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WHEN DOES RANK$(A+B)$=RANK(A)+RANK(B)?

BY
G. MARSAGLIA AND G. P. H. STYAN

In a recent note in the *Bulletin*, Murphy [5] gave a short proof that for complex $m\times n$ matrices A and B, $r(A+B)=r(A)+r(B)$ if the rows of A are orthogonal to the rows of B and the columns of A are orthogonal to the columns of B. His proof was elegant and simple, an improvement on an earlier proof of the same result by Meyer [4].

While there is no dispute now with the proofs of this result, we would like to suggest that the orthogonality condition is far too strong and is, in fact, misleading. Additivity of rank is related in a simple way to the intersection of the row and column spaces of A and B, as we will now show.

Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be the column spaces of A and B, let $\mathscr{R}_1$ and $\mathscr{R}_2$ be their row spaces. Let

$$c = \text{dimension}(\mathscr{C}_1 \cap \mathscr{C}_2), \qquad d = \text{dimension}(\mathscr{R}_1 \cap \mathscr{R}_2).$$

THEOREM. $r(A+B)=r(A)+r(B)$ *if and only if* $c=d=0$, *that is,*

$$\textit{dimension}(\mathscr{C}_1 \cap \mathscr{C}_2) = \textit{dimension}(\mathscr{R}_1 \cap \mathscr{R}_2) = 0.$$

This theorem appears as a corollary to a more general result of Marsaglia [2] who proved that

$$r(A)+r(B)-c-d \leq r(A+B) \leq r(A)+r(B)-\max(c, d).$$

Khatri [1] also gives a theorem from which this inequality may be derived. Simpler proofs and related inequalities are given by Marsaglia and Styan [3]. But the importance of the particular case $r(A+B)=r(A)+r(B)$ merits special attention, and so we provide a short proof that the conditions are $c=d=0$.

Proof of Theorem. First, the condition $c=d=0$ is necessary, as these two strings of inequalities show:

$$r(A+B) \leq r[(A, B)] = r(A)+r(B)-c \leq r(A)+r(B)$$
$$r(A+B) \leq r\left[\begin{pmatrix} A \\ B \end{pmatrix}\right] = r(A)+r(B)-d \leq r(A)+r(B).$$

To show $c=d=0$ is sufficient, we use *full rank decompositions* of A and B:

$A = C_1R_1$; $r(A) = r(C_1) = r(R_1) = a$; A is $m\times n$, C_1 is $m\times a$, R_1 is $a\times n$;
$B = C_2R_2$; $r(B) = r(C_2) = r(R_2) = b$; B is $m\times n$, C_2 is $m\times b$, R_2 is $b\times n$.

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Such representations exist since, for example, R_1 can be any matrix whose rows are a basis of the row space of A and then $A=C_1R_1$ for some C_1 and $r(A)=r(C_1R_1)\le\min[r(C_1), r(R_1)]\le a=r(A)$. We now write

$$A+B = C_1R_1+C_2R_2 = (C_1, C_2)\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = CR,$$

say. Then $c=0$ implies that all the $a+b$ columns of C are linearly independent and so C has a left inverse such that $LC=I$. Thus when $c=0$,

$$r(A+B) = r(CR) \ge r(LCR) = r(R) = r(A)+r(B)-d \ge r(A+B)-d.$$

If in addition $d=0$ the whole string collapses and $r(A+B)=r(A)+r(B)$.

References

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McGill University,
Montreal, Quebec