup $\langle b_1, b_t \rangle$ contains an element c with $c^{-1}b_t c = b_t^c \in G(0)$. Let $a_i = b_i^c$ and $t + 1$ be a fixed point of a_i in $\Gamma(t) - \{t - 1\}$. For each i with $1 \leq i \leq t - 1$ there exists an element $c_i \in \langle b_i, a_i \rangle$ with $a_i = c_i^{-1}b_i c_i \in G(t + 1)$. For $1 \leq i \leq t$ we have $a_i \in G(0, \dots, t, t + 1) \cap \bar{G}(i)$ and $|a_i|_i = f_i$. \square

(2.2) *Every s -path is good. If $(0, \dots, s, s + 1)$ is a good $(s + 1)$ -path, then $G(0, \dots, s) \leq G(s + 1)$. If $f_s \neq 1$, then $s + 1$ is the only vertex in $\Gamma(s) - \{s - 1\}$ such that $(0, \dots, s, s + 1)$ is good.*

Proof. For every vertex x there exists, according to (2.1), at least one good s -path beginning at x . Since $G(x)$ acts transitively on $W_s(x)$, the first claim follows. Let $a \in G(0, \dots, s, s + 1) \cap \bar{G}(s)$ be an element with $|a|_s = f_s$. Suppose there exists an element $b \in G(0, \dots, s) - G(s + 1)$. Then $\langle a, b \rangle \leq G(0, \dots, s)$ acts transitively on $\Gamma(s) - \{s - 1\}$ (for if $f_s = 1$, then $\langle b \rangle$ itself must act transitively on $\Gamma(s) - \{s - 1\}$), contradicting the hypothesis that $G(0)$ acts intransitively on $W_{s+1}(0)$. In particular, if $(0, \dots, s, y)$ is a good path and $f_s \neq 1$, then there exists an element in $G(0, \dots, s)$ whose only fixed point in $\Gamma(s) - \{s - 1\}$ is y ; thus $y = s + 1$. \square

If we take any 1-path and start extending it to an arbitrarily long good path, the resulting path, since Γ is finite, contains, after a while, no new vertices. Thus we may choose, once and for all, an infinitely long path $W = (\dots, -1, 0, 1, 2, \dots)$ such that for each i there exists an element $h_i \in G(W) \cap \bar{G}(i)$ with $|h_i|_i = f_i$.

(2.3) $G_i(1) = 1$ if $s = 2$ and $G_i(1, 2) \cap G(0, \dots, s) = 1$ otherwise.

Proof. Let $A = G_i(1, 2) \cap G(0, \dots, s)$. Since $h_s \in G(1, \dots, s)$, $G(1, \dots, s)$ acts primitively on $\Gamma(s) - \{s - 1\}$. Since $G_i(1) \cap G(1, \dots, s) \trianglelefteq G(1, \dots, s)$ and $G_i(1) \cap G(1, \dots, s) \leq G(0, \dots, s)$ acts intransitively on $\Gamma(s) - \{s - 1\}$, we have $G_i(1) \cap G(1, \dots, s) \leq G_i(s)$ and in particular $G_i(1) \leq A$ if $s = 2$.

Similarly, $G_1(s) \cap G(s, \dots, 2, x_2) \leq G_1(x_2)$, $x_2 \in \Gamma(2) - \{1, 3\}$ arbitrary. By (2.2), $G(0, \dots, s) \leq G(-s + 4, \dots, s)$ and thus $A \leq G_1(x_2) \cap G(x_2, 2, 1, 0, \dots, -s + 4) \leq G_1(-s + 4)$, hence $A \leq G_1(-s + 4) \cap G(-s + 4, \dots, 3) \leq G_1(3)$. Choose any $y \in \Gamma(s) - \{s - 1\}$. Then $A \leq G_1(2) \cap G(2, \dots, s, y) \leq G_1(y)$, $A \leq G_1(y) \cap G(y, s, \dots, 3, x_3)$, $x_3 \in \Gamma(3) - \{2, 4\}$ arbitrary, thus $A \leq G_1(x_3) \cap G(x_3, 3, 2, \dots, -s + 5) \leq G_1(-s + 5)$, $A \leq G_1(-s + 5) \cap G(-s + 5, \dots, 4) \leq G_1(4)$. Also, $A \leq G_1(3) \cap G(3, \dots, s, y) \cap G_1(y) \leq G_1(z)$, $z \in \Gamma(y) - \{s\}$ arbitrary. It should now be clear that $A \leq G_1(1, \dots, s, y, z, \dots, w)$ for every path $(1, \dots, s, y, z, \dots, w)$ of arbitrary length beginning with $(1, \dots, s)$. Since Γ is connected, it follows that $A = 1$. \square

To prove (1.1), we have only to show now that $s \leq 5$ or $s = 7$. From now on we assume that $s \geq 3$.

(2.4) $G_1(1, 2)$ is a p -group. For each $t \geq 3$ and each i with $1 \leq i \leq t - 2$, $G_1(i, i + 1) \cap G(0, \dots, t) = G_1(1, \dots, t - 1)$.

Proof. By (2.3), $G_1(1, 2)$ acts semi-regularly on the set of s -paths beginning with $(0, \dots, 3)$ and thus $|G_1(1, 2)| \leq q^{s-3}$. To prove the second claim, we note that $G_1(1, 2) \leq G_1(2) \leq G(2, 3)$ and thus $G_1(1, 2)^{r(3)} \leq O_p(G(2, 3))^{r(3)}$ so that $G_1(1, 2) \cap G(4) = G_1(1, 2, 3)$. \square

(2.5) If $2 \leq t \leq s - 1$, then $G_1(1, \dots, t - 1)$ acts transitively on $\Gamma(t) - \{t - 1\}$.

Proof. Let x_1 and x_2 be any two vertices in $\Gamma(t) - \{t - 1\}$. There exists an element $a_i \in G(0, \dots, t, x_i) \cap \bar{G}(t)$ with $|a_i|_t = f_t$ ($i = 1, 2$). If $f_t \neq 1$, the commutator group $\langle a_1, a_2 \rangle' \leq \bar{G}(0, \dots, t)$ of $\langle a_1, a_2 \rangle \leq G(0, \dots, t)$, therefore any p -Sylow group of $\langle a_1, a_2 \rangle'$ and therefore $G_1(1, \dots, t - 1)$ act transitively on $\Gamma(t) - \{t - 1\}$. If $f_t = 1$, then $q \leq 3$ so that $G_1(1, \dots, t - 1) \in \text{Syl}_p(G(0, \dots, t))$ and the claim follows directly from the fact that $G(0, \dots, t)$ acts transitively on $\Gamma(t) - \{t - 1\}$. \square

From now on, we set $m = (s/2) - 1$ when s is even and $m = (s - 3)/2$ when s is odd.

(2.6) If $s \geq 4$, then $ZO_p(G(0, 1)) \leq G_m(0, 1)$, where $ZO_p(G(0, 1))$ denotes the center of $O_p(G(0, 1))$.

Proof. By (2.5), $G_1(0, 1) \neq 1$ and thus $O_p(G(0, 1)) \neq 1$. Let b be a non-trivial element in $ZO_p(G(0, 1))$. If $w \in \Gamma(1) - \{0\}$ is arbitrary, then $G_1(1, w) \leq O_p(G(0, 1))$ and thus $G_1(1, w) = G_1(1, (w)b)$, so that $b \in G(w)$ by (2.5).

Thus $b \in G_1(1)$ and similarly, $b \in G_1(0)$. Since $G_1(0, 1) \cap G(0, \dots, s-1) = 1$, there exists an $n < s$ such that $b \in G(0, \dots, n) - G(n+1)$. By (2.5), there exists a nontrivial element $a \in G_1(1, \dots, s-2) \leq O_p(G(0, 1))$. Since $b \in ZO_p(G(0, 1))$, we have $a \in G_1(s-2, s-3, \dots, n, (n+1)b, \dots, (s-2)b)$. By (2.3), the length of the path $(s-2, s-3, \dots, n, (n+1)b, \dots, (s-2)b)$ is at most $s-3$. Therefore $s-1 \leq 2n$. \square

(2.7) Suppose $s \notin \{2, 3, 4, 5, 7\}$. Then s is odd, $ZO_p(G(0, 1)) \leq G_{m+1}(0)$ or $ZO_p(G(0, 1)) \leq G_{m+1}(1)$ and G operates intransitively on the vertex set $V(\Gamma)$.

Proof. We assume first that there exists an element $b \in ZO_p(G(0, 1)) - G_1(m+1)$. Then $[b, ZO_p(G(m+1, m+2))] \leq G_1(-m+2, \dots, 2m)$ because of (2.6). Since $s \notin \{2, 3, 4, 5, 7\}$, the length of $(-m+2, \dots, 2m)$ is at least $s-2$. By (2.3), it follows that $[b, ZO_p(G(m+1, m+2))] = 1$ and therefore $ZO_p(G(m+1, m+2)) = ZO_p(G(m+1, (m+2)b))$, so that $ZO_p \cdot (G(m+1, m+2)) \trianglelefteq \langle G(m+1, m+2), G(m+1, (m+2)b) \rangle = G(m+1)$. By (2.6), we have $ZO_p(G(m+1, m+2)) \leq G_{m+1}(m+1)$.

On the other hand, if $ZO_p(G(0, 1)) \leq G_1(m+1)$, then $ZO_p(G(0, 1)) \leq G_{m+1}(1)$ since $ZO_p(G(0, 1)) \trianglelefteq G(0, 1)$ and $G(0, 1)$ acts transitively on the set of $(m+1)$ -paths beginning with $(0, 1)$. Therefore $ZO_p(G(0, 1)) \leq G_{m+1}(u)$ for $u = 0$ or 1 .

Suppose that $ZO_p(G(0, 1)) \leq G_{m+1}(0)$. Since $G_{m+1}(0) \leq G_1(-m, \dots, m)$, we have $2m \leq s-3$ so that s is odd and $G_{m+1}(0) \cap G_1(m+1) = 1$. If G contains an element c which exchanges 0 and 1 , then $ZO_p(G(0, 1)) = ZO_p(G(0, 1))^c \leq G_{m+1}(0)^c = G_{m+1}(1)$ and thus $ZO_p(G(0, 1)) \leq G_{m+1}(0) \cap G_1(m+1) = 1$, a contradiction. Therefore G acts intransitively on $V(\Gamma)$. \square

(2.8) $s \in \{2, 3, 4, 5, 7, 9, 13\}$.

Proof. We may assume that s is odd, $s \geq 9$ and $G_{m+1}(0) \neq 1$. Since $G_{m+1}(0) \neq 1$, $G_{m+1}(i) \neq 1$ for every even i . There exists an element $c \in G_{m+1}(0) - G_1(m+1)$. Suppose first that $s \equiv 3 \pmod{4}$ and thus $G_{m+1}(m+2) \neq 1$. Since $[c, G_{m+1}(m+2)] \leq G_1(-m+2, \dots, 2m) - G_1(2m+1)$, we have $3m-2 \leq s-3$, hence $s \leq 7$. Therefore $s \equiv 1 \pmod{4}$. It follows that $G_{m+1}(m+3) \neq 1$ and thus $[c, G_{m+1}(m+3)] \leq G_1(-m+4, \dots, 2m-1) - G_1(2m)$ so that $3m-5 \leq s-3$, hence $s \leq 13$. \square

Before going on to §3, we prove more lemmas needed later.

(2.9) If $s \in \{5, 7, 9, 13\}$, then $G_{m+1}(u) \leq ZO_p(G(0, 1))$ for $u = 0$ and 1 and

$G_{m+1}(u) \neq 1$ for $u = 0$ or 1 (or both); if $G_{m+1}(u) \neq 1$, then $|G_{m+1}(u)| = q$.

Proof. Let $u = 0$ or 1 . Since $G_{m+1}(u) \trianglelefteq O_p(G(0, 1))$, either $ZO_p(G(0, 1)) \cap G_{m+1}(u) \neq 1$ or $G_{m+1}(u) = 1$. Suppose that $ZO_p(G(0, 1)) \cap G_{m+1}(u)$ contains a nontrivial element b . Then $G_{m+1}(u) = \langle h_{m+u+1}^{-j} b h_{m+u+1}^j \mid 0 \leq j < f_{m+u+1} \rangle$ since $G_{m+1}(u) \cap G_1(m + u + 1) = 1$ and $G_{m+1}(u)^{\Gamma(m+u+1)} \leq O_p(G(m + u, m + u + 1))^{\Gamma(m+u+1)}$. It follows that $|G_{m+1}(u)| = q$ and $G_{m+1}(u) \leq ZO_p(G(0, 1))$ since h_{m+u+1} normalizes $ZO_p(G(0, 1))$.

It remains only to show that $G_{m+1}(u) \neq 1$ for $u = 0$ or 1 . Thus we suppose instead that $G_{m+1}(x) = 1$ for every vertex x . By (2.7), $s = 5$ or 7 . Let $s = 5$. Then $ZO_p(G(3, 4)) \leq G(2) - G_1(2)$ since otherwise $ZO_p(G(3, 4)) \leq G_2(3)$. Since h_2 normalizes $ZO_p(G(3, 4))$, $ZO_p(G(3, 4))$ acts transitively on $\Gamma(2) - \{3\}$. Since $ZO_p(G(3, 4))$ centralizes $G_1(1, 2, 3)$, we have $G_1(1, 2, 3) \leq G_2(2) = 1$, in contradiction to (2.5). Thus $s = 7$ and $ZO_p(G(i, i + 1))$ acts transitively on $\Gamma(i + 3) - \{i + 2\}$ for every i . Since $ZO_p(G(1, 2))$ centralizes $G_1(1, \dots, 5)$, we have $G_1(1, \dots, 5) \leq G_2(4)$. Since $ZO_p(G(0, 1))$ centralizes $G_1(1, \dots, 5)$, it follows that $G_1(1, \dots, 5) \leq G_3(3) = 1$, again a contradiction. \square

Thus we may suppose, from now on, that $G_{m+1}(i) \neq 1$ for every even i whenever $s \in \{5, 7, 9, 13\}$.

(2.10) *Let $s \in \{5, 7, 9, 13\}$ and $p = 2$. Then there exists an element $a \in G_1((s - 1)/2) \cap G(0, \dots, 2(s - 1)) \cap G_1(3(s - 1)/2)$ with $|a|_{s-1} = q - 1$.*

Proof. We may suppose that $q \neq 2$. Let x_1 and x_2 be any two vertices in $\Gamma(s - 1) - \{s - 2, s\}$. By (2.5), there exists for $j = 1, 2$ an element $g_j \in O_2(G(x_j, s - 1))$ such that $(i)g_j = 2(s - 1) - i$ for $s - 1 \leq i \leq 2(s - 1)$. Since $O_2(G(x_j, s - 1))$ induces an elementary abelian 2-group on $\Gamma(s - 1)$, we have $(s - 2)g_j = s$. Therefore both $(0, \dots, 2(s - 1))$ and $(2(s - 1), \dots, 0)g_j = (0, \dots, s, (s - 3)g_j, \dots, (0)g_j)$ are good paths. By (2.2), $(i)g_j = 2(s - 1) - i$ also for $0 \leq i \leq s - 3$. Let $a = g_1 g_2$. Then $|a|_{s-1} = q - 1$. By (2.9), $G_{m+1}(s - 1) \leq ZO_2(G(x_1, s - 1)) \cap ZO_2(G(x_2, s - 1))$ and thus $[a, G_{m+1}(s - 1)] = 1$. Since $s - 1$ is even, $G_{m+1}(s - 1)$ acts transitively on $\Gamma((s - 1)/2) - \{(s + 1)/2\}$. Since $a \in G((s - 3)/2)$, $a \in G_1((s - 1)/2)$. Similarly, $a \in G_1(3(s - 1)/2)$. \square

It is in the proof of the next lemma that we require [6, (8.2.11)].

(2.11) *If $p \neq 2$ and $s \geq 4$, then $|\overline{G}(W)| = |\overline{G}(\dots, -1, 0, 1, 2, \dots)|$ is even.*

Proof. We first suppose that we can choose $u \in \{0, 1\}$ such that f_u is even. The reader should check the following simple fact:

(*) Let $q - 1 = 2^k w$ with w odd. If σ is an arbitrary element in the stabilizer $P\Gamma L(2, q)_\infty$ of $\infty \in PG(1, q)$ but not in $PGL(2, q)$ whose order is a power of two, then $|\sigma| \mid 2^k$ and either $k = 2$, $|\sigma| = 4$ and $\sigma^2 \in PGL(2, q)$ or $k \geq 3$ and $\sigma^{2^{k-2}} \in PGL(2, q)$.

We choose an odd $n \in N$ such that $|h_u|/n$ is a power of two. It follows from (*) that $h_u^{nf_u/2}$ or $h_u^{nf_u} \in \bar{G}(W) - \{1\}$.

It remains to show that f_u is even for $u = 0$ or 1 . To show this, it will be necessary to make only a few minor changes in the proof of [8, (6.3)]: Suppose that both f_0 and f_1 are odd. Then $q \equiv 3 \pmod{4}$, $\bar{G}(u)^{\Gamma(u)} \cong PSL(2, q)$ for $u = 0$ and 1 and $|G(0, 1)|$ is odd. Thus a 2-Sylow group of $\bar{G}(u)$ is isomorphic to a 2-Sylow group of $PSL(2, q)$, so that $\bar{G}(u)$ is p -stable for $u = 0$ and 1 (see [6, (2.8.3), (8.1.2)]). Let $u = 0$ or 1 and $C = C_{\bar{G}(u)}(O_p(\bar{G}(u)))$, the centralizer of $O_p(\bar{G}(u))$ in $\bar{G}(u)$, and $c \in C$. Let $w \in \Gamma(u)$. Since $G_1(u, w) \leq O_p(\bar{G}(u))$, we have $G_1(u, w) = G_1(u, w)c$. By (2.5) and the hypothesis $s \geq 4$, $G_1(u, w) \not\leq G_1(z)$ for $z \in \Gamma(u) - \{w\}$. Therefore $c \in G_1(u)$, since w was arbitrary. Now let z and w be any two neighbors of u . Since $G_1(u, w)^{\Gamma(z)} = O_p(G(u, z)^{\Gamma(z)})$, we have $C^{\Gamma(z)} \leq O_p(G(u, z)^{\Gamma(z)})$. Therefore we can find elements $d \in G_1(u, w)$ and $e \in G_1(u, z)$ such that $cd = e$ and thus $c = ed^{-1} \in O_p(\bar{G}(u))$, so that $C \leq O_p(\bar{G}(u))$. Thus $O_p(\bar{G}(u)) = 1$ and $\bar{G}(u)$ is p -constrained (see [6, p. 268]).

Let $S \in \text{Syl}_p(\bar{G}(0))$. By [6, (8.2.11)], we have $J(S) \trianglelefteq \bar{G}(0)$. We may assume that $S \leq \bar{G}(1)$ and thus $S \in \text{Syl}_p(\bar{G}(1))$. Therefore $J(S) \trianglelefteq \langle \bar{G}(0), \bar{G}(1) \rangle$. Since Γ is connected, $\langle \bar{G}(0), \bar{G}(1) \rangle$ acts transitively on the set of edges of Γ and thus $J(S) = 1$, a contradiction. \square

(2.12) *If $s = 3$, then $q(q - 1)/(q - 1, 2) \mid |G_1(u) \cap \bar{G}(1 - u)|$ for $u = 0$ and 1 .*

Proof. Let $u = 0$ or 1 and $A = \langle G_1(w) \mid w \in \Gamma(u) \rangle$. Let $y \in \Gamma(u)$. Then $[G_1(u), G_1(y)] \leq G_1(u, y)$ and thus, by (2.3), $[A, G_1(u)] = 1$. By (2.5), $G_1(y)$ acts transitively on $\Gamma(u) - \{y\}$, so that $A^{\Gamma(u)} \cong PSL(2, q)$. Let a be an element in $A \cap \bar{G}(u) \cap G(y)$ such that $|a|_u = (q - 1)/(q - 1, 2)$ and $(|a|, p) = 1$. Since $[a, G_1(u)] \leq [A, G_1(u)] = 1$ and $G_1(u)$ acts transitively on $\Gamma(y) - \{u\}$, we have $a \in G_1(y)$. \square

(2.13) *Let $s = 3, q = 3, G(x)^{\Gamma(x)} \cong PGL(2, 3)$ and $|G_1(x)| = 3$ for every vertex x . Let $u = 0$ or 1 and y_1 and y_2 be vertices such that $(u, u + 1, u + 2, y_1,$*

y_2) is a good 4-path. Then $(y_2, y_1, u + 2, u + 3, u + 4)$ is also good.

Proof. Let $u = 0$ (the proof is the same when $u = 1$), $A = \langle G_1(w) \mid w \in \Gamma(2) \rangle$ and $B = \langle A, G_1(2) \rangle$; we have $[A, G_1(2)] = 1$ and $|B| = |B^{\Gamma(2)}| \cdot |G_1(2)| = 36$. Let $G_1(2) = \langle h \rangle$ and $g_1 = 1, g_2, \dots, g_{12}$ be elements of B inducing different permutations on $\Gamma(2)$ which we may choose such that $|g_i| = 2$ for $2 \leq i \leq 4$. Then three divides the order of every element in $B = \{g_i h^j \mid 1 \leq i \leq 12; 0 \leq j \leq 2\}$ except g_i for $1 \leq i \leq 4$. Thus B contains just one 2-Sylow group S . It follows that $A = \langle S, G_1(1) \rangle$, therefore $|A| = |S| \cdot |G_1(1)| = 12$ and, in particular, $A \cap G_1(2) = 1$.

Since $(0, \dots, 4)$ is good, there exists an involution $b \in G(0, \dots, 4)$. For $i = 1$ and 3, there exists an element $c_i \in G_i(i)$ mapping $4 - i$ to y_1 . Since $(0, \dots, 4)c_3 = ((0)c_3, y_1, 1, 2, 3)$ is good, we may assume that $(0)c_3 \neq y_2$. On the other hand, since both $(0, \dots, 4)c_1 = (0, 1, 2, y_1, (4)c_1)$ and $(0, 1, 2, y_1, y_2)$ are good, we have $(4)c_1 = y_2$ by (2.2). Let c be the element in $G_1(y_1)$ mapping 1 onto 3 and $d = cc^{-1}(c, cc_3^{-1})^b c_3 c$. Then $d \in A \cap G_1(2) = 1$. But $b^{c_1 c^{-1}} \in G(2, y_1, y_2)$ and $b^{c_3 c} \in G(2, y_1, (0)c_3)$ so that $d = b^{c_1 c^{-1}} b^{c_3 c} \notin G_1(y_1)$, a contradiction. \square

3. The case $s = 9$

Since $G_4(2) \leq Z O_p(G(2, 3))$ and $G_4(2)$ acts transitively on $\Gamma(6) - \{5\}$, it follows that $G_1(2, \dots, 8) \leq G_2(6)$. Choose an arbitrary element $b_{10} \in G_4(10)^* = G_4(10) - \{1\}$. For any $b_5 \in G_1(2, \dots, 8)^*$, we have $[b_5, b_{10}] \in G_1(5, \dots, 11) - G_1(12)$, therefore $[b_5, b_{10}] \notin G_1(4)$ and hence $b_5 \notin G_1((4)b_{10}^{-1})$. Let b_2 be the element in $G_4(2)$ with $(5)b_{10}^{-1} b_2 = 7$. Since $[G_4(2), G_1(2, \dots, 8)] = 1$, $b_5 = b_5^{b_2} \in G_1(2, \dots, 8) - G_1((4)b_{10}^{-1} b_2)$. Thus $G_1(2, \dots, 8) \cap G_1((4)b_{10}^{-1} b_2) = 1$.

- (3.1) a) *There exist elements $b_i \in G_1(i - 3, \dots, i + 3)^*$ for $i = 3, 4$ and 5 such that $[b_3, b_5] = b_4$.*
- b) *If $b_4 \in G_4(4)^*$ and $b_9 \in G_1(6, \dots, 12)^*$, then there exists an element $b_6 \in G_4(6)^*$ such that $[b_4, b_9] = b_6$.*
- c) *If $b_4 \in G_4(4)^*$ and $b_{10} \in G_4(10)^*$, then there exist elements $b_6 \in G_4(6)^*$, $b_7 \in G_1(4, \dots, 10)$ and $b_8 \in G_4(8)^*$ such that $[b_4, b_{10}] = b_6 b_7 b_8$.*
- d) *If $b_7 \in G_1(4, \dots, 10)^*$ and $b_{11} \in G_1(8, \dots, 14)^*$, then there exist elements $b_8 \in G_4(8)^*$, $b_9 \in G_1(6, \dots, 12)$ and $b_{10} \in G_4(10)^*$ such that $[b_7, b_{11}] = b_8 b_9 b_{10}$.*

Proof. a) We have seen that there exists a vertex $x \in \Gamma(7)$ such that $G_1(2, \dots, 8) \cap G_1(x) = 1$. Let b_3 be the element in $G_1(0, \dots, 6)$ such that $(8)b_3^{-1} = x$. Then $[b_3, b_5] \in G_4(4)^*$ for every $b_5 \in G_1(2, \dots, 8)^*$. b) is left to

the reader. c) We have $[b_4, b_{10}] \in G_1(5, \dots, 9) - G_4(4) - G_4(10)$. There exist elements $b_6 \in G_4(6)^*$ and $b_8 \in G_4(8)^*$ such that $[b_4, b_{10}]b_6^{-1}b_8^{-1} \in G_1(4, \dots, 10)$. Since $[G_4(6), G_4(8)] = [G_4(6), G_1(4, \dots, 10)] = 1$, the claim follows. d) is now clear. \square

We now suppose that $p \neq 2$. By (2,11), $\bar{G}(W)$ contains an involution a . Let $\zeta(i) = (-1)^{|a|^{i+1}}$ for each i .

(3.2) For every even i :

- A) $\zeta(i) = \zeta(i - 1)\zeta(i + 1)$
- B) $\zeta(i) = \zeta(i - 2)\zeta(i + 3)$
- C) $\zeta(i)\zeta(i + 6) = \zeta(i + 2) = \zeta(i + 4)$

Proof. A) Choose b_3, b_4 and b_5 as in (3.1.a). Since $G_1(2, \dots, 8)^{\Gamma(1)} = O_p(G(1, 2)^{\Gamma(1)})$, $G_1(2, \dots, 8)^{\Gamma(9)} = O_p(G(8, 9)^{\Gamma(9)})$ and $G_1(1, \dots, 8) = G_1(2, \dots, 9) = 1$, we have $b_5^a = b_5^{\zeta(1)} = b_5^{\zeta(9)}$ and, in particular, $\zeta(1) = \zeta(9)$. Similarly, $b_3^a = b_3^{\zeta(-1)} = b_3^{\zeta(7)}$ and $b_4^a = b_4^{\zeta(0)} = b_4^{\zeta(8)}$. We have $[b_3^{\zeta(-1)}, b_5^{\zeta(1)}] = [b_3, b_5]^{\zeta(-1)\zeta(1)}$ because $[b_3, b_4] = [b_4, b_5] = 1$. Therefore $b_4^{\zeta(1)\zeta(-1)} = b_4^a = b_4^{\zeta(0)}$ and thus $\zeta(1)\zeta(-1) = \zeta(0)$. For arbitrary even i , we find, as in (3.1.a), elements $b_{i+j} \in G_1(i + j - 3, \dots, i + j + 3)^*$ for $j = 3, 4$ and 5 such that $[b_{i+3}, b_{i+5}] = b_{i+4}$ and proceed as before. B) follows analogously from (3.1.b). C) Choose b_i for $i = 4, 6, 7, 8$ and 10 as in (3.1.c). Then $b_8^{\zeta(2)}b_7^{\zeta(3)}b_8^{\zeta(4)} = (b_6b_7b_8)^a = [b_4^a, b_{10}^a] = [b_4^{\zeta(0)}, b_{10}^{\zeta(6)}] = (b_6b_7b_8)^{\zeta(0)\zeta(6)} = b_6^{\zeta(0)\zeta(6)}b_7^{\zeta(0)\zeta(6)}b_8^{\zeta(0)\zeta(6)}$ since $[G_4(j), G_1(k - 3, \dots, k + 3)] = 1$ whenever j is even and $|j - k| \leq 4$. Thus $b_8^{\zeta(0)\zeta(6) - \zeta(2)} \in G_1(10)$. It follows that $\zeta(0)\zeta(6) = \zeta(2)$. Similarly, $\zeta(0)\zeta(6) = \zeta(4)$. \square

By (3.2.C), $\zeta(i) = \zeta(0)$ for every even i . By (3.2.B), it follows that $\zeta(i) = 1$ for i odd. Therefore, by (3.2.A), $\zeta(2) = 1$ and thus $a \in G_1(1, 2)$. By (2.4), it follows that $a = 1$, a contradiction.

Thus $p = 2$. First let $q = 2$. For each i let b_i be the nontrivial element in $G_1(i - 3, \dots, i + 3)$. Since there exists a vertex $x \in \Gamma(7)$ such that $G_1(2, \dots, 8) \cap G_1(x) = 1$ and $|\Gamma(7)| = 3$, it follows that $b_5 \notin G_1((6)b_{11})$. Similarly, $b_{11} \notin G_1((10)b_5)$. Thus $[b_5, b_{11}] \in G_1(7, 8, 9) - G_1(6) - G_1(10)$, so that $b_6b_{10}[b_5, b_{11}] \in G_1(6, \dots, 10)$ and $(b_6b_{10}[b_5, b_{11}])^2 \in G_1(5, \dots, 11)$. Since $[G_4(i), G_1(7, 8, 9)] = 1$ for $i = 6$ and 10 , $(b_6b_{10}[b_5, b_{11}])^2 = [b_5, b_{11}]^2$. If $[b_5, b_{11}]^2 = 1$, then $[b_5, [b_5, b_{11}]] = 1$ and therefore $b_5 \in G_1(2, \dots, 6, (5) [b_5, b_{11}], \dots, (2)[b_5, b_{11}])$, in contradiction to (2.3). Therefore $[b_5, b_{11}]^2 \neq 1$ and, in particular, $[b_5, b_{11}]^2 \notin G(3)$. Since $[G_4(2), G_1(2, \dots, 8)] = 1$, we have $b_5 \in G_1((8)b_2)$. If $(8)b_2$

$= (4)[b_5, b_{11}]$, then $b_5 \in G((3)[b_5, b_{11}])$ and thus $(3)[b_5, b_{11}]^2 = (3)[b_5, b_{11}]b_{11}b_5b_{11} = 3$. It follows that $(8)b_2 \neq (4)[b_5, b_{11}]$. Since $[b_5, b_{10}] \in G(4, \dots, 12) - G(13)$, we have $[b_5, b_{10}] \notin G(3)$, so that $b_5 \notin G_1((4)b_{10})$ and therefore $(4)b_{10} \neq (8)b_2$. Thus $(4)b_{10} = (4)[b_5, b_{11}]$. Hence $(b_{10}[b_5, b_{11}])^2 \in G(3)$. Since $[b_{10}, G_1(6, 7, 8)] = 1$, $[b_5, b_{11}]^2 \in G(3)$, a contradiction.

When $q > 2$, a different argument is required.

(3.3) *Let $p = 2$. For every i there exists an element $e_i \in G_i(i) \cap G(i, \dots, i + 8) \cap G_1(i + 8)$ with $|e_i|_j = q - 1$ for $i < j < i + 8$.*

Proof. By (2.10), there exists an element $a \in G_1(4) \cap G(4, \dots, 12) \cap G_1(12) \leq G(W)$ with $|a|_8 = q - 1$. Thus $q - 1 \mid |a|$. Since $a^{1015} \in G_1(4, 5)$, we have $|a| = |a|_5$ by (2.3). If $\sigma \in PGL(2, q)_\infty$ and $q - 1 \mid |\sigma|$, then $|\sigma| = q - 1$. It follows that $|a|_5 = q - 1$. Similarly, $|a|_{11} = q - 1$.

For each i let $a_i = a^{101i}$. Then $[a_i, G_1(i + 1, \dots, i + 7)] = 1$ and thus $a_i \in G_1(i + 8)$. It follows that $a_i \in G_1(j)$ whenever $j \equiv i \pmod{8}$.

By (3.1.c), we can find elements $b_i \in G_4(i)^*$ for $i = 0, 2, 4$ and 6 and an element $b_3 \in G_1(0, \dots, 6)$ such that $[b_0, b_6] = b_2b_3b_4$. Since $[b_0, b_6] = [b_0^{a^{10}}, b_6^{a^{10}}] = b_2^{a^{10}}b_3^{a^{10}}b_4^{a^{10}}, b_4^{a^{10}}b_4^{-1} \in G_1(0)$ and thus $[b_4, a_{10}] = 1$. Since $[b_i, a^j] = 1$ implies $|a|_8 = q - 1 \mid j$, we conclude that $|a|_{10} = q - 1$. Similarly, $|a|_6 = q - 1$. By (3.1.b), we can find $b_i \in G_1(i - 3, \dots, i + 3)^*$ for $i = 8, 10$ and 13 such that $[b_8, b_{13}] = b_{10}$. Then $b_{10}^{a^9} = [b_8^{a^9}, b_{13}^{a^9}] = [b_8, b_{13}] = b_{10}$ and therefore $|a|_6 = q - 1 \mid |a|_9$. It follows that $|a|_9 = q - 1$ and similarly $|a|_7 = q - 1$. Thus the claim is proven for i even.

Let c be an element in $G_1(2) \cap G(2, \dots, 10) \cap G_1(10)$ with $|c|_i = q - 1$ for $3 \leq i \leq 9$. We can choose c such that $d = ac \in G_1(3)$; let $d_i = d^{101i}$ for each i . Since $[d, G_1(4, \dots, 10)] = 1, d \in G_1(11)$. Since $a \in G_1(4)$ and $c \in G_1(10)$, we have $|d|_4 = |d|_{10} = q - 1$. By (3.1.a), we can find elements $b_i \in G(i - 3, \dots, i + 3)^*$ for $i = 7, 8$ and 9 such that $[b_7, b_9] = b_8$. Then $b_8^{d_5} = [b_7^{d_5}, b_9^{d_5}] = [b_7, b_9] = b_8$ and thus $|d|_4 = q - 1 \mid |d|_5$ so that $|d|_5 = q - 1$. Similarly, $|d|_9 = q - 1$. By (3.1.b), we can find elements $b_i \in G_1(i - 3, \dots, i + 3)^*$ for $i = 7, 10$ and 12 such that $[b_7, b_{12}] = b_{10}$. Then $b_{10}^{d_5} = [b_7^{d_5}, b_{12}^{d_5}] = [b_7, b_{12}] = b_{10}$ and so $|d|_6 \mid |d|_8$. Similarly, we have $|d|_8 \mid |d|_6$ and therefore $|d|_6 = |d|_8$. If we pick b_i ($i = 4, 6, 7, 8, 10$) as in (3.1.c), then $(b_6b_7b_8)^{d_5} = [b_4^{d_5}, b_{10}^{d_5}] = [b_4, b_{10}] = b_6b_7b_8$ and so $b_6^{d_5}b_6 \in G_4(6) \cap G_1(10) = 1$, thus $|d|_{10} = q - 1 \mid |d|_6 = |d|_8$. Finally, let b_i with $7 \leq i \leq 11$ be as in (3.1.d). Then $(b_8b_9b_{10})^{d_7} = [b_7^{d_7}, b_{11}^{d_7}] = [b_7, b_{11}] = b_8b_9b_{10}$ and therefore $b_8^{d_7}b_8 \in G_4(8) \cap G_1(12) = 1$, so that $|d|_4 = q - 1 \mid |d|_7$. \square

We are now in a position to obtain a contradiction by constructing a generalized 8-gon of order (q, q) . We will save space, however, by postponing this until later, where we include it as one case in the construction crucial to the proof of (1.2).

4. The case $s = 13$

This time we suppose first that $p = 2$. If $b_0 \in G_6(0)^*$ and $b_{10} \in G_6(10)^*$, then $[b_0, b_{10}] \in G_1(3, \dots, 7) - G_1(2) - G_1(8)$. If $-2 \leq i \leq 6$, then $\partial(0, (i)[b_0, b_{10}]) \leq 6$, so that $(i)[b_0, b_{10}]b_0 = (i)[b_0, b_{10}]$ and thus $[b_0, b_{10}]^2 \in G(i)$. If $4 \leq i \leq 12$, then $\partial(10, (i)[b_0, b_{10}]) \leq 6$, so that $(i)b_0b_{10}b_0 = (i)[b_0, b_{10}]b_{10} = (i)[b_0, b_{10}]$ and thus $[b_0, b_{10}]^2 \in G(i)$. Therefore $[b_0, b_{10}]^2 \in G(-2, \dots, 12) \cap G_1(3, \dots, 7) = 1$. It follows that $[b_0, [b_0, b_{10}]] = 1$ and hence $b_0 \in G_1(-5, \dots, 1, 2, (1)[b_0, b_{10}], \dots, (-5)[b_0, b_{10}]) = 1$. Contradiction.

Thus $p \geq 3$.

(4.1) a) *If $b_0 \in G_6(0)^*$ and $b_7 \in G_1(2, \dots, 12)^*$, then there exists an element $b_2 \in G_6(2)^*$ such that $[b_0, b_7] = b_2$.*

b) *If $b_0 \in G_6(0)^*$ and $b_8 \in G_6(8)^*$, then there exists an element $b_4 \in G_6(4)^*$ such that $[b_0, b_8] = b_4$.*

c) *If $b_0 \in G_6(0)^*$ and $b_9 \in G_1(4, \dots, 14)^*$, then there exist elements $b_i \in G_1(i - 5, \dots, i + 5)$ for $i = 2, 3, 4, 5$ and 6 with $b_6 \neq 1$ such that $[b_0, b_9] = b_2b_3b_4b_5b_6$.*

Proof. We leave a) and b) to the reader and turn to part c). Since $[G_1(4, \dots, 14), G_6(12)] = 1$ and $G_6(12)$ acts transitively on $\Gamma(6) - \{7\}$, we have $G_1(4, \dots, 14) \leq G_2(6)$. Thus $[b_0, b_9] \in G_1(1, \dots, 7) - G_1(0)$. There exist $b_2 \in G_1(-3, \dots, 7)$ and $b_6 \in G_1(1, \dots, 11)^*$ such that $[b_0, b_9]b_2^{-1}b_6^{-1} \in G_1(0, \dots, 8)$ and thus $b_i \in G_1(i - 5, \dots, i + 5)$ for $i = 3, 4$ and 5 such that $[b_0, b_9] \cdot b_2^{-1}b_6^{-1}b_5^{-1}b_3^{-1} = b_4$. Since $[b_2, b_i] = [b_4, b_i] = 1$ for $2 \leq i \leq 6$, we have $[b_0, b_9] = b_2b_3b_4b_5b_6$. \square

By (2.11), there exists an involution a in $\bar{G}(W)$. Let $\zeta(i) = (-1)^{|a_i|+1}$ for each i .

(4.2) *For every even i :*

- A) $\zeta(i - 1) = \zeta(i + 4)\zeta(i + 6)$ and $\zeta(i + 7) = \zeta(i)\zeta(i + 2)$
- B) $\zeta(i) = \zeta(i + 4)\zeta(i + 8)$
- C) $\zeta(i) = \zeta(i + 6)$ if $\zeta(i + 3) = 1$.

Proof. A) We may take $i = 2$. If b_0, b_2 and b_7 are as in (4.1.a), then $b_2^{\zeta(6)} = b_2^a = [b_0^a, b_7^a] = [b_0^{\zeta(6)}, b_7^{\zeta(1)}] = [b_0, b_7]^{\zeta(1)\zeta(6)}$ since $[b_2, b_0] = [b_2, b_7] = 1$.

Thus $\zeta(8) = \zeta(1)\zeta(6)$. By (3.1.a), we can find elements $b_i \in G_1(i - 5, \dots, i + 5)^*$ for $i = 3, 8$ and 10 such that $[b_3, b_{10}] = b_8$. Then $b_8^{\zeta(2)} = [b_3, b_{10}]^a = [b_3^{\zeta(9)}, b_{10}^{\zeta(4)}] = b_8^{\zeta(9)\zeta(4)}$. B) follows analogously from (4.1.b). For C) we assume $i = 0$ and $\zeta(3) = 1$. If b_i with $i = 0, 2, 3, 4, 5, 6$, and 9 are as in (4.1.c), then $(b_2b_3b_4b_5b_6)^a = [b_0^a, b_9^a] = [b_0^{\zeta(6)}, b_9] = [b_0, b_9]^{\zeta(6)}$ since $[b_0, b_i] = 1$ for $2 \leq i \leq 6$. Since $[b_i, b_6] = 1$ for $2 \leq i \leq 5$, we have $(b_2b_3b_4b_5b_6)^{\zeta(6)} = (b_2b_3b_4b_5)^{\zeta(6)}b_6^{\zeta(6)}$ and therefore $b_6^{\zeta(6)-\zeta(6)} = b_6^a b_6^{-\zeta(6)} \in \langle b_2, b_3, b_4, b_5 \rangle \leq G_1(0)$, so that $\zeta(0) = \zeta(6)$. \square

Suppose that $\zeta(3) = 1$. By (2.4), we have $\zeta(2) = \zeta(4) = -1$. By (4.2.C), $\zeta(0) = \zeta(6)$. By (4.2.B), $\zeta(8) = \zeta(0)\zeta(4) = -\zeta(0)$ and $\zeta(10) = \zeta(2)\zeta(6) = -\zeta(0)$. By (4.2.A), $\zeta(9) = \zeta(2)\zeta(4) = 1$, $\zeta(1) = \zeta(6)\zeta(8) = -1$ and $\zeta(11) = \zeta(4)\zeta(6) = -\zeta(0)$. Since $a \notin G_1(8, 9)$, we have $\zeta(8) = -\zeta(0) = -1$ and therefore $\zeta(6) = 1$. Since $a \notin G_1(5, 6)$ and $a \notin G_1(6, 7)$, we have $\zeta(5) = \zeta(7) = -1$.

We now choose elements b_i with $i = 0, 2, \dots, 6, 9$ as in (4.1.c). Since $\zeta(3) = \zeta(6) = 1$, we have $b_2 \cdots b_6 = [b_0, b_9] = [b_0^a, b_9^a] = (b_2 \cdots b_6)^a = b_2^{\zeta(8)}b_3^{\zeta(9)} \cdot b_4^{\zeta(10)}b_5^{\zeta(11)}b_6^{\zeta(0)} = b_2^{-1}b_3b_4^{-1}b_5^{-1}b_6$. Thus $b_5^2 \in \langle b_2, b_3, b_4 \rangle \leq G_1(-1)$, so that $b_5 = 1$. Therefore $b_4^2 \in \langle b_2, b_3 \rangle \leq G_1(-2)$, so that $b_4 = 1$ and thus $b_2 = 1$. There exists an element $g \in G$ with $(0, \dots, 13)g = (2, \dots, 15)$. Since $\zeta(1) = \zeta(2) = -1, f_i > 1$ for every i and thus, by (2.2), $(i)g = i + 2$ for every i . If $c = gag^{-1}$, then $b_3^{-1}b_6^{-1} = b_3^c b_6^c = [b_0^c, b_9^c] = [b_0^{-1}, b_9^{-1}]$. From $[b_0, b_9] = b_3b_6$ it follows that $[b_0^{-1}, b_9^{-1}] = b_9b_0b_3b_6b_0^{-1}b_9^{-1}$. Since $[b_0, b_i] = 1$ for $i = 0$ and 9 and $[b_0, b_3] = 1$, we have $b_3^{-1}b_6^{-1} = [b_0^{-1}, b_9^{-1}] = b_9b_3b_9^{-1}b_6$ and thus $b_3^{-2}b_6^{-2} = b_3^{-1}b_9b_3b_9^{-1} \in G_1(9)$. Therefore $b_3^{-2} \in G_1(-2, \dots, 9) = 1$, so that $b_3 = 1, b_6^{-2} = b_9b_9^{-1} = 1$ and thus $b_9 = 1$. Contradiction. It follows that $\zeta(3) = -1$ and thus $\zeta(i) = -1$ for every odd i .

From (4.2.A) we have that $\zeta(i) = -\zeta(i + 2)$ for every even i . Thus either $\zeta(6) = \zeta(10) = \zeta(14) = -1$ or $\zeta(8) = \zeta(12) = \zeta(16) = -1$, in contradiction to (4.2.B).

5. Proof of (1.2): Preliminaries

(5.1) *Let $q \neq 2, s \in \{4, 5, 7\}, p \neq 2$ if $s = 4$ and $G(x)^{\Gamma(x)} \cong PGL(2, 3)$ for every vertex x when $s = 5$ and $q = 3$. Let $u = 0$ or 1 . Then $G_1(u) \cap G(W) \cap \bar{G}(u + i) \not\leq G_1(u + i)$ for every i with $1 \leq i \leq s - 2$ excluding $i = (s - 1)/2$ if $q = 3$ and $s = 5$ or 7 and $i = 2$ and 4 if $q = 4$ and $s = 7$.*

Proof. Suppose $G_1(u) \cap G(W) \not\leq G_1(u + i)$ for some i . Since h_{u+i} normalizes $G_1(u) \cap G(W)$, it follows that $G_1(u) \cap G(W) \cap \bar{G}(u + i) \not\leq G_1(u$

+ i). It thus suffices to prove $G_i(u) \cap G(W) \not\leq G_i(u + i)$ to conclude that $G_1(u) \cap G(W) \cap \bar{G}(u + i) \not\leq G_1(u + i)$. We choose, once and for all, an element $g \in G$ such that $(0, \dots, s)g = (2, \dots, s + 2)$ and, in case $p \neq 2$, an involution $a \in \bar{G}(W)$; let $\zeta(i) = (-1)^{|a|i+1}$ for every i .

Suppose first that $s = 4$ and $p \neq 2$. Then $b_i^a = b_i^{\zeta(i+2)}$ for every i and every $b_i \in G_i(i, i + 1)$. For each w there exist elements $b_i \in G_i(i, i + 1)^*$ for $i = w, w + 1$ and $w + 2$ such that $[b_w, b_{w+2}] = b_{w+1}$. Then $b_{w+1}^{\zeta(w+3)} = b_{w+1}^a = [b_w, b_{w+2}]^a = [b_w^{\zeta(w+2)}, b_{w+2}^{\zeta(w+4)}] = [b_w, b_{w+2}]^{\zeta(w+2)\zeta(w+4)}$ since $[b_{w+1}, b_w] = [b_{w+1}, b_{w+2}] = 1$. Thus $\zeta(w + 3) = \zeta(w + 2)\zeta(w + 4)$. Thus there exists a k such that $\zeta(i) = 1$ iff $i \equiv k \pmod{3}$. In particular, $f_i > 1$ for every i so that, by (2.2), $(i)g = i + 2$ for every i . Therefore $ag^{-1}ag \in G_1(i)$ iff $i \equiv k + 1 \pmod{3}$.

Now let $s = 5$. Since, by assumption, $f_i > 1$ for every i , we have $(i)g = i + 2$ for every i . We claim that it would suffice to show that $G(W) \cap G_i(u) \neq 1$ for $u = 0$ or 1 when $q > 3$ and for $u = 0$ and 1 when $q = 3$. Let, for instance, $H = G(W) \cap G_1(0)$ and suppose that $H \neq 1$. If $a \in H$, then $[a, G_1(1, 2, 3)] = 1$ and thus $a \in G_1(4)$. Thus $H \leq G_1(i)$ for every $i \equiv 0 \pmod{4}$. By (2.4), we have $H \not\leq G_1(i)$ for every odd i . Let $\bar{H} = H \cap \bar{G}(1)$. By the remarks at the beginning of this proof, $\bar{H} \neq 1$. Since for each i , $[\bar{H}, h_i] \leq G_i(0, 1) \cap G(W) = 1$, $\bar{H} \leq \bar{G}(W)$ and thus $\bar{H} = H \cap \bar{G}(W) = H \cap \bar{G}(i)$ for each odd i . Suppose that $q > 3$ and $\bar{H} \leq G_1(2)$ so that $\bar{H} = \bar{G}(W) \cap G_1(i)$ for every even i . Let Σ be the graph with $V(\Sigma) = \{(0)n \mid n \in N_G(\bar{H})\}$ and $E(\Sigma) = \{\{x, y\} \mid x, y \in V(\Sigma) \text{ and } \partial(x, y) = 2\}$ and let S be the subgroup of $\text{aut}(\Sigma)$ induced by $N_G(\bar{H})$. Since $G(i, \dots, i + 4) \leq N_G(\bar{H})$ for every even i , Σ is $(S, 3)$ -transitive and $PSL(2, q) \cong S(x)^{S(x)}$ for every $x \in V(\Sigma)$. By (2.12), $(q - 1)/(q - 1, 2)$ divides $|(S_1(0) \cap \bar{S}(2) \cap S(4))^{S(2)}|$ and hence $|(H \cap \bar{G}(2))^{r(2)}|$, too. Choose an element d in $H \cap \bar{G}(2)$ with $|d|_2 = (q - 1)/(q - 1, 2)$. Then $d^r \in H \cap \bar{G}(W)$ (where $q = p^r$) and, since $r < |d|_2$, $d^r \notin G_1(2)$. This contradicts the assumption that $\bar{H} \leq G_1(2)$. It follows that there exists an element $c \in \bar{H}$ not in $G_1(2)$. By (2.3), $|c| = |c|_{-1} = |c|_1$ and so $|c|_1 = |g^{-1}cg|_1$. Since $\bar{G}(W)^{r(1)}$ is cyclic, $\langle c \rangle$ and $\langle g^{-1}cg \rangle$ induce the same permutation group on $\Gamma(1)$. Hence there exists an integer j relatively prime to $|c|$ such that $c^j g^{-1}cg \in G_1(1)$. Since $g^{-1}cg \in G_1(2)$, $|c^j g^{-1}cg|_2 = |c^j|_2 \neq 1$. Hence $G_1(1) \cap G(W) \neq 1$ and we can proceed as before. If we start by assuming $G_1(1) \cap G(W) \neq 1$, the proof is the same.

When $p = 2$, $H \neq 1$ follows from (2.10). Let $p \neq 2$. There exist elements $b_i \in G_1(i - 1, i, i + 1)^*$ for $0 \leq i \leq 3$ such that $[b_0, b_3] = b_1 b_2$. Let

$c_2 = [b_1, b_3] \in G_i(1, 2, 3) = G_2(2)$. Suppose that $\zeta(i) = -1$ for every i . Then $c_2^{-1} = c_2^a = [b_1, b_3]^a = [b_1^{-1}, b_3^{-1}] = [b_1, b_3]$ since $[c_2, b_i] = 1$ for $i = 1$ and 3 . Thus $c_2 = 1$. It follows that $[b_i, [b_0, b_3]] = 1$ for $i = 0$ and 3 , so that $b_1^{-1}b_2^{-1} = [b_0, b_3]^a = [b_0^{-1}, b_3^{-1}] = [b_0, b_3] = b_1b_2$. Therefore $b_1b_2 = 1$, so that $b_1 \in G_1 \cdot (0, 1, 2, 3) = 1$, a contradiction. We are thus finished with the case $s = 5$ when $q > 3$. Let $q = 3$. If $\zeta(1) = 1$, then $b_1^{\zeta(3)}b_2^{\zeta(0)} = b_1^a b_2^a = [b_0, b_3]^a = [b_0^{\zeta(2)}, b_3] = [b_0, b_3]^{\zeta(2)} = b_1^{\zeta(2)}b_2^{\zeta(2)}$ since $[b_0, b_i] = 1$ for $i = 1$ and 2 . Thus $\zeta(0) = \zeta(2) = \zeta(3)$. Since $a \notin G_i(0, 1)$, $\zeta(0) = -1$. Therefore $ag^{-1}ag \in G_1(2) - G_1(3)$. Thus we may suppose that $G(W) \cap G_i(i) = 1$ for every odd i . Since W is good and, by assumption, $f_0 = 2$, we may, by replacing a if necessary, assume that $\zeta(0) = -1$. Then $c_2^{-1} = c_2^a = [b_1^a, b_3^a] = [b_1^{-1}, b_3^{-1}] = [b_1, b_3]$ so that $[b_1, b_3] = 1$. Thus $b_1^{-1}b_2^{-1} = [b_0^a, b_3^a] = [b_0^{\zeta(2)}, b_3^{-1}] = [b_0, b_3]^{-\zeta(2)} = b_1^{-\zeta(2)}b_2^{-\zeta(2)}$, so that $\zeta(2) = 1$. Therefore $ag^{-1}ag \in G_i(1) - G_i(2)$, a contradiction.

Now let $s = 7$. This time we claim that it suffices to show that $G_1(u) \cap G(W) \neq 1$ for $u = 0$ or 1 when $q = 3$ or $q \geq 5$ and for $u = 0$ and 1 when $q = 4$. Let, for instance, $H = G(W) \cap G_1(1)$ and suppose that $H \neq 1$. Since $[H, G_1(2, \dots, 6)] = 1$, $H \leq G_1(7)$ and thus $H = G(W) \cap G_i(i)$ for every $i \equiv 1 \pmod{6}$ and $H \not\leq G_i(i)$ for every $i \equiv 0$ or $2 \pmod{6}$. If $H \leq G_1(4)$ and thus $H = G(W) \cap G_i(i)$ for every $i \equiv 1 \pmod{3}$, we obtain a contradiction from (2.12) as in the case $s = 5$ (when $q \neq 3$). Let $\bar{H} = H \cap \bar{G}(2)$. As in the case $s = 5$, $\bar{H} = H \cap \bar{G}(W) = H \cap \bar{G}(i)$ for every $i \equiv 0$ or $2 \pmod{6}$. Suppose that $\bar{H} \leq G_1(3)$. Let c be an element with $(i)c = 8 - i$ for $1 \leq i \leq 7$. Since $\bar{H} = \bar{G}(1, \dots, 7) \cap G_1(1) = \bar{G}(1, \dots, 7) \cap G_1(7)$, c normalizes \bar{H} . Thus $\bar{H} \leq G_1(5)$ and hence $\bar{H} = \bar{G}(W) \cap G_1(i)$ for every odd i . Let Σ be the graph with $V(\Sigma) = \{(1)n \mid n \in N_c(\bar{H})\}$ and $E(\Sigma) = \{\{x, y\} \mid x, y \in V(\Sigma) \text{ and } \partial(x, y) = 2\}$ and let S be the subgroup of $\text{aut}(\Sigma)$ induced by $N_c(\bar{H})$. Then $PSL(2, q) \cong S(x)^{\Sigma(x)}$ for every $x \in V(\Sigma)$ and Σ is locally $(S, 4)$ -transitive. We may thus conclude that $(q - 1)/(q - 1, 3)$ divides $|(S_1(1) \cap \bar{S}(3) \cap S(5))^{\Sigma(3)}|$ from the very theorem (i.e., (1.2)) we are busy proving, paying attention that we never use the case $s = 7$ in the proof of the case $s = 4$. This contradicts the assumption that $\bar{H} \leq G_1(3)$ as in the case $s = 5$ if $q \neq 4$. In particular, $f_i > 1$ for every i and thus $(i)g = i + 2$ for every i . Exactly as in the case $s = 5$, we can find an element $c \in \bar{H}$ and an integer j such that $c^j g^{-1} c g \in G_i(0) \cap G(W)^*$ (if $q \neq 4$). Thus we can proceed as before.

If $p = 2$, $G_1(1) \cap G(W) \neq 1$ follows from (2.10). Suppose $q = 4$. If $a = (h_1)^2$,

then $a \in \bar{G}(W)$ and $a \notin G_1(1)$. There exists an element $b \in G_1(1) \cap G(W)$ such that $ab \in G_1(0)$. Hence $G_1(0) \cap G(W) \neq 1$. Finally, suppose that $p \neq 2$. Suppose $\zeta(i) = -1$ for every i . There exist elements $b_i \in G_3(i)^*$ for $i = 0, 2$ and 4 such that $[b_0, b_4] = b_2$. Thus $b_2^{\zeta(5)} = b_2^2 = [b_0, b_4]^a = [b_0^{\zeta(3)}, b_4^{\zeta(7)}] = [b_0, b_4]^{\zeta(3)\zeta(7)}$ since $[b_2, b_i] = 1$ for $i = 0$ and 4 . Thus $-1 = \zeta(5) = \zeta(3) \cdot \zeta(7) = +1$. Contradiction. \square

In the next lemma, we include the case $s = 9, p = 2$ and $q \geq 4$, continuing from where we left off in § 3.

(5.2) *Let $q > 2, s \in \{4, 5, 7\}$ or $s = 9$ and $p = 2$ and $G(x)^{\Gamma(x)} \cong PGL(2, 3)$ for every vertex x when $q = 3$ and $s = 5$. Let $u = 0$ or 1 and y_1, \dots, y_{s-1} be vertices with $y_1 \neq u + s$ such that $(u, u + 1, \dots, u + s - 1, y_1, \dots, y_{s-1})$ is a good $2(s - 1)$ -path. Then $(y_{s-1}, \dots, y_1, u + s - 1, u + s, \dots, u + 2(s - 1))$ is a good $2(s - 1)$ -path.*

Proof. By (2.1), there exist vertices y'_2, \dots, y'_{s-1} such that $(y'_{s-1}, \dots, y'_2, y_1, u + s - 1, u + s, \dots, u + 2(s - 1))$ is a good $2(s - 1)$ -path.

We first assume that $s = 4$ and $p = 2$. By (2.5), $\langle G_1(y_1, y_2), G_1(3 + u, y_1) \rangle$ contains an element a with $(1 + u, 2 + u)a = (5 + u, 4 + u)$. Since $[G_1(y_1, y_2), G_1(3 + u, y_1)] \leq G_1(3 + u, y_1, y_2) = 1$, a is an involution. By (2.2), a exchanges u and $6 + u$. Thus a exchanges y_2 and y'_2 . But $a \in G_1(y_1)$ so that $y_2 = y'_2$. Now taking $(6 + u, 5 + u, 4 + u, 3 + u, y_1, y_2, y'_3)$ in place of $(u, u + 1, \dots, u + 6)$, $2 + u$ in place of y_1 and $1 + u$ in place of y_2 , we conclude that $(1 + u, 2 + u, 3 + u, y_1, y_2, y'_3)$ is good. Since $(1 + u, 2 + u, 3 + u, y_1, y_2, y_3)$ is also good, it follows from (2.2) that $y_3 = y'_3$.

We may thus assume that $p \neq 2$ if $s = 4$. By (3.3) and (5.1), there exists an element $a \in G_1(u + s - 1) \cap \bar{G}(y_1) \cap G(y_2) - G_1(y_1)$ with $(|a|, p) = 1$. Since $(|a|, p) = 1$, there exists an $(s - 1)$ -path $(x_u, x_{u+1}, \dots, x_{u+s-2}, x_{u+s-1})$ with $x_{u+s-1} = u + s - 1, x_{u+s-2} \neq y_1$ and $a \in G(x_u, x_{u+1}, \dots, x_{u+s-2}, x_{u+s-1})$. Since Γ is locally (G, s) -transitive, we may assume that $x_i = i$ for $u + 1 \leq i \leq u + s - 2$. By (2.2), $x_u = u$ since $f_x > 1$ for every vertex x , by assumption when $s = 5$ and by (5.1) when $s \in \{4, 7\}$. Since $a \in G(u, \dots, u + s), a \in G(u, \dots, u + 2(s - 1))$ and thus $a \in G(y'_2)$. But y_2 is the only fixed point of a in $\Gamma(y_1) - \{u + s - 1\}$. Thus $y_2 = y'_2$. Again using (3.3) and (5.1), we can find an element in $G(u, \dots, u + s - 1) \cap G_1(u + s - 1) \cap G(y_1, y_2, y_3) \cap \bar{G}(y_2) - G_1(y_2)$, so that $y_3 = y'_3$. Continuing, we obtain $y_i = y'_i$ for $1 \leq i \leq s - 1$ except when $q = 3$ and $s \in \{5, 7\}$ or $q = 4$ and $s = 7$.

If $q = 3$ and $s \in \{5, 7\}$, we have only $y_i = y'_i$ for $1 \leq i \leq v$ where $v = (s - 1)/2$ from (5.1). If we knew that $G_i(u) \not\leq G(W) \cap G_i(u + v)$ (which, however, a posteriori is not the case), we would be finished as before. Thus we may assume that $G_i(u) \cap G(W) = G_i(i) \cap G(W)$ for every $i \equiv u \pmod{v}$. Let $H = G_i(u) \cap G(W)$, $S = N_G(H)/H$ and Σ be the graph with $V(\Sigma) = \{(u)n | n \in N_G(H)\}$ and $E(\Sigma) = \{\{x, y\} | x, y \in V(\Sigma) \text{ and } \partial(x, y) = v\}$. The graph Σ is locally $(S, 3)$ -transitive. Since $S(x)^{x(x)} \cong PGL(2, 3)$ and $|S_i(x)| = 3$ for every vertex, there exists, by (2.13), an involution in $S(y_v, \dots, y_1, u + 2v, u + 3v, u + 4v)$. Thus there exists an element in $G(y_v, \dots, y_1, u + s - 1, u + s, \dots, u + 2(s - 1))$ whose only fixed point in $\Gamma(y_v) - \{y_{v-1}\}$ is y_{v+1} . Thus $y_{v+1} = y'_{v+1}$. Using (5.1), we can then conclude that $y_i = y'_i$ for $v + 2 \leq i \leq s - 1$.

If $q = 4$ and $s = 7$, we may assume that $G_i(u) \cap G(W) = G_i(i) \cap G(W)$ for every $i \equiv u \pmod{2}$. Let $H = G_i(u) \cap G(W)$, $S = N_G(H)/H$ and Σ be the graph with $V(\Sigma) = \{(u)n | n \in N_G(H)\}$ and $E(\Sigma) = \{\{x, y\} | x, y \in V(\Sigma) \text{ and } \partial(x, y) = 2\}$. The graph Σ is $(S, 4)$ -transitive. By the case $s = 4$ of the lemma we are busy proving, $(y_6, y_4, y_2, u + 6, u + 8, u + 10, u + 12)$ is a good 6-path in Σ . It follows that $(y_6, y_5, \dots, y_2, y_1, u + 6, u + 7, \dots, u + 12)$ is a good 12-path in Γ . \square

6. Proof of (1.2): The construction

We assume that $q \neq 2, f_x = 2$ for every vertex x when $s = 5$ and $q = 3$ and $s \in \{4, 5, 7\}$ or $s = 9$ and $p = 2$. For each $i \in N$ and each vertex x , let $\Gamma_i(x) = \{y | \partial(x, y) \leq i\}$. We point out that the girth of Γ is at least $2(s - 1)$ (see, for instance, [10, p. 61]). Let $F = \Gamma_{s-2}(0) \cup \Gamma_{s-2}(1)$ and Π be the undirected graph with vertex set $V(\Pi) = F$ and $\{x, y\} \in E(\Pi)$ iff x or y or both are in $\Gamma_{s-3}(0) \cup \Gamma_{s-3}(1)$ and $x \in \Gamma(y)$ or there exists a good $(2s - 3)$ -path (x_0, \dots, x_{2s-3}) with $x_{s-2} = 0, x_{s-1} = 1$ and either $x_0 = x$ and $x_{2s-3} = y$ or $x_0 = y$ and $x_{2s-3} = x$. By (2.2), Π is regular of valency $q + 1$. Let $P = \text{aut}(\Pi)$.

Let a be any element in $G(1) - G(0)$. We define a permutation \hat{a} of F as follows: If $x \in \Gamma_{s-2}(1)$, we set $(x)a = (x)\hat{a}$. If $x \in F - \Gamma_{s-2}(1)$, we set $(x)\hat{a} = (x_{2(s-1)})a$, where $(x_0, \dots, x_{2(s-1)})$ is the uniquely determined $2(s - 1)$ -path with $x_0 = x, x_{s-2} = 0, x_{s-1} = 1$ and $x_s = (0)a^{-1}$. It is straightforward to check, using (5.2), that \hat{a} is an element of P . Thus $P(1) \not\leq P(0)$. Similarly, $P(0) \not\leq P(1)$.

If $a \in G(\{0, 1\})$, then clearly the permutation which a induces on F is

an element of P . Since, for $u = 0$ and 1 , $P(u) \not\leq P(1 - u)$, it follows that $P(u)$ acts transitively on $\Pi(u)$. Since Π is connected, P acts transitively on $E(\Pi)$. Thus the girth of Π is $2(s - 1)$ and Π is the incidence graph of a generalized $(s - 1)$ -gon of order (q, q) . By [3], $s \in \{4, 5, 7\}$. Since, by (2.5) and (2.9), P contains sufficiently many “generalized elations”, it follows from [5, Theorem 1.8], [7, Theorem 2] and [12, (4.4)] that $\Pi \cong \Gamma_{s-1,q}$ and $P \cong G_{s-1,q}$.

Let $u = 0$ or 1 . We have seen that for each $a \in G(u)$ there exists an element $\hat{a} \in P(u)$ such that a and \hat{a} agree on $\Gamma_{s-2}(u)$. The map τ mapping a onto \hat{a} is an injective homomorphism from $G(u)$ into $P(u)$. For each $w \in \Gamma(u)$, an element $a \in G(u, w)$ lies in $O_p(G(u, w))$ iff for $i = u$ and w , a induces a permutation on $\Gamma(i)$ contained in $O_p(G(u, w)^{\Gamma(i)})$. Thus τ maps $O_p(G(u, w))$ into $O_p(P(u, w))$. But, by (2.3) and (2.5), $|O_p(G(u, w))| = q^{s-1} = |O_p(P(u, w))|$. Theorem (1.2) follows now from the next lemma whose proof is left to the reader:

(6.1) *Let $n = s - 1$ and (X, Y) be a 1-path in $\Gamma_{n,q}$. For $U = X$ and Y , let $\tilde{G}_{n,q}(U) = \langle O_p(G_{n,q}(U, W)) \mid W \in \Gamma_{n,q}(U) \rangle$. Then $\tilde{G}_{n,q}(U) \leq H_{n,q}(U)$ for $U = X$ and Y and $H_{n,q}(X, Y) = \langle \tilde{G}_{n,q}(X) \cap G_{n,q}(Y), G_{n,q}(X) \cap \tilde{G}_{n,q}(Y) \rangle$.*

7. Proof of (1.3)

When $q = 2$, we are in the unfortunate situation that every path is a good path, so that the construction used in the proof of (1.2) does not work. We leave undecided the question whether (1.2)—with an appropriate clause for the exceptional case $s = 4$ and $G(k) \cong \hat{G}_{4,2}(K)$ —nevertheless remains true when $q = 2$.

First let $s = 4$ and, for every i , b_i be the nontrivial element in $G_i(i, i + 1)$. Then $[b_i, b_{i+2}] = b_{i+1}$ for every i . We have $|b_0 b_3|_2 = 3$. Thus $(b_0 b_3)^3 \in G_1(2) = \langle b_1, b_2 \rangle$ and therefore $(b_0 b_3)^6 = 1$. Suppose $(b_0 b_3)^3 \neq 1$. Let $a \in G$ be an element with $(0, \dots, 4)a = (0, 1, 2, (1)b_3, (0)b_3)$. Then $b_3^a = b_0^a$ and hence $((b_0 b_3)^3)^a = (b_0 b_3 b_0 b_3)^3 = (b_0 b_3)^6 = 1$, a contradiction. Thus $G(x) \cong \langle t_0, t_1, t_2, t_3 \mid t_i^2 = 1 \text{ for } 0 \leq i \leq 3; [t_i, t_j] = 1 \text{ if } |i - j| = 1; [t_i, t_{i+2}] = t_{i+1} \text{ for } i = 0 \text{ and } 1; (t_0 t_3)^3 = 1 \rangle$ for every vertex x . If Γ is $(G, 4)$ -transitive, then there exists an element $c \in G$ with $(0, 1, \dots, 4)c = (5, 4, \dots, 1)$. Thus $c^2 \in G(1, \dots, 4) = \langle b_2 \rangle$ and $cb_i c = b_{4-i}$ for $1 \leq i \leq 3$. We have $G(\{2, 3\}) \cong G_{3,2}(\{X, Y\})$ if $c^2 = 1$ and $G(\{2, 3\}) \cong \hat{G}_{4,2}(K)$ otherwise.

Let $s = 5$ and, for every i , b_i be the nontrivial element in $G_i(i - 1,$

$i, i + 1)$. Then $[b_i, b_{i+3}] = b_{i+1}b_{i+2}$. Since $G_1(i - 1, i, i + 1) = G_2(i)$ for every even i , we have $[b_i, b_j] = 1$ when $|i - j| \leq 2, i$ even. Suppose $[b_1, b_3] = b_2$. Then $[b_0, b_3] = (b_0b_3)^2 = b_1b_2 = b_1(b_1b_3)^2 = b_3b_1b_3$ and thus $b_0b_3b_0 = b_3b_1$. Squaring both sides, we have $1 = (b_3b_1)^2 = b_2$, a contradiction. Thus $[b_i, b_j] = 1$ when $|i - j| \leq 2, i$ arbitrary. If Γ is $(G, 5)$ -transitive, then there exists an element $c \in G$ with $(0, \dots, 5)c = (5, \dots, 0)$ and thus $c^2 = 1$ and $cb_ic = b_{i-i}$ for $1 \leq i \leq 4$. Thus the structure of $G(\{2, 3\})$ is completely determined. We have $(b_1b_3)^3 \in G_1(3) = \langle b_2, b_3, b_4 \rangle$ and thus $(b_1b_3)^6 = 1$. Suppose that $(b_1b_3)^3 \neq 1$. Let $b'_5 = b_3b_1b_3$. Then $(b_1b'_5)^3 = 1$. There exists, however, an element $a \in G$ with $(1, \dots, 5)a = (1, 2, 3, (2)b_5, (1)b_5)$ and thus $b_1^a = b_1$ and $b_5^a = b'_5$ since $[b_i, b_{i+2}] = 1$ and thus $G_2(i) = \langle b_i \rangle$ for every i . Thus $(b_1b'_5)^a = b_1b'_5$. Contradiction. It follows that $(b_1b_3)^3 = 1$. Similarly, $(b_0b_4)^3 = 1$. Thus $G(x) \cong \langle t_0, \dots, t_4 \mid t_i^2 = 1 \text{ for } 0 \leq i \leq 4; [t_i, t_j] = 1 \text{ if } |i - j| \leq 2, [t_i, t_{i+3}] = t_{i+1}t_{i+2} \text{ for } i = 0 \text{ and } 1; (t_0t_4)^3 = 1 \rangle$ for every vertex x .

Now let $s = 7$ and, for every i, b_i be the nontrivial element in $G_1(i - 2, \dots, i + 2)$. For every even $i, G_1(i - 2, \dots, i + 2) = G_3(i)$ and thus $[b_i, b_j] = 1$ when $|i - j| \leq 3$ and $[b_i, b_{i+4}] = b_{i+2}$. Also, there exist u and $v \in \{0, 1\}$ such that $[b_0, b_6] = b_1b_2^u b_3^v b_4$. Hence $b_5b_0b_5 = b_0b_1b_2^u b_3^v b_4$. Squaring both sides, we have $(b_0b_1b_2^u b_3^v b_4)^2 = 1$. Since $(b_0b_4)^2 = b_2, [b_2, b_i] = 1$ for $0 \leq i \leq 4$ and $[b_i, b_j] = 1$ for $i \in \{1, 3\}$ and $j \in \{0, 4\}$, we have $b_2(b_1b_3^v)^2 = 1$. Thus $v = 1$ and $(b_1b_3)^2 = b_2$. Therefore $(b_i b_{i+2})^2 = b_{i+1}$ for every odd i . In particular, $b_i \notin G_2(i - 1)$ and $b_i \notin G_2(i + 1)$ whenever i is odd. It follows that $[b_1, b_5] \in G(1, \dots, 5) - G_1(1) - G_1(5)$ and thus $[b_1, b_5] = b_2b_3^w b_4$ with $w \in \{0, 1\}$. Therefore $b_5b_1b_5 = b_1b_2b_3^w b_4$. Squaring both sides, we have $1 = (b_1b_3^w)^2$ and thus $w = 0$.

Suppose $(b_0b_6)^3 \in G_1(3) = \langle b_1, \dots, b_5 \rangle$ has even order. Let $b'_6 = b_6b_0b_6$. Then $|b_0b'_6| = |b_0b_6|/2$. There exists, however, an element $a \in G$ with $(0, \dots, 6)a = (0, \dots, 3, (2)b_6, \dots, (0)b_6)$ and thus $(b_0b'_6)^a = b_0b'_6$. Contradiction.

Let (x_0, \dots, x_8) be an arbitrary 8-path. Since $|G(x_1, \dots, x_7)| = 2$, there exist exactly two elements g_1 and g_2 such that $(x_1, \dots, x_i)g_i = (x_7, \dots, x_1)$ for $i = 1$ and 2 . If d is any involution in $G(x_i) - G_1(x_i)$, then there exists a 6-path (y_1, \dots, y_7) with $y_4 = x_4$ such that $(y_i)d = y_{8-i}$ for $1 \leq i \leq 7$. Since G contains an element mapping (y_1, \dots, y_7) onto (x_1, \dots, x_7) , g_1 and g_2 must be involutions. If $(x_8)g_1 = (x_8)g_2$, then $g_1g_2 \in G(x_0, \dots, x_8) = 1$, a contradiction. Thus $(x_0, \dots, x_8)g_i = (x_8, \dots, x_0)$ for $i = 1$ or 2 .

Thus there exists an element g mapping $(-1, \dots, 7)$ onto $(7, \dots, -1)$. Since $G_1(i - 2, \dots, i + 2) = G_3(i)$ for even $i, G_1(i - 2, \dots, i + 2)^g = G_1(4 - i,$

$\dots, 8 - i)$ for $0 \leq i \leq 6$ and thus $[b_1, b_6] = [b_5, b_0]^9 = (b_4 b_3 b_2^9 b_1)^9 = b_2 b_3 b_4^9 b_5$. Therefore, for $u = 0$ or 1 , $G(3) = \langle b_0, \dots, b_6 \rangle \cong H_u$ where $H_u = \langle t_0, \dots, t_6 | t_i^2 = 1$ for $0 \leq i \leq 6$; $[t_i, t_j] = 1$ for $|i - j| \leq 3$, i even; $[t_i, t_{i+4}] = t_{i+2}$ for $i = 0$ and 2 ; $[t_i, t_{i+2}] = t_{i+1}$ for $i = 1$ and 3 ; $[t_1, t_5] = t_2 t_4$; $[t_0, t_5] = t_1 t_2^9 t_3 t_4$; $[t_1, t_6] = t_2 t_3 t_4^9 t_5$; $(t_0 t_6)^3 = 1 \rangle$. The map $\xi: \{t_0, \dots, t_6\} \rightarrow H_0$ given by $(t_1)\xi = t_1 t_2$ and $(t_i)\xi = t_i$ for $0 \leq i \leq 6$, $i \neq 1$, induces an isomorphism from H_1 onto H_0 . Thus the structure of $G(3)$ (and therefore also that of $G(2, 3) = \langle b_0, \dots, b_5 \rangle$) is uniquely determined. Since $b_1 \notin G_2(0)$ and $G_3(1) \leq G_1(-1, \dots, 3) = \langle b_1 \rangle$, $G_3(1) = 1$. Thus Γ cannot be $(G, 7)$ -transitive.

Let h be the involution mapping $(0, \dots, 8)$ onto $(8, \dots, 0)$. Let $x_i = i$ for $0 \leq i \leq 9$, $x_{-1} = (9)h$ and c_i be the nontrivial element in $G_1(x_{i-2}, \dots, x_{i+2})$ for $i = 1$ and 7 . Suppose that $|c_1 c_7|$ is even. If we set $y_i = x_i$ for $-1 \leq i \leq 4$, $y_i = (x_{8-i})c_7$ for $5 \leq i \leq 9$ and let d_i be the nontrivial element in $G_1(y_{i-2}, \dots, y_{i+2})$ for $i = 1$ and 7 , then $|d_1 d_7| = |c_1 c_7|/2$. In addition, $(y_i)c_7 = y_{8-i}$ for $-1 \leq i \leq 9$. Repeating, if necessary, we obtain a 10-path (z_{-1}, \dots, z_9) with $z_4 = 4$ such that there exists an involution a with $(z_i)a = z_{8-i}$ for $-1 \leq i \leq 9$ and $|e_1 e_7| = 3$ where e_i is the nontrivial element in $G_1(z_{i-2}, \dots, z_{i+2})$ for $1 \leq i \leq 7$. There exists a $w \in \{0, 1\}$ such that $[e_2, e_7] = e_3 e_4^w e_5 e_6$. Thus $[e_1, e_6] = [e_7, e_2]^a = (e_6 e_5 e_4^w e_3)^a = e_2 e_3 e_4^w e_5$. Therefore $G(4) = \langle e_1, \dots, e_7 \rangle \cong J_w$ where $J_w = \langle t_1, \dots, t_7 | t_i^2 = 1$ for $1 \leq i \leq 7$; $[t_i, t_j] = 1$ for $|i - j| \leq 3$, i even; $[t_2, t_6] = t_4$; $[t_i, t_{i+2}] = t_{i+1}$ for $i = 1, 3$ and 5 ; $[t_i, t_{i+4}] = t_{i+1} t_{i+3}$ for $i = 1$ and 3 ; $[t_1, t_6] = t_2 t_3 t_4^w t_5$; $[t_2, t_7] = t_3 t_4^w t_5 t_6$; $(t_1 t_7)^3 = 1 \rangle$. The map $\theta: \{t_1, \dots, t_7\} \rightarrow J_0$ given by $(t_5)\theta = t_1 t_5$ and $(t_i)\theta = t_i$ for $1 \leq i \leq 7$, $i \neq 5$, induces an isomorphism from J_1 onto J_0 . Thus the structure of $G(4)$ is uniquely determined.

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