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DERIVED INTERTWINING NORMS FOR REDUCIBLE SPHERICAL PRINCIPAL SERIES

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Abstract

We use the second derivative of intertwining operators to realize a unitary structure for the irreducible subrepresentations in the reducible spherical principal series of $U(1, n)$. These representations can also be realized as the kernels of certain invariant first-order differential operators acting on sections of homogeneous bundles over the hyperboloid $(U(1) \times U(n))\backslash U(1, n)$.

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1. Introduction

We describe a unitary structure on the irreducible subrepresentations of the reducible spherical principal series of the representations of the reductive group $U(1, n)$. This is achieved by taking the second derivative of the intertwining operator, restricted to the kernel of its first derivative. We give an elementary argument based on the binomial theorem, as was done by Hansen [9] in his proof of the unitarity of the complementary series. The case of $SU(1, 2)$ had been treated by Fabec [3] using the non-compact picture and Fourier analysis on the Heisenberg group.

We also present a connection between these irreducible representations and another realization, in the kernel of an invariant first-order differential operator acting on

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sections of a homogeneous bundle over the hyperboloid $(U(1)\times U(n))\backslash U(1,n)$. This is done by a Cauchy-Szegö map.

2. Notation

Fix an integer $n\geq 2$ and view the elements of $\mathbb{C}^{n+1}$ as rows with entries indexed by $\{0,1,\dots,n\}$. Equip $\mathbb{C}^{n+1}$ with the form

$$\langle\zeta,\xi\rangle=\zeta_0\overline{\xi_0}-\sum_{j=1}^{n}\zeta_j\overline{\xi_j},\qquad \forall\zeta,\xi\in\mathbb{C}^{n+1}.$$

We will be examining some representations of the reductive group

$$G^1=U(1,n)=\left\{g\in GL(n+1,\mathbb{C})\ :\ \langle\zeta,\xi\rangle=\langle\zeta g,\xi g\rangle,\ \forall\zeta,\xi\in\mathbb{C}^{n+1}\right\}.$$

This group acts on $\mathbb{C}^{n+1}$ by multiplication on the right, and hence on the cone

$$\Phi=\left\{\xi\in\mathbb{C}^{n+1}\ :\ \xi\neq 0,\ \langle\xi,\xi\rangle=0\right\}.$$

Let us consider the point $\xi^0=(1,0,\dots,0,-1)$ as an origin in Φ. Similarly, G^1 acts on the hyperboloid $\{\xi\ :\ \langle\xi,\xi\rangle=1\}$, with $\xi^1=(1,0,\dots,0)$ as an origin. The isotropy subgroup of ξ^1 is

$$K^1=\left\{\begin{pmatrix}1&0\\0&u\end{pmatrix}\ :\ u\in U(n)\right\}.$$

We now list some important subgroups of G^1:

$$A=\left\{a(y)=\begin{pmatrix}\cosh(y)&0&\sinh(y)\\0&I_{n-1}&0\\\sinh(y)&0&\cosh(y)\end{pmatrix}\ :\ y\in\mathbb{R}\right\}\cong\mathbb{R};$$

the Heisenberg group

$$N=\left\{n(z,t)=\begin{pmatrix}1-it+\frac{1}{2}|z|^2&z&it-\frac{1}{2}|z|^2\\z^*&I_{n-1}&-z^*\\-it+\frac{1}{2}|z|^2&z&1+it-\frac{1}{2}|z|^2\end{pmatrix}\ :\ z\in\mathbb{C}^n,t\in\mathbb{R}\right\};$$

the centre

$$L^1=\left\{\ell_\theta=e^{-i\theta}I_{n+1}\ :\ \theta\in\mathbb{R}\right\}\cong\mathbb{T};$$

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the image of N under the Cartan involution,

$$V = \left\{ v(z,t) = \begin{pmatrix} 1 + it + \frac{1}{2}|z|^2 & z & it + \frac{1}{2}|z|^2 \\ z^* & I_{n-1} & z^* \\ -it - \frac{1}{2}|z|^2 & -z & 1 - it - \frac{1}{2}|z|^2 \end{pmatrix} : z \in \mathbb{C}^n, t \in \mathbb{R} \right\};$$

and the centralizer of A in K^1,

$$M^1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{pmatrix} : u \in U(n-1) \right\}.$$

It is known that $G^1 = L^1 ANK^1$; see [7, 9]. The semisimple subgroup $G = SU(1,n)$ is $G = ANK$, where $K = L^1K^1 \cap SU(n+1)$. Note that $\xi^0 a(y) = e^{-y}\xi^0$ and $\xi^0 n(z,t) = \xi^0$.

The subgroup M^1N fixes ξ^0 while the orbit

$$\xi^0 V = \left\{ \xi^0 v(z,t) = \left(1 + 2it + |z|^2, 2z, -1 + 2it + |z|^2\right) : z \in \mathbb{C}^n, t \in \mathbb{R} \right\}$$

is an open dense subset of the intersection $\Phi \cap \{\xi \in \mathbb{C}^{n+1} : (\xi - \xi^0)(\xi^0)^* = 0\}$.

The unit sphere S^{2n-1} in $\mathbb{C}^n$ will be identified with the intersection

$$\mathbb{S} = \Phi \cap \{\xi \in \mathbb{C}^{n+1} : \xi_0 = 1\}$$

via

$$(1,x) \in \Phi \cap \{\xi \in \mathbb{C}^{n+1} : \xi_0 = 1\} \quad \text{if and only if} \quad x \in \mathbb{C}^n \text{ and } |x| = 1.$$

We will often write x to mean the element $(1,x)$ in $\mathbb{S}$. The compact group K^1 acts transitively on $\mathbb{S}$; this is the action of $U(n)$ on S^{2n-1}. Normalize the $U(n)$-invariant measure μ on S^{2n-1} to have total mass one.

Following Hansen [9], we define a function $c : \mathbb{C}^{n+1} \times G^1 \to \mathbb{C}$ by

$$c(\xi, g) = \text{ the } 0^{\text{th}} \text{ coordinate of } \xi g.$$

For every ξ in the cone Φ we have that $(c(\xi,g))^{-1}(\xi g)$ is in the sphere $\mathbb{S}$. This leads to an action of G^1 on S^{2n-1}, namely, $(x,g) \mapsto x \cdot g$ with

$$(1) \qquad (1,x)g = (c(\xi,g))^{-1}(1, x\cdot g), \qquad \forall x \in S^{2n-1}, g \in G^1.$$

Note that $c(\xi, g) = 1$ for all $\xi \in \mathbb{S}$ and $k \in K^1$. If $x \in S^{2n-1}$ and $g \in G^1$, write $c(x,g)$ to mean $c((1,x),g)$. Lemma 4 in [9] states that if g and g' are in G^1 and x is in S^{2n-1}, then

$$(2) \qquad c(x, gg') = c(x\cdot g, g')c(x,g).$$

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In particular, taking $g' = g^{-1}$ shows that

(3) $$1 = c(x,1) = c\left(x\cdot g, g^{-1}\right)c(x,g) \quad \text{and so} \quad c(x,g)^{-1} = c\left(x\cdot g, g^{-1}\right).$$

The effect this has on the measure μ is described in [9, Lemma 21].

LEMMA 1. *If $g \in G^1$ and F is a continuous function on S^{2n-1} then*

$$\int_{S^{2n-1}} F(x\cdot g^{-1})\,d\mu(x) = \int_{S^{2n-1}} F(x)|c(x,g)|^{-2n}\,d\mu(x).$$

3. Homogeneous functions

A function $f : \Phi \to \mathbb{C}$ is said to be homogeneous of degree α if

(4) $$f(s\xi) = |s|^{\alpha} f(\xi), \qquad \forall s \in \mathbb{C}\setminus\{0\}, \xi \in \Phi.$$

When we compose a homogeneous function with multiplication by an element of G^1, we have

(5) $$f(s\xi g) = |s|^{\alpha} f(\xi g), \qquad \forall s \in \mathbb{C}\setminus\{0\}, \xi \in \Phi, g \in G^1.$$

Furthermore, a homogeneous function is determined by its restriction to the sphere,

(6) $$f(\xi_0, \xi_1, \dots, \xi_n) = |\xi_0|^{\alpha} f(1, \xi_1/\xi_0, \dots, \xi_n/\xi_0), \qquad \forall \xi \in \Phi.$$

When f is homogeneous of degree α, we can combine (5) and (6) to obtain

(7) $$f((1,x)g) = f(c(x,g)(1, x\cdot g)) = |c(x,g)|^{\alpha} f(1, x\cdot g),$$

for every $g \in G^1$ and $x \in S^{2n-1}$. Furthermore, when we take into account Lemma 1, we have that

(8)
$$\begin{aligned}\int_{S^{2n-1}} |f((1,x)g)|^2\,d\mu(x) &= \int_{S^{2n-1}} |c(x,g)|^{2\Re(\alpha)}\,|f(1,x\cdot g)|^2\,d\mu(x)\\ &= \int_{S^{2n-1}} |c(x\cdot g, g^{-1})|^{-2\Re(\alpha)}\,|f(1,x\cdot g)|^2\,d\mu(x)\\ &= \int_{S^{2n-1}} |c(x,g^{-1})|^{-2\Re(\alpha)}|c(x,g^{-1})|^{-2n}\,|f(1,x)|^2\,d\mu(x).\end{aligned}$$

This will equal $\|f\|^2_{L^2(S^{2n-1})}$ if and only if $\Re(\alpha) = -n$.

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DEFINITION 3.1. For each complex number ν let $\mathbf{I}_\nu$ denote the space of functions on Φ which are homogeneous of degree $-n-\nu$, measurable when restricted to S^{2n-1}, and satisfying

$$\int_{S^{2n-1}} |f(1,x)|^2 \, d\mu(x) < \infty.$$

We equip this space with an action of G^1,

$$U(\nu, g) f(\xi) = f(\xi g), \qquad \forall \xi \in \Phi, g \in G^1.$$

This is called a *spherical principal series representation* of G^1. When restricted to functions on the sphere $\mathbb{S}$, this is

$$U(\nu, g) f(1,x) = |c(x,g)|^{-n-\nu} f(1, x \cdot g), \qquad \forall x \in S^{2n-1}, g \in G^1.$$

Equation (8) gives a well-known criterion for the unitarity of the principal series.

LEMMA 2. *For each complex number ν there is a G^1-invariant pairing between $(\mathbf{I}_\nu, U(\nu,\cdot))$ and $(\mathbf{I}_{-\nu}, U(-\nu,\cdot))$ given by*

$$(f_1, f_2) \mapsto \int_{S^{2n-1}} f_1(x) f_2(x) \, d\mu(x), \qquad \forall f_1 \in \mathbf{I}_\nu, f_2 \in \mathbf{I}_{-\nu}.$$

COROLLARY 1. *When $\mathbf{I}_\nu$ is equipped with the $L^2(S^{2n-1})$ norm, the representation $(\mathbf{I}_\nu, U(\nu,\cdot))$ is unitary if and only if ν is purely imaginary.*

4. Intertwining operators

Lemma 2 gives a direct way of producing an intertwining operator from $\mathbf{I}_\nu$ to $\mathbf{I}_{-\nu}$.

DEFINITION 4.1. For each $\nu \in \mathbb{C}$ let $\mathscr{P}_\nu : \Phi \times \Phi \to \mathbb{C}$ be the kernel

$$\mathscr{P}_\nu(\xi, \zeta) = |\langle \xi, \zeta \rangle|^{-n+\nu}, \qquad \forall \xi, \zeta \in \Phi.$$

For $\Re(\nu) > 0$ the function $\xi \mapsto \mathscr{P}_\nu(\xi, \zeta)$ is an element of $\mathbf{I}_{-\nu}$ and so the operator

$$A(\nu) f(\zeta) = \int_{S^{2n-1}} f(1,x) \mathscr{P}_\nu((1,x), \zeta) \, d\mu(x) \tag{9}$$

defines a G^1-invariant linear operator from $\mathbf{I}_\nu$ to $\mathbf{I}_{-\nu}$.

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That is, $A(\nu) : \mathbf{I}_\nu \longrightarrow \mathbf{I}_{-\nu}$ has the intertwining property

$$U(-\nu, g)A(\nu)f = A(\nu)U(\nu, g)f, \qquad \forall g \in G^1,\ f \in \mathbf{I}_\nu, \tag{10}$$

and it is self-adjoint as an operator acting on $L^2(S^{2n-1})$ when $\nu \in \mathbb{R}$. The boundedness for $\Re(\nu) > 0$ is a standard result: see Proposition 7.8 in [12].

For real ν, equip $\mathbf{I}_\nu$ with the G^1-invariant hermitian form

$$(f_1 \mid f_2)_\nu = \int_{S^{2n-1}} A(\nu)f_1(1, x)\overline{f_2(1, x)}\, d\mu(x). \tag{11}$$

The question is then to find $\nu > 0$ for which $(\cdot|\cdot)_\nu$ is a positive-definite hermitian form, so that the representation $(\mathbf{I}_\nu, U(\nu, \cdot))$ can be equipped with a unitary structure. This unitary structure is called the *complementary series*.

As is done in [9], we use the binomial expansion of $\mathscr{P}_\nu$, restricted to $\mathbb{S} \times \mathbb{S}$. That is, if x and y are in S^{2n-1}, we have

$$|\langle(1, x), (1, y)\rangle|^{-n+\nu} = |1 - xy^*|^{-n+\nu} = (1 - xy^*)^{(\nu-n)/2}(1 - yx^*)^{(\nu-n)/2}. \tag{12}$$

When we expand the right hand side, we find that

$$\mathscr{P}_\nu((1, x), (1, y)) = \sum_{k,j\geq 0} \binom{(\nu-n)/2}{j}\binom{(\nu-n)/2}{k}(-1)^{j+k}(xy^*)^j(yx^*)^k. \tag{13}$$

Lemma 17 in [9] states that if f is a continuous function on S^{2n-1} and if

$$\int_{S^{2n-1}}\int_{S^{2n-1}} f(x)\overline{f(y)}(xy^*)^j(yx^*)^k\, d\mu(x)d\mu(y) = 0 \qquad \forall j, k \geq 0, \qquad \text{then } f = 0.$$

It is also known that each of the kernels $(x, y) \mapsto (xy^*)^j(yx^*)^k$ is of positive type on the sphere. This can be traced back to the theory of positive-definite kernels due to Kreĭn [18]. That is, for every continuous function h on S^{2n-1},

$$\int_{S^{2n-1}}\int_{S^{2n-1}} h(x)\overline{h(y)}(xy^*)^j(yx^*)^k\, d\mu(x)d\mu(y) \geq 0 \qquad \forall j, k \geq 0.$$

It follows that $(\cdot|\cdot)_\nu$ will be positive-definite whenever

$$(-1)^j\binom{(\nu-n)/2}{j} > 0, \qquad \forall j \geq 0. \tag{14}$$

This inequality is true when $0 < \nu < n$; see [9, Proposition 22].

PROPOSITION 1. *If $0 < \nu < n$, the representation $(\mathbf{I}_\nu, U(\nu, \cdot))$ can be equipped with a unitary structure $(\cdot|\cdot)_\nu$.*

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When $(\nu - n)/2$ is a non-negative integer the expansion (13) terminates and we see that $A(\nu)$ has a non-trivial kernel. Suppose that $\nu = n + 2J - 2$ for some integer $J \geq 1$. Then

$$(15) \quad \binom{(\nu-n)/2}{j} = \binom{J-1}{j} = 0 \quad \forall j \geq J, \text{ and } \binom{J-1}{j} \neq 0 \text{ for } 0 \leq j < J.$$

LEMMA 3. *If J is a positive integer, the kernel of $A(n+2J-2)$ is the subspace*

$$\left\{ f \in \mathbf{I}_{n+2J} : \int_{S^{2n-1}} f(1,x)(x^*y)^j (y^*x)^k \, d\mu(x) = 0, \quad \forall y \in S^{2n-1}, 0 \leq j,k < J \right\}.$$

5. Spherical harmonics

It is well known that $L^2(S^{2n-1})$ has a decomposition into irreducible $U(n)$-invariant subspaces,

$$L^2(S^{2n-1}) = \bigoplus\nolimits_{p,q\geq 0} \mathscr{H}_{p,q} \qquad \text{(Hilbert space direct sum)}$$

where $\mathscr{H}_{p,q}$ is the restriction to S^{2n-1} of the polynomials in x and x^* on $\mathbb{C}^n$ which are harmonic and bihomogeneous of degree (p,q). See for example [16] and [23, Section 12]. The dimension of $\mathscr{H}_{p,q}$ is

$$(16) \quad d(p,q) := \dim\left(\mathscr{H}_{p,q}\right) = \frac{(n+p+q-1)(n+p-2)!(n+q-2)!}{p!q!(n-1)!(n-2)!}.$$

If $\{Y_1, \dots, Y_{d(p,q)}\}$ is an orthonormal basis of $\mathscr{H}_{p,q}$ in $L^2(S^{2n-1})$ then

$$(17) \quad \sum_{j=1}^{d(p,q)} Y_j(x)\overline{Y_j(y)} = d(p,q) R^{n-2}_{p,q}(xy^*), \qquad \forall x, y \in S^{2n-1},$$

where $R^{n-2}_{p,q}$ is the normalized disk polynomial

$$(18) \quad R^{n-2}_{p,q}\left(re^{i\theta}\right) := r^{|p-q|} e^{i(p-q)\theta} P^{n-2,|p-q|}_{p\wedge q}(2r^2-1) / P^{n-2,|p-q|}_{p\wedge q}(1).$$

If F is any bihomogeneous polynomial of degree (p,q) on $\mathbb{C}^n$, then

$$(19) \quad F(x) = \sum_{j=0}^{p\wedge q} |x|^{2j} Y_j(x), \quad \text{with } Y_j \in \mathscr{H}_{p-j,q-j}.$$

As a special case of (19) and (18) we have the following.

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LEMMA 4. *For each pair of integers $j, k \geq 0$ and $y \in S^{2n-1}$, the polynomial $x \mapsto (xy^*)^j(yx^*)^k$ is an element of $\bigoplus_{\ell=0}^{j\wedge k} \mathscr{H}_{j-\ell,k-\ell}$. Furthermore, it has a non-trivial component in $\mathscr{H}_{j,k}$.*

We recall some elementary properties of binomial coefficients.

LEMMA 5. *For each $m \in \mathbb{N}$, the function*

$$s \mapsto \binom{s}{m} = \frac{1}{m!}\prod_{j=0}^{m-1}(s-j)$$

is zero for all $s \in \mathbb{N}$ with $s < m$. In addition, for integers $k < m$

$$\lim_{s\to k}\frac{1}{s-k}\binom{s}{m} = \lim_{s\to k}\frac{1}{m!}\left(\prod_{j=0}^{k-1}(s-j)\right)\left(\prod_{j=k+1}^{m-1}(s-j)\right) = \frac{k!}{m!}(-1)^{m-k-1}(m-1-k)!$$

Fix a pair $p, q \geq 0$ and consider the action of $A(\nu)$ on $\mathscr{H}_{p,q}$. That is, for $Y_{p,q} \in \mathscr{H}_{p,q}$ we have

$$A(\nu)Y_{p,q}(y) = \int_{S^{2n-1}} Y_{p,q}(x)\mathscr{P}_\nu((1,x),(1,y))\,d\mu(x).$$

Using equations (12) and (13) and Lemma 4, we can rewrite this as

(20)

$$A(\nu)Y_{p,q}(y) =$$
$$\int_{S^{2n-1}} Y_{p,q}(x)\left(\sum_{\ell=0}^{\infty}(-1)^{p+q}\binom{(\nu-n)/2}{p+\ell}\binom{(\nu-n)/2}{q+\ell}(xy^*)^{p+\ell}(yx^*)^{q+\ell}\right)d\mu(x).$$

Lemma 5 shows that this vanishes if $(\nu-n)/2 = J-1$, for some integer $J > 0$, and either $p \geq J$ or $q \geq J$. As a function of ν this is a zero of order 1 if $p \geq J, 0 \leq q < J$ or $q \geq J, 0 \leq p < J$. It is a zero of order 2 if $p \geq J$ and $q \geq J$.

PROPOSITION 2. *As a subspace of $L^2(S^{2n-1})$, the kernel of $A(n+2J-2)$ is the Hilbert space direct sum $\bigoplus_{p\geq J \text{ or } q\geq J}\mathscr{H}_{p,q}$.*

We adopt the convention of writing $\sum_{p,q\geq 0} Y_{p,q}(F)$ for the spherical harmonic expansion of a function F on the sphere, with $Y_{p,q}(F) \in \mathscr{H}_{p,q}$ for all $p, q \geq 0$.

6. Jantzen filtrations

The following definition is taken from Vogan [28] and provides a way of dealing with those $\mathbf{I}_\nu$ where the intertwining operator $A(\nu)$ has non-trivial kernel.

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Suppose that E is a finite-dimensional vector space equipped with a real-analytic family of hermitian forms $\langle\cdot|\cdot\rangle_z$, for $-\delta < z < \delta$. In addition, assume that $\langle\cdot|\cdot\rangle_z$ is non-degenerate for $0 < |z| < \delta$. The *Jantzen filtration* of E is the sequence

$$E = E_0 \supset E_1 \supset \cdots \supset E_N = \{0\}$$

defined by the following condition:

for $n \geq 0$, an element $\xi \in E$ is in E_n if and only if for every $\epsilon > 0$ and $f_\xi : \{z \in \mathbb{C} : |z| < \epsilon\} \to E$ with

(1) $f_\xi(0) = \xi$;

(2) $z \mapsto \langle f_\xi(z)|\xi'\rangle_z$ vanishes to order $n \geq 0$ at $z = 0$ for all $\xi' \in E$.

For ξ and ξ' in E take f_ξ and $f_{\xi'}$ as above and set

$$\langle \xi | \xi' \rangle^n = \lim_{z\to 0} \frac{1}{z^n} \langle f_\xi(z) | f_{\xi'}(z)\rangle_z .$$

PROPOSITION 3 (Jantzen-Vogan). *For each $n \geq 0$, $\langle\cdot|\cdot\rangle^n$ is an hermitian form on E_n with radical E_{n+1}:*

- $\mathrm{Rad}\left(\langle\cdot|\cdot\rangle^0\right) = E_1$ *and*
- $\langle\cdot|\cdot\rangle^n$ *is a non-degenerate hermitian form on E_n/E_{n+1}.*

The algebraic direct sum $\mathscr{E}$ of the spherical harmonic spaces $\mathscr{H}_{p,q}$ $(p, q \geq 0)$ is the $(\mathfrak{g}^1, K^1)$-module of K^1-finite vectors in each of the spaces $\mathbf{I}_\nu$, restricted to S^{2n-1}. When we fix elements $f_1, f_2 \in \mathscr{E}$, and consider

$$\nu \mapsto (A(\nu) f_1 | f_2) \quad \text{(the } L^2(S^{2n-1}) \text{ inner product)}$$

we have an analytic function on the right half plane. The action of the principal series $\nu \mapsto U(\nu, g) f|_{S^{2n-1}}$ is an analytic function $\mathscr{E} \longrightarrow C^\infty(S^{2n-1})$ for each $g \in G^1$. Furthermore, there is the intertwining property,

$$(A(\nu) f_1 | f_2) = (A(\nu) U(\nu, g) f_1 | U(\nu, g) f_2), \qquad \forall \Re(\nu) > 0, g \in G.$$

Taking the derivative of this with respect to ν,

$$\begin{aligned}\left(A'(\nu) f_1 \,|\, f_2\right) = &\left(A'(\nu) U(\nu, g) f_1 \,|U(\nu, g) f_2\right) + \left(A(\nu) U'(\nu, g) f_1 \,|U(\nu, g) f_2\right) \\ &+ \left(A(\nu) U(\nu, g) f_1 \,\middle|\, U'(\nu, g) f_2\right).\end{aligned}$$

If we assume that f_1 and f_2 are in $\ker A(\nu)$ and that ν is real, this reduces to

$$\left(A'(\nu) f_1 \,|\, f_2\right) = \left(A'(\nu) U(\nu, g) f_1 \,|U(\nu, g) f_2\right).$$

Note that $\ker(A(\nu))$ is a G-invariant subspace of $\mathbf{I}_\nu$ and we have just seen that the form $(A'(\nu) f_1 \,| f_2)$ on $\ker(A(\nu))$ is G-invariant there.

LEMMA 6. *If ν is real and positive, the hermitian form $(f_1, f_2) \mapsto (A'(\nu)f_1|f_2)$ is G^1-invariant on* $\ker(A(\nu))$. *Furthermore, the nullspace of this form is a G^1-invariant subspace of* $\ker(A(\nu))$.

This argument can be repeated to obtain a descending chain of $(\mathfrak{g}^1, K^1)$-submodules of $\mathscr{E}$, equipped with the representation $U(\nu, \cdot)$.

- $\mathscr{N}_0(\nu) = \{f \in \mathscr{E} \ : \ (A(\nu)f|f) = 0\}$;
- For $j \geq 1$, set $\mathscr{N}_j(\nu) = \left\{f \in \mathscr{N}_{j-1}(\nu) \ : \ (A^{(j)}(\nu)f|f) = 0\right\}$.
- The hermitian form $(f_1, f_2) \mapsto (A^{(j)}(\nu)f_1|f_2)$ is $(\mathfrak{g}^1, K^1)$-invariant on $\mathscr{N}_{j-1}(\nu)$.

This is an example of a Jantzen filtration, as defined in [28]. The idea of differentiating the intertwining operator predates the naming of the process, since it has been used by Molchanov [21 and 22] and Faraut [5]. Fabec used this for $SO_0(1,4)$ in [4] and for $G = SU(2,1)$ in [3], with the non-compact picture of principal series. A similar technique, differentiating the Poisson transform, had been used by Faraut [6] and Schlichtkrull [24]. Residues of intertwining operators were used by Shintani [25] and Johnson and Wallach [10].

We have seen earlier, in equation (20), that for each integer $J > 0$ there is the filtration

$$\mathscr{E} \supset \mathscr{N}_0(n+2J-2) \supset \mathscr{N}_1(n+2J-2) \supset \{0\}.$$

In addition, we saw that

$$\mathscr{N}_0(n+2J-2) = \sum_{p \geq J \text{ or } q \geq J} \mathscr{H}_{p,q} \quad \text{(algebraic direct sum)}$$

and

$$\mathscr{N}_1(n+2J-2) = \sum_{p \geq J \text{ and } q \geq J} \mathscr{H}_{p,q}.$$

The argument above shows that $(f_1, f_2) \mapsto (A'(n+2J-2)f_1|f_2)$ is a $(\mathfrak{g}^1, K^1)$-invariant hermitian form on $\mathscr{N}_0(n+2J-2)$ and $(f_1, f_2) \mapsto (A''(n+2J-2)f_1|f_2)$ is a $(\mathfrak{g}^1, K^1)$-invariant hermitian form on $\mathscr{N}_1(n+2J-2)$. Going back to Lemma 5, we have that

(21)
$$\begin{aligned}
&\left(A'(n+2J-2)f_1\middle|\, f_2\right) = \\
&\int_{S^{2n-1}} f_1(x)\overline{f_2(y)} \left(\sum_{j=J}^{\infty}\sum_{k=0}^{J-1} \frac{(J-1)!}{j!}(-1)^{k-J}(j-J)!\binom{J-1}{k}(xy^*)^j(yx^*)^k\right. \\
&\left.+\sum_{j=0}^{J-1}\sum_{k=J}^{\infty}\frac{(J-1)!}{k!}(-1)^{j-J}(k-J)!\binom{J-1}{j}(xy^*)^j(yx^*)^k\right) d\mu(x)d\mu(y),
\end{aligned}$$

for all $f_1, f_2 \in \mathcal{N}_0(n+2J-2)$, and $f_1, f_2 \in \mathcal{N}_1(n+2J-2)$ implies that

(22)
$$\left(A''(n+2J-2)f_1 \mid f_2\right) = \int_{S^{2n-1}} f_1(x)\overline{f_2(y)}\left(\sum_{j=J}^{\infty}\sum_{k=J}^{\infty}\frac{((J-1)!)^2}{j!k!}(j-J)!(k-J)!(xy^*)^j(yx^*)^k\right) d\mu(x)d\mu(y).$$

To summarize our preceeding discussion, we have arrived at the following theorem.

THEOREM 1. *Suppose that J is a positive integer. Then the subspace*

$$\ker(A(n-2+2J) \cap \ker(A'(n-2+2J)))$$

of $\mathbf{I}_{n-2+2J}$ consists of those elements F which when restricted to S^{2n-1} have spherical harmonic expansions

$$F(x) = \sum_{p,q \geq J} Y_{p,q}(F)(x), \qquad \forall x \in S^{2n-1}.$$

This is an $SU(1,n)$-invariant subspace carrying a unitary structure given by the positive-definite hermitian pairing $(f_1, f_2) \mapsto (A''(n-2+2J)f_1|f_2)$. The completion of this subspace with respect to this norm provides a unitary representation of $SU(1,n)$.

In [10, Theorem 6.3], Johnson and Wallach had described a unitary structure on irreducible quotients of spherical principal series at places where $A(\nu)$ has a pole, with negative values of ν. They modified the intertwining operators by multiplying by a meromorphic function which compensated for the singularities at these poles. Our method could be thought of as being dual to theirs.

7. The results of Johnson and Wallach

Theorem 1 tells us that the derived intertwining norm provides a G-invariant unitary structure on $\ker(A(n-2+2J))$. Now we will describe its value on each of the constituent K-types in that representation. Since each intertwining operator $A(\nu)$ commutes with the action of the unitary group on functions on the sphere, it acts as a multiplier of spherical harmonic expansion,

$$A(\nu)\sum_{p,q\geq 0} Y_{p,q} = \sum_{p,q} A_{p,q}(\nu) Y_{p,q},$$

where the coefficients $A_{p,q}(\nu)$ are meromorphic functions of ν. In [10] Johnson and Wallach calculated the values of $A_{p,q}(\nu)/A_{0,0}(\nu)$. On page 400 of [6] Faraut refers to other sources for calculations of these coefficients.

LEMMA 7. *For non-negative integers p and q and a complex number ν,*

$$a_{p,q}(\nu) = \frac{A_{p,q}(\nu)}{A_{0,0}(\nu)} = \prod_{j=1}^{p}\left(\frac{n-2+2j-\nu}{n-2+2j+\nu}\right)\prod_{k=1}^{q}\left(\frac{n-2+2k-\nu}{n-2+2k+\nu}\right).$$

This formula comes from [10, Theorem 6.1], with changes appropriate to the fact that they use a different normalization of principal series representations.

It is then straighforward to determine the Fourier coefficients of $A''(\nu)$. For $p \geq 1$ and $\nu \in \mathbb{R}$, set $f_p(\nu) = \prod_{j=1}^{p}(n-2+2j-\nu)$. The lemma above says that

$$a_{p,q}(\nu) = \frac{f_p(\nu)f_q(\nu)}{f_p(-\nu)f_q(-\nu)}.$$

In particular, if J is a non-negative integer and $\nu = n-2+2J$, then the pairs (p,q) for which both $a_{p,q}(x) = 0$ and $a'_{p,q}(x) = 0$ are $\{(p,q) : p \geq J \text{ and } q \geq J\}$. From Theorem 1, we see that we must determine $a''_{p,q}(n-2+2J)$ for all $p \geq J$ and $q \geq J$. If $\nu = n-2+2J$ then

$$a''_{p,q}(n-2+2J) = 2\frac{f'_p(n-2+2J)f'_q(n-2+2J)}{f_p(2-n-2J)f_q(2-n-2J)}.$$

THEOREM 2. *If J is a non-negative integer and if f is an element of*

$$\ker(A(n-2+2J)) \cap \ker(A'(n-2+2J)) \subset \mathbf{I}_{n-2+2J}$$

then $(A''(n-2+2J)f \mid f)$ is equal to

$$c_{J,n}\sum_{p,q\geq J}\frac{(p-J)!(q-J)!}{(n-2+p+J)!(n-2+q+J)!}\|Y_{p,q}(f)\|^2_{L^2(S^{2n-1})},$$

where the positive constant $c_{J,n}$ depends only on J and n.

Note that $(p-J)!/(n-2+p+J)! = \Gamma(p-J+1)/\Gamma(p+J+n-1) \approx p^{2-n-2J}$ as $p \to \infty$. The asymptotics of the gamma function show that this representation space is a Sobolev type space of distributions on $M\backslash K$, with an additional orthogonality or moment condition. For class one exceptional representations of $SO_0(1,n)$, Theorem 9 of [2] gives such a structure in the non-compact picture.

8. Cauchy-Szegö operators

For the subgroups A, N, and K defined in Section 1 there is the Iwasawa decomposition $G = ANK$. The **Iwasawa projections** $\mathrm{H} : G \to \mathbb{R}$, $\mathrm{N} : G \to N$, and $\mathrm{K} : G \to K$ are given in the expression $g = a(\mathrm{H}(g))\mathrm{N}(g)\mathrm{K}(g)$, for all $g \in G$. For each complex number λ let $a \mapsto a^\lambda$ be the character on A such that $a(t)^\lambda = e^{t\lambda}$. Furthermore, set $\rho = n$.

DEFINITION 8.1. Suppose that (σ, H_σ) is an irreducible unitary representation of M and that ν is a complex number. The nonunitary principal series representation coming from these parameters is the action of G by right translation on the space

$$\mathbf{I}_{\sigma,\nu} = \left\{ f \ : \ \begin{array}{ll} f : G \to H_\sigma \quad \text{is measurable, } f|_K \in L^2(K, H_\sigma), & \text{and} \\ f(ma(t)ng) = e^{(\rho+\nu)t}\sigma(m)f(g), & \forall ma(t)n \in MAN, g \in G \end{array} \right\}.$$

We will write the action of G on $\mathbf{I}_{\sigma,\nu}$ by

$$U_\sigma(\nu, g)f(x) = f(xg), \qquad \forall x, g \in G, f \in \mathbf{I}_{\sigma,\nu}. \tag{23}$$

Elements of $\mathbf{I}_{\sigma,\nu}$ are said to be $\sigma \otimes (\rho + \nu)$-*covariant functions* on G and this action of G is said to be a *principal series representation.* The space $\mathbf{I}_{\sigma,\nu}$ carries an inner product which comes from integrating over K,

$$(f_1|f_2) = \int_K (f_1(k)|f_2(k))_\sigma dk, \qquad \forall f_1, f_2 \in \mathbf{I}_{\sigma,\nu}.$$

As a representation of K, $\mathbf{I}_{\sigma,\nu}$ is equivalent to the subspace of $L^2(K, H_\sigma)$ given by

$$\left\{ f \in L^2(K, H_\sigma) : f(mk) = \sigma(m)f(k), \quad \text{a.e.} \quad m \in M, k \in K \right\},$$

equipped with right translation by elements of K. The purpose of the additional factor ρ in the definition of the principal series is so that this inner-product is G-invariant when ν is purely imaginary.

When σ is the trivial representation of M on $H_1 = \mathbb{C}$ then $\left(\mathbf{I}_{1,\nu}, U_1(\nu, \cdot)\right)$ is the spherical principal series representation $\mathbf{I}_\nu$ which we studied in previous sections.

The principal series representations come about by inducing finite-dimensional irreducible representations of MAN up to G. There is a similar construction which starts with finite-dimensional representations of K.

DEFINITION 8.2. Suppose that $(\tau, \mathscr{H}_\tau)$ is a finite-dimensional representation of the compact group K. The space of τ-covariants on G is

$$C^\infty(G, \tau) = \left\{ f \ : \ \begin{array}{l} f : G \to \mathscr{H}_\tau \quad \text{is smooth and} \\ f(kg) = \tau(k)f(g) \quad \forall k \in K, g \in G \end{array} \right\}.$$

This space carries a representation of G, acting by right translation.

In general this is a realization of the space of smooth sections of the homogeneous bundle over $K\backslash G$ with fibre $\mathscr{H}_\tau$. When $(\tau, \mathscr{H}_\tau) = (1, \mathbb{C})$, then this is really the action of G on the space of smooth complex-valued functions on the symmetric space $K\backslash G$.

Whenever (σ, H_σ) is an M-invariant subspace of $(\tau|_M, \mathscr{H}_\tau)$, we can produce G-equivariant operators from $(\mathbf{I}_{\sigma,\nu}, U_\sigma(\nu, \cdot))$ into $C^\infty(G, \tau)$, for each $\nu \in \mathbb{C}$. These operators depend on the M-equivariant imbedding of H_σ into $\mathscr{H}_\tau$.

DEFINITION 8.3. Suppose that (σ, H_σ) is an irreducible representation of M, that $(\tau, \mathscr{H}_\tau)$ is a finite-dimensional representation of K, and that $R : H_\sigma \to \mathscr{H}_\tau$ is a non-zero M-equivariant linear map. For each $\nu \in \mathbb{C}$ we define the *Cauchy-Szegö map* with data (σ, τ, R, ν) to be the G-equivariant operator $\mathfrak{S}_\nu : \mathbf{I}_{\sigma,\nu} \to C^\infty(G, \tau)$ defined by

$$\mathfrak{S}_\nu f(g) = \int_K \tau(k)^{-1} R f(kg)\, dk, \qquad \forall f \in \mathbf{I}_{\sigma,\nu}, g \in G.$$

This is the definition given in [20], except that here we suppress the dependence on the choice of A and positive noncompact roots. These are also called τ-quotient maps and vector valued Poisson transforms; see [19 and 27]. The Cauchy-Szegö map with data (σ, τ, R, ν) has the important property that it always preserves the K-type τ; see [20, Lemma 3.5.1].

LEMMA 8. *For all (σ, τ, R, ν) as above, the image $\mathfrak{S}_\nu\left(\mathbf{I}_{\sigma,\nu}\right)$ in $C^\infty(G, \tau)$ contains the K-type $(\tau, \mathscr{H}_\tau)$ with multiplicity greater than or equal to one.*

Since $R \in \operatorname{Hom}_M(H_\sigma, \mathscr{H}_\tau)$, Frobenius reciprocity guarantees that there is is a K-equivariant map

$$B(\nu) : \mathscr{H}_\tau \longrightarrow \mathbf{I}_{\sigma,\nu}, \quad \forall \nu,$$

where $B(\nu)\xi(g) = e^{(\rho+\nu)\mathrm{H}(g)} R^*(\tau(\mathrm{K}(g))\xi)$ for all $g \in G$ and $\xi \in \mathscr{H}_\tau$. The adjoint $B(\nu)^*$ is the Cauchy-Szegö map with data (σ, τ, R, ν). To see this, take $f \in \mathbf{I}_{\sigma,\nu}$ and $\xi \in \mathscr{H}_\tau$. Then

$$(B(\nu)^* f | \xi)_\tau = (f | B(\nu)\xi) = \int_K (f(k) | R^* \tau(k)\xi)_\sigma \, dk.$$

The right-hand side can be rearranged to equal

$$\int_K \left(\tau(k)^{-1} R f(k) | \xi\right)_\sigma dk = (\mathfrak{S}_\nu f(1) | \xi)_\tau .$$

This means that for all $f \in \mathbf{I}_{\sigma,\nu}$ and $g \in G$,

$$\mathfrak{S}_\nu f(g) = B(\nu)^* U_\sigma(\nu, g) f.$$

It is known (see [13, Proposition 38]) that if ν is purely imaginary, then there exists a unitary intertwining operator $\mathscr{A}_\nu : \mathbf{I}_{\sigma,\nu} \longrightarrow \mathbf{I}_{\sigma,-\nu}$. Since τ has multiplicity one in all $\mathbf{I}_{\sigma,\nu}$, there must be non-zero complex numbers $\mathscr{C}(\nu)$ such that $\mathscr{A}_\nu B(\nu)\xi = \mathscr{C}(\nu) B(-\nu)\xi$ for all imaginary ν and all $\xi \in \mathscr{H}_\tau$. Normalize the operators $\mathscr{A}_\nu$ in such a way that $\mathscr{C}(\nu) = 1$. Then

$$B(\nu)^* U_\sigma(\nu, g) B(\nu) = B(-\nu)^* \mathscr{A}_\nu U_\sigma(\nu, g) \mathscr{A}_\nu^{-1} B(-\nu) = B(-\nu)^* U_\sigma(-\nu, g) B(-\nu)$$

for all purely imaginary ν. For each $\xi \in \mathscr{H}_\tau$ and $g \in G$, the map

$$\nu \mapsto B(\nu)^* U_\sigma(\nu, g) B(\nu)\xi$$

is analytic on the complex plane. Hence,

$$B(\nu)^* U_\sigma(\nu, g) B(\nu)\xi = B(-\nu)^* U_\sigma(-\nu, g) B(-\nu)\xi, \qquad \forall \xi \in \mathscr{H}_\tau, g \in G,$$

for all complex numbers ν. We have proved the following theorem.

THEOREM 3. *Suppose (σ, H_σ) occurs with multiplicity one in $(\tau|_M, \mathscr{H}_\tau)$ and that $R : H_\sigma \to \mathscr{H}_\tau$ is the M-equivariant imbedding. For each $\nu \in \mathbb{C}$, the image of the K-type τ under the Cauchy-Szegö map with data (σ, τ, R, ν) is equal to the image under the Cauchy-Szegö map with data $(\sigma, \tau, R, -\nu)$.*

Suppose that $\mathscr{N}_\tau(\nu)$ is the G-invariant subspace of $\mathbf{I}_{\sigma,\nu}$ generated by the K-type τ, that is, the G-invariant span of $B(\nu)\mathscr{H}_\tau$. Then we have shown that

$$\mathfrak{S}_\nu(\mathscr{N}_\tau(\nu)) = \mathfrak{S}_{-\nu}(\mathscr{N}_\tau(-\nu)) \subset C^\infty(G, \tau).$$

Lemma 8 shows that these images are not zero.

9. The invariant differential operator $\eth_J$

In this section we will fix an integer $J \geq 2$ and let $(\tau_{J,J}, \mathscr{H}_{J,J})$ denote the irreducible unitary representation of K on the space of harmonic polynomials on $\mathbb{C}^n$ which are bihomogeneous of degree (J, J). In [20, Section 6] we constructed a first order G-invariant differential operator $\eth_J$ acting on $C^\infty(G, \tau_{J,J})$ with the following properties:

(1) $\eth_J$ is elliptic;
(2) The K-types in the kernel of $\eth_J$ are contained in the set

$$\left\{\left(\tau_{p,q}, \mathscr{H}_{p,q}\right) : p \geq J \text{ and } q \geq J\right\};$$

(3) Let $R_J : \mathbb{C} \to \mathscr{H}_{J,J}$ denote the inclusion of the line spanned by an M-fixed unit vector in $\mathscr{H}_{J,J}$. Then the Cauchy-Szegö map with data $(1, \tau_{J,J}, R_J, 2-n-2J)$ takes $\mathbf{I}_{2-n-2J}$ into $\ker(\eth_J)$.

For each K-type τ and real number ν let $\mathscr{N}_\tau(\nu)$ denote the G-invariant subspace of $\mathbf{I}_\nu$ generated by the K-type τ. We have already seen that

$$\mathfrak{S}_{n-2+2J}(\mathscr{N}_{\tau_{J,J}}(n-2+2J)) = \mathfrak{S}_{2-n-2J}(\mathscr{N}_{\tau_{J,J}}(2-n-2J)),$$

and we combine this with the preceeding comment to obtain the following realization of the irreducible subrepresentations of reducible spherical principal series.

THEOREM 4. *For each integer $J \geq 2$, the Cauchy-Szegö map with data*

$$(1, \tau_{J,J}, R_J, n-2+2J)$$

is an injective G-equivariant map

$$\mathfrak{S}_{n-2+2J} : \ker(A(n-2+2J)) \cap \ker(A'(n-2+2J)) \to \ker(\eth_J).$$

When restricted to the space of K-finite vectors, this is an isomorphism of $(\mathfrak{g}, K)$*-modules.*

The fact that this operator is one-to-one follows from the irreducibility of

$$\ker(A(n-2+2J)) \cap \ker(A'(n-2+2J)).$$

In particular, this identifies the unitary structure on $\ker(\eth_J)_K$, the $(\mathfrak{g}, K)$-module of K-finite elements of $\ker(\eth_J)$.

There is also a realization of this unitary structure via the boundary behaviour described in Blank's work [1]. Theorem 6.6.1 in [20] exhibits another Cauchy-Szegö map into $\ker(\eth_J)$, coming from taking (σ_J, H_{σ_J}) to be the irreducible M-invariant subspace of $\mathscr{H}_{J,J}$ generated by a highest weight vector. For this (σ_J, H_{σ_J}), and R the inclusion map, the Cauchy-Szegö map with data $(\sigma_J, \tau_{J,J}, R, n-2)$ takes $\mathbf{I}_{\sigma,n-2}$ into $\ker(\eth_J)$. This is based on the results of Knapp and Wallach [15]. This is a surjective map of $(\mathfrak{g}, K)$-modules,

$$\ker(\eth_J)_K = \mathfrak{S}_{n-2}\left(\mathbf{I}_{\sigma_J,n-2}\right)_K = \mathfrak{S}_{n-2+2J}\,(\ker(A(n-2+2J)))_K$$

and so for each $F \in \ker(\eth_J)_K$, there is a K-finite $f \in \mathbf{I}_{\sigma_J,n-2}$ such that $F = \mathfrak{S}_{n-2}f$. Let $A(\sigma_J, n-2) : \mathbf{I}_{\sigma_J,n-2} \to \mathbf{I}_{\sigma_J,2-n}$ be the standard intertwining operator, as defined in [12]. The hermitian form $(A(\sigma_J, n-2)f|f)$ is a G-invariant on $\mathbf{I}_{\sigma_J,n-2}$. We now see that this form is positive-definite on the subquotient generated by the K-type $\tau_{J,J}$ in $\mathbf{I}_{\sigma_J,n-2}$. This gives an example of a process which starts with certain scalar-valued functions on the sphere, produces vector-valued functions on the hyperboloid which are in the kernel of an elliptic differential operator and then these functions have vector-valued boundary values on the sphere. For other examples of work in this area, see [1, 11 and 8].

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