SHUFFLE ALGEBRAS FOR QUIVERS AND R-MATRICES

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Abstract We define slope subalgebras in the shuffle algebra associated to a (doubled) quiver, thus yielding a factorization of the universal R-matrix of the double of the shuffle algebra in question. We conjecture that this factorization matches the one defined by [1, 18, 32, 33, 34] using Nakajima quiver varieties.

1. Introduction

Fix a quiver Q with vertex set I and edge set E; edge loops and multiple edges are allowed. We consider a certain Hopf algebra

$$\mathcal{A} = \mathcal{A}^+ \otimes (\text{Cartan subalgebra}) \otimes \mathcal{A}^-,$$

where $\mathcal{A}^+$ is the shuffle algebra associated to the (double of the) quiver Q and $\mathcal{A}^-$ is its opposite. When Q is a finite (resp., affine) Dynkin diagram, the algebra $\mathcal{A}$ is the quantum loop (resp., quantum toroidal) algebra. In general, the shuffle algebra $\mathcal{A}^+$ matches the localized K-theoretic Hall algebra of the quiver Q [31].

The main purpose of the present paper is to define and study *slope* subalgebras

$$\mathcal{B}_{\boldsymbol{m}}^{\pm} \subset \mathcal{A}^{\pm}$$

for any $\boldsymbol{m} \in \mathbb{Q}^I$, and produce a Hopf algebra

$$\mathcal{B}_{\boldsymbol{m}} = \mathcal{B}_{\boldsymbol{m}}^+ \otimes (\text{Cartan subalgebra}) \otimes \mathcal{B}_{\boldsymbol{m}}^-.$$

For nontrivial reasons, there exist inclusions $\mathcal{B}_{\boldsymbol{m}} \subset \mathcal{A}$ which preserve the product and the Hopf pairing but not the coproduct and antipode. As $\mathcal{A}$ and $\mathcal{B}_{\boldsymbol{m}}$ arise as Drinfeld

doubles, we may consider their universal R-matrices[1]

$$\mathcal{R}' \in \mathcal{A} \,\widehat{\otimes}\, \mathcal{A} \qquad \text{and} \qquad \mathcal{R}'_{\boldsymbol{m}} \in \mathcal{B}_{\boldsymbol{m}} \,\widehat{\otimes}\, \mathcal{B}_{\boldsymbol{m}}.$$

Our main result, proved by combining Corollaries 3.20 and 3.21, is the following:

Theorem 1.1. *For any $\boldsymbol{m} \in \mathbb{Q}^I$ and $\boldsymbol{\theta} \in \mathbb{Q}_+^I$, the multiplication map induces an isomorphism*

$$\overset{\rightarrow}{\bigotimes_{r \in \mathbb{Q}}} \mathcal{B}_{\boldsymbol{m}+r\boldsymbol{\theta}}^{\pm} \xrightarrow{\sim} \mathcal{A}^{\pm} \tag{1.1}$$

(the arrow $\rightarrow$ refers to taking the product in increasing order of r) which preserves the Hopf pairings on the two sides, and thus leads to a factorization

$$\mathcal{R}' = \overset{\rightarrow}{\prod_{r \in \mathbb{Q}}} \mathcal{R}'_{\boldsymbol{m}+r\boldsymbol{\theta}} \tag{1.2}$$

of the (off-diagonal part of the) universal R-matrix.

When Q is a cyclic quiver, Theorem 1.1 was proved in [27, 26]. The isomorphism (1.1) is inspired by the one constructed by Burban and Schiffmann [3] in the elliptic Hall algebra (which is isomorphic to $\mathcal{A}^+$ when Q is the Jordan quiver, namely one vertex and one loop). Meanwhile, the product formula (1.2) generalizes well-known formulas for R-matrices of finite and affine-type quantum groups [6, 14, 13, 16, 17, 37].

In §2, we recall general facts about the shuffle algebra $\mathcal{A}^+$. In §3, we define the slope subalgebras $\mathcal{B}_{\boldsymbol{m}}$ and prove Theorem 1.1. In §4, we present connections (as well as conjectures and open questions) between our slope subalgebras and other concepts in the field such as Kac polynomials, cohomological and K-theoretic Hall algebras (particularly the connection between $\mathcal{B}_{\boldsymbol{0}}$ and the Lie algebra of BPS states studied in [4, 5]), and the conjectural connection between formulas (1.1) and (1.2) and the analogous formulas for quantum groups defined via geometric R-matrices [1, 18, 32, 33, 34] in the context of Nakajima quiver varieties.

It is likely that Theorem 1.1 can be generalized to the case of quivers with potential, although working out all the details would probably be a very nontrivial and interesting task (see [35] for the setting of such a generalization; note, however, that Theorem 6.3 there provides an isomorphism of a different nature from formula (1.1)).

2. The shuffle algebra of a (doubled) quiver

2.1.   A quiver is a finite oriented graph Q with vertex set I and edge set E; edge loops and multiple edges are allowed. We will work over the field

$$\mathbb{F} = \mathbb{Q}(q, t_e)_{e \in E}.$$

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[1]The symbol $\widehat{\otimes}$ refers to the fact that the universal R-matrices lie in certain completions of the algebras in question, as they are given by infinite sums. Meanwhile, the primes refer to the fact that $\mathcal{R}'$ is only the 'off-diagonal' part of the universal R-matrix; see equations (2.28) and (3.24)

We will write elements of $\mathbb{N}^I$ as $\boldsymbol{n} = (n_i \geq 0)_{i \in I}$.[2] For such an $\boldsymbol{n}$, let us define

$$\boldsymbol{n}! = \prod_{i \in I} n_i!.$$

Consider the vector space

$$\mathcal{V} = \bigoplus_{\boldsymbol{n} = (n_i)_{i \in I} \in \mathbb{N}^I} \mathbb{F}\left[\ldots, z_{i1}^{\pm 1}, \ldots, z_{in_i}^{\pm 1}, \ldots\right]^{\text{sym}}, \qquad (2.1)$$

where 'sym' refers to Laurent polynomials which are symmetric in $z_{i1}, \ldots, z_{in_i}$, for each $i \in I$ separately. We will make $\mathcal{V}$ into an associative algebra using the following *shuffle product* (which originated with a construction of [9] involving elliptic algebras, though the setting at hand is closer to the one studied in [7, 8, 39] and other works):

$$F\left(\ldots, z_{i1}, \ldots, z_{in_i}, \ldots\right) * F'\left(\ldots, z_{i1}, \ldots, z_{in_i'}, \ldots\right) =$$

$$\text{Sym}\left[\frac{F\left(\ldots, z_{i1}, \ldots, z_{in_i}, \ldots\right) F'\left(\ldots, z_{i,n_i+1}, \ldots, z_{i,n_i+n_i'}, \ldots\right)}{\boldsymbol{n}! \cdot \boldsymbol{n}'!} \prod_{\substack{1 \leq a \leq n_i \\ n_j < b \leq n_j + n_j'}}^{i,j \in I} \zeta_{ij}\left(\frac{z_{ia}}{z_{jb}}\right)\right],$$

$$(2.2)$$

where 'Sym' denotes symmetrization with respect to the variables $z_{i1}, \ldots, z_{i,n_i+n_i'}$ for each $i \in I$ separately, and for any $i, j \in I$ we define the following function:

$$\zeta_{ij}(x) = \left(\frac{1 - xq^{-1}}{1 - x}\right)^{\delta_j^i} \prod_{e = \overrightarrow{ij} \in E} \left(\frac{1}{t_e} - x\right) \prod_{e = \overrightarrow{ji} \in E} \left(1 - \frac{t_e}{qx}\right). \qquad (2.3)$$

Note that although the right-hand side of equation (2.2) seemingly has simple poles at $z_{ia} - z_{ib}$ for all $i \in I$ and all $a < b$, these poles vanish when we take the symmetrization, as the orders of such poles in a symmetric rational function must be even.

Definition 2.2. The *shuffle algebra* is defined as the subset

$$\mathcal{S} \subset \mathcal{V}$$

of Laurent polynomials $F\left(\ldots, z_{i1}, \ldots, z_{in_i}, \ldots\right)$ that satisfy the 'wheel conditions'

$$F\big|_{z_{ia} = \frac{qz_{jb}}{t_e} = qz_{ic}} = F\big|_{z_{ja} = t_e z_{ib} = qz_{jc}} = 0 \qquad (2.4)$$

for all edges $e = \overrightarrow{ij}$ and all $a \neq c$ (and further, $a \neq b \neq c$ if $i = j$).

It is easy to show that $\mathcal{S}$ is a subalgebra of $\mathcal{V}$ – that is, that it is closed under the shuffle product (see [23, Proposition 2.1] for the proof in the particular case of the Jordan quiver, which already incorporates all the ideas that one needs in the general case).

Theorem 2.3 ([31, Theorem 1.2].). *As an $\mathbb{F}$-algebra, $\mathcal{S}$ is generated by $\left\{z_{i1}^d\right\}_{i \in I}^{d \in \mathbb{Z}}$.*

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[2]Although nonstandard, it will be convenient for us to include 0 in the set $\mathbb{N}$.

2.4. The algebra $\mathcal{S}$ is $\mathbb{N}^I \times \mathbb{Z}$ graded via

$$\deg F = (\boldsymbol{n}, d) \tag{2.5}$$

if F lies in the $\boldsymbol{n}$th direct summand of equation (2.1), and has homogeneous degree d. The components of the degree will be called 'horizontal' and 'vertical', respectively:

$$\mathrm{hdeg} F = \boldsymbol{n}, \qquad \mathrm{vdeg} F = d. \tag{2.6}$$

We will denote the graded pieces of the shuffle algebra by

$$\mathcal{S} = \bigoplus_{\boldsymbol{n} \in \mathbb{N}^I} \mathcal{S}_{\boldsymbol{n}} = \bigoplus_{(\boldsymbol{n}, d) \in \mathbb{N}^I \times \mathbb{Z}} \mathcal{S}_{\boldsymbol{n}, d}. \tag{2.7}$$

For any $\boldsymbol{k} \in \mathbb{Z}^I$, we have a *shift automorphism*

$$\mathcal{S} \xrightarrow{\tau_{\boldsymbol{k}}} \mathcal{S}, \quad F(\ldots, z_{ia}, \ldots) \mapsto F(\ldots, z_{ia}, \ldots) \prod_{i \in I, a \geq 1} z_{ia}^{k_i}. \tag{2.8}$$

These notions also apply to the opposite algebra $\mathcal{S}^{\mathrm{op}}$, although we make slightly different conventions. For one thing, we set the grading on $\mathcal{S}^{\mathrm{op}}$ to

$$\deg G = (-\boldsymbol{n}, d) \tag{2.9}$$

if G lies in the $\boldsymbol{n}$th direct summand of equation (2.1) and has homogeneous degree d. As for the analogue of the shift automorphism (2.8), we make the following convention:

$$\mathcal{S}^{\mathrm{op}} \xrightarrow{\tau_{\boldsymbol{k}}} \mathcal{S}^{\mathrm{op}}, \quad G(\ldots, z_{ia}, \ldots) \mapsto G(\ldots, z_{ia}, \ldots) \prod_{i \in I, a \geq 1} z_{ia}^{-k_i}. \tag{2.10}$$

2.5. We will now recall the well-known Hopf algebra structure on the shuffle algebra (see [23, 39, 43] for incarnations of this construction in settings such as ours). As usually, the Hopf algebra is actually the double extended shuffle algebra,[3] namely,

$$\mathcal{A} = \mathcal{S} \otimes \mathbb{F}\left[h_{i, \pm 0}^{\pm 1}, h_{i, \pm 1}, h_{i, \pm 2}, \ldots\right]_{i \in I} \otimes \mathcal{S}^{\mathrm{op}} \Big/ \text{relations (2.13)–(2.15)}. \tag{2.11}$$

Since the algebras $\mathcal{S}$ and $\mathcal{S}^{\mathrm{op}}$ are generated by

$$e_{i,d} = z_{i1}^d \in \mathcal{S} \qquad \text{and} \qquad f_{i,d} = z_{i1}^d \in \mathcal{S}^{\mathrm{op}},$$

it suffices to present the defining relations, as well as the Hopf algebra structure, on the generators. More precisely, if we package the generators into formal series

$$e_i(z) = \sum_{d \in \mathbb{Z}} \frac{e_{i,d}}{z^d}, \qquad f_i(z) = \sum_{d \in \mathbb{Z}} \frac{f_{i,d}}{z^d}, \qquad h_i^{\pm}(w) = \sum_{d=0}^{\infty} \frac{h_{i, \pm d}}{w^{\pm d}}, \tag{2.12}$$

then we set

$$e_i(z) h_j^{\pm}(w) = h_j^{\pm}(w) e_i(z) \frac{\zeta_{ij}\left(\frac{z}{w}\right)}{\zeta_{ji}\left(\frac{w}{z}\right)} \tag{2.13}$$

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[3]It is possible to enlarge $\mathcal{A}$ by introducing a central element which measures the failure of the h^+s and the h^-s to commute, but we will not need it.

$$f_i(z)h_j^{\pm}(w) = h_j^{\pm}(w)f_i(z)\frac{\zeta_{ji}\left(\frac{w}{z}\right)}{\zeta_{ij}\left(\frac{z}{w}\right)} \tag{2.14}$$

(the rational functions in the right-hand sides of these expressions are expanded as power series in $w^{\mp 1}$) and

$$[e_{i,d}, f_{j,k}] = \delta_j^i \cdot \gamma_i \begin{cases} -h_{i,d+k} & \text{if } d+k > 0, \\ h_{i,-0} - h_{i,+0} & \text{if } d+k = 0, \\ h_{i,d+k} & \text{if } d+k < 0, \end{cases} \tag{2.15}$$

where[4]

$$\gamma_i = \frac{\prod_{e=\overrightarrow{ii}}\left[\left(\frac{1}{t_e}-1\right)\left(1-\frac{t_e}{q}\right)\right]}{1-\frac{1}{q}}. \tag{2.16}$$

It is easy to see that the grading of equations (2.5) and (2.9) extends to the whole of $\mathcal{A}$ by setting

$$\deg h_{i,\pm d} = (0, \pm d)$$

for all $i \in I, d \geq 0$. The shift automorphisms (2.8) and (2.10) extend to automorphisms

$$\tau_{\boldsymbol{k}} : \mathcal{A} \to \mathcal{A} \tag{2.17}$$

by setting $\tau_{\boldsymbol{k}}(h_{i,\pm d}) = h_{i,\pm d}$ for all $i \in I$ and $d \in \mathbb{N}$.

2.6. To write down the (topological) coproduct on $\mathcal{A}$, consider the subalgebras

$$\mathcal{A}^+ = \mathcal{S} \qquad \text{and} \qquad \mathcal{A}^- = \mathcal{S}^{\mathrm{op}}$$

and the extended subalgebras

$$\mathcal{A}^{\geq} = \mathcal{A}^+ \otimes \mathbb{F}\left[h_{i,+0}^{\pm 1}, h_{i,1}, h_{i,2}, \dots\right]_{i \in I}$$
$$\mathcal{A}^{\leq} = \mathcal{A}^- \otimes \mathbb{F}\left[h_{i,-0}^{\pm 1}, h_{i,-1}, h_{i,-2}, \dots\right]_{i \in I}.$$

The reason for these extended subalgebras is that $\mathcal{A}^+$ (resp., $\mathcal{A}^-$) does not admit a coproduct, but $\mathcal{A}^{\geq}$ (resp., $\mathcal{A}^{\leq}$) does, according to the formulas

$$\Delta\left(h_i^{\pm}(z)\right) = h_i^{\pm}(z) \otimes h_i^{\pm}(z) \tag{2.18}$$

$$\Delta\left(e_i(z)\right) = e_i(z) \otimes 1 + h_i^+(z) \otimes e_i(z) \tag{2.19}$$

$$\Delta\left(f_i(z)\right) = f_i(z) \otimes h_i^-(z) + 1 \otimes f_i(z). \tag{2.20}$$

There are unique antipode maps $S : \mathcal{A}^{\geq} \to \mathcal{A}^{\geq}$ and $S : \mathcal{A}^{\leq} \to \mathcal{A}^{\leq}$ which are determined by these topological coproducts, and so we leave their computation to the interested reader (the antipode will not feature in this paper).

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[4]This differs slightly from the conventions of [31], where the constant γ_i did not appear in the analogue of equation (2.15); this can be explained by simply rescaling the generators $f_{i,d}$ by γ_i.

2.7. It is straightforward to show that the Hopf algebra structures on $\mathcal{A}^{\geq}$ and $\mathcal{A}^{\leq}$ defined extend to the entire $\mathcal{A}$. An alternative way to see this is to note that $\mathcal{A}$ is the Drinfeld double of $\mathcal{A}^{\geq}$ and $\mathcal{A}^{\leq}$. Indeed, consider the Hopf pairing

$$\langle \cdot, \cdot \rangle : \mathcal{A}^{\geq} \otimes \mathcal{A}^{\leq} \longrightarrow \mathbb{F}, \tag{2.21}$$

which is defined by the formulas

$$\left\langle h_i^+(z), h_j^-(w) \right\rangle = \frac{\zeta_{ij}\left(\frac{z}{w}\right)}{\zeta_{ji}\left(\frac{w}{z}\right)} \tag{2.22}$$

(the right-hand side is expanded as $|z| \gg |w|$) and

$$\langle e_{i,d}, f_{j,k} \rangle = \delta_j^i \gamma_i \delta_{d+k}^0. \tag{2.23}$$

All other pairings between the es, fs, and hs vanish, from which we deduce that the pairing (2.21) only pairs nontrivially elements of opposite degrees. From equations (2.22) and (2.23), one can then deduce the pairing on any elements by applying the properties

$$\langle a, b_1 b_2 \rangle = \langle \Delta(a), b_1 \otimes b_2 \rangle \tag{2.24}$$

$$\langle a_1 a_2, b \rangle = \langle a_1 \otimes a_2, \Delta^{\mathrm{op}}(b) \rangle \tag{2.25}$$

for all $a, a_1, a_2 \in \mathcal{A}^{\geq}$ and $b, b_1, b_2 \in \mathcal{A}^{\leq}$. We remark that the pairing also satisfies the property

$$\langle S(a), S(b) \rangle = \langle a, b \rangle$$

with respect to the antipode for all $a \in \mathcal{A}^{\geq}$ and $b \in \mathcal{A}^{\leq}$, but we will not need this fact. The pairing (2.21) was shown to be nondegenerate in [31, Proposition 3.3], although this is also easily seen from our formulas (2.37) and (2.38). Therefore, one can make the vector space

$$\mathcal{A} = \mathcal{A}^{\geq} \otimes \mathcal{A}^{\leq} \tag{2.26}$$

into a Hopf algebra using the well-known Drinfeld double construction, as follows. First, make equation (2.26) into an algebra by requiring that $\mathcal{A}^{\geq} = \mathcal{A}^{\geq} \otimes 1 \subset \mathcal{A}$ and $\mathcal{A}^{\leq} = 1 \otimes \mathcal{A}^{\leq} \subset \mathcal{A}$ be algebra homomorphisms and the multiplication of elements coming from the two tensor factors in the equation be governed by the relation

$$a_1 b_1 \langle a_2, b_2 \rangle = b_2 a_2 \langle a_1, b_1 \rangle \tag{2.27}$$

for any $a \in \mathcal{A}^{\geq} \subset \mathcal{A}, b \in \mathcal{A}^{\leq} \subset \mathcal{A}$.[5] It is straightforward to show that the resulting algebra structure on $\mathcal{A}$ of equation (2.26) matches the one introduced in §2.5. As for the coalgebra structure and antipode on equation (2.26), they are uniquely determined by the respective structures on the two tensor factors $\mathcal{A}^{\geq}$ and $\mathcal{A}^{\leq}$, and multiplicativity.

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[5] We use the Sweedler notation $\Delta(a) = a_1 \otimes a_2$ and $\Delta(b) = b_1 \otimes b_2$ for the coproduct, with the summation sign implied.

2.8. Since the Hopf algebra $\mathcal{A}$ is a Drinfeld double, it has a universal R-matrix

$$\mathcal{R} \in \mathcal{A}^{\geq} \widehat{\otimes} \mathcal{A}^{\leq} \subset \mathcal{A} \widehat{\otimes} \mathcal{A}$$

(the completion is necessary because our coproduct is topological). Specifically, $\mathcal{R}$ is the canonical tensor of the pairing (2.21), and it takes the form[6]

$$\mathcal{R} = \mathcal{R}' \cdot [\text{a sum of products involving the } h_{i,\pm d}], \tag{2.28}$$

where $\mathcal{R}'$ is the canonical tensor of the restriction of the pairing (2.21) to

$$\langle \cdot, \cdot \rangle : \mathcal{A}^+ \otimes \mathcal{A}^- \longrightarrow \mathbb{F}. \tag{2.29}$$

In other words, we have

$$\mathcal{R}' = 1 + \sum_{i \in I} \sum_{d \in \mathbb{Z}} \frac{e_{i,d} \otimes f_{i,-d}}{\gamma_i} + \cdots, \tag{2.30}$$

where the ellipsis denotes terms which are quadratic, cubic, etc., in the es and the fs. In what follows, we will construct a factorization of $\mathcal{R}'$ as an infinite product of R-matrices arising from 'slope subalgebras', generalizing the treatment of cyclic quivers in [27, 26]. Such factorizations are inspired by the analogous constructions pertaining to quantum groups from [6, 14, 13, 16, 17, 37] (which coincide with our construction for simply laced quantum affine groups) and with the constructions of geometric R-matrices from [1, 18, 32, 33, 34] (see §4).

2.9. In what follows, we will need to present the bialgebra structure of §§2.5–2.7 in shuffle-algebra language. More precisely, Theorem 2.3 implies that formulas (2.13), (2.14), (2.19), (2.20), and (2.23) extend from the generators $e_{i,d}$ (resp., $f_{i,d}$) to the entire shuffle algebra $\mathcal{S} = \mathcal{A}^+$ (resp., $\mathcal{S}^{\mathrm{op}} = \mathcal{A}^-$). All the statements in this subsection are straightforward, and left as exercises to the interested reader (equivalently, they were proved in [31, §§3 and 4]). Formulas (2.13) and (2.14) imply that

$$Fh_j^{\pm}(w) = h_j^{\pm}(w)F \prod_{\substack{i \in I \\ 1 \leq a \leq n_i}} \frac{\zeta_{ij}\left(\frac{z_{ia}}{w}\right)}{\zeta_{ji}\left(\frac{w}{z_{ia}}\right)} \tag{2.31}$$

$$Gh_j^{\pm}(w) = h_j^{\pm}(w)G \prod_{\substack{i \in I \\ 1 \leq a \leq n_i}} \frac{\zeta_{ji}\left(\frac{w}{z_{ia}}\right)}{\zeta_{ij}\left(\frac{z_{ia}}{w}\right)} \tag{2.32}$$

for any $F \in \mathcal{S}_{\boldsymbol{n}}$ and any $G \in \mathcal{S}_{\boldsymbol{n}}^{\mathrm{op}}$ (the rational functions in the right-hand sides are expanded as power series in $w^{\mp 1}$). In particular, by extracting the coefficient of w^0 from these formulas, we obtain the following:

$$Fh_{j,+0} = h_{j,+0}F \cdot \prod_{i \in I} \left(q^{\delta_j^i} \prod_{e = \overrightarrow{ij}} \frac{1}{t_e} \prod_{e = \overrightarrow{ji}} \frac{t_e}{q} \right)^{n_i} \tag{2.33}$$

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[6]For a survey of this formula in the particular case of the Jordan quiver, we refer the reader to [30], where we recall the standard difficulties in properly defining this product.

$$Gh_{j,-0} = h_{j,-0}G \cdot \prod_{i \in I} \left(\frac{1}{q^{\delta^i}} \prod_{e=\overrightarrow{ij}} \frac{q}{t_e} \prod_{e=\overrightarrow{ji}} t_e \right)^{n_i}. \tag{2.34}$$

As for formulas (2.19) and (2.20), they imply the following for any $F \in \mathcal{S}_{\boldsymbol{n}}$ and $G \in \mathcal{S}_{\boldsymbol{n}}^{\mathrm{op}}$:

$$\Delta(F) = \sum_{\{0 \le k_i \le n_i\}_{i \in I}} \frac{\prod_{k_j < b \le n_j}^{j \in I} h_j^+(z_{jb}) \, F(\ldots, z_{i1}, \ldots, z_{ik_i} \otimes z_{i,k_i+1}, \ldots, z_{in_i}, \ldots)}{\prod_{1 \le a \le k_i}^{i \in I} \prod_{k_j < b \le n_j}^{j \in I} \zeta_{ji} \left(\frac{z_{jb}}{z_{ia}} \right)} \tag{2.35}$$

$$\Delta(G) = \sum_{\{0 \le k_i \le n_i\}_{i \in I}} \frac{G(\ldots, z_{i1}, \ldots, z_{ik_i} \otimes z_{i,k_i+1}, \ldots, z_{in_i}, \ldots) \prod_{1 \le a \le k_i}^{i \in I} h_i^-(z_{ia})}{\prod_{1 \le a \le k_i}^{i \in I} \prod_{k_j < b \le n_j}^{i \in I} \zeta_{ij} \left(\frac{z_{ia}}{z_{jb}} \right)}. \tag{2.36}$$

To make sense of the right-hand side of these equations, we expand the denominators as power series in the range $|z_{ia}| \ll |z_{jb}|$ and place all the powers of z_{ia} to the left of the $\otimes$ sign and all the powers of z_{jb} to the right of the $\otimes$ sign (for all $i,j \in I, 1 \le a \le k_i, k_j < b \le n_j$).

Finally, formulas (2.22) and (2.23) together with the defining properties (2.24) and (2.25) of a bialgebra pairing imply (see [31, formulas (3.2) and (3.30)], respectively) that

$$\langle F, f_{i_1,d_1} * \cdots * f_{i_n,d_n} \rangle = \int_{|z_1| \ll \cdots \ll |z_n|} \frac{z_1^{d_1} \cdots z_n^{d_n} F(z_1, \ldots, z_n)}{\prod_{1 \le a < b \le n} \zeta_{i_a i_b} \left(\frac{z_a}{z_b} \right)} \prod_{a=1}^n \frac{dz_a}{2\pi i z_a} \tag{2.37}$$

$$\langle e_{i_1,d_1} * \cdots * e_{i_n,d_n}, G \rangle = \int_{|z_1| \gg \cdots \gg |z_n|} \frac{z_1^{d_1} \cdots z_n^{d_n} G(z_1, \ldots, z_n)}{\prod_{1 \le a < b \le n} \zeta_{i_b i_a} \left(\frac{z_b}{z_a} \right)} \prod_{a=1}^n \frac{dz_a}{2\pi i z_a} \tag{2.38}$$

for any $F \in \mathcal{S}$ (resp., $G \in \mathcal{S}^{\mathrm{op}}$) and any $i_1, \ldots, i_n \in I, d_1, \ldots, d_n \in \mathbb{Z}$ such that the shuffle elements being paired in any $\langle \cdot, \cdot \rangle$ of these equations have opposite degrees. In order for formula (2.37) to make sense, one needs to plug the variable z_a into a variable of the form $z_{i_a \bullet}$ of F, where the choice of $\bullet$ does not matter due to the symmetry of F. The analogous remark applies to formula (2.38).

3. Slope subalgebras and factorizations of *R*-matrices

3.1. We will consider $\mathbb{N}^I \subset \mathbb{Z}^I \subset \mathbb{Q}^I$. Recall that $\mathbb{N}^I$ includes the element $\boldsymbol{0} = (0, \ldots, 0)$, according to our convention that $0 \in \mathbb{N}$, as well as the elements

$$\boldsymbol{\varsigma}^i = \underbrace{(0, \ldots, 0, 1, 0, \ldots, 0)}_{\text{1 on } i\text{th position}}, \quad \forall i \in I.$$

We will consider two operations on $\mathbb{N}^I \subset \mathbb{Z}^I \subset \mathbb{Q}^I$, namely the dot product

$$\boldsymbol{k} \cdot \boldsymbol{l} = \sum_{i \in I} k_i l_i \tag{3.1}$$

and the bilinear form

$$\langle \boldsymbol{k}, \boldsymbol{l} \rangle = \sum_{i,j \in I} k_i l_j \#_{\overrightarrow{ij}} \tag{3.2}$$

for any $\boldsymbol{k} = (k_i)_{i \in I}$ and $\boldsymbol{l} = (l_i)_{i \in I}$, where

$$\#_{\overrightarrow{ij}} = \text{the number of arrows } \overrightarrow{ij} \text{ in } Q. \tag{3.3}$$

For any $\boldsymbol{k} = (k_i)_{i \in I}$ and $\boldsymbol{n} = (n_i)_{i \in I}$, we will write

$$\boldsymbol{0} \le \boldsymbol{k} \le \boldsymbol{n} \tag{3.4}$$

if $0 \le k_i \le n_i$ for all $i \in I$ (we will use the notation $\boldsymbol{0} < \boldsymbol{k} < \boldsymbol{n}$ if we wish to further indicate that $\boldsymbol{k} \ne \boldsymbol{0}$ and $\boldsymbol{k} \ne \boldsymbol{n}$). Finally, let

$$|\boldsymbol{n}| = \sum_{i \in I} n_i. \tag{3.5}$$

3.2. The following is the key notion of this section:

Definition 3.3. Set $\boldsymbol{m} \in \mathbb{Q}^I$. We will say that $F \in \mathcal{A}^+$ has *slope* $\le \boldsymbol{m}$ if

$$\lim_{\xi \to \infty} \frac{F\left(\dots, \xi z_{i1}, \dots, \xi z_{ik_i}, z_{i,k_i+1}, \dots, z_{in_i}, \dots\right)}{\xi^{\boldsymbol{m} \cdot \boldsymbol{k} + \langle \boldsymbol{k}, \boldsymbol{n} - \boldsymbol{k} \rangle}} \tag{3.6}$$

is finite for all $\boldsymbol{0} \le \boldsymbol{k} \le \boldsymbol{n}$. Similarly, we will say that $G \in \mathcal{A}^-$ has *slope* $\le \boldsymbol{m}$ if

$$\lim_{\xi \to 0} \frac{G\left(\dots, \xi z_{i1}, \dots, \xi z_{ik_i}, z_{i,k_i+1}, \dots, z_{in_i}, \dots\right)}{\xi^{-\boldsymbol{m} \cdot \boldsymbol{k} - \langle \boldsymbol{n} - \boldsymbol{k}, \boldsymbol{k} \rangle}} \tag{3.7}$$

is finite for all $\boldsymbol{0} \le \boldsymbol{k} \le \boldsymbol{n}$.

We will also say that $F \in \mathcal{A}^+$ and $G \in \mathcal{A}^-$ have *naive slope* $\le \boldsymbol{m}$ if

$$\text{vdeg} F \le \boldsymbol{m} \cdot \text{hdeg} F \tag{3.8}$$

$$\text{vdeg} G \ge \boldsymbol{m} \cdot \text{hdeg} G. \tag{3.9}$$

The $\boldsymbol{k} = \boldsymbol{n}$ case of formulas (3.6) and (3.7) shows that having slope $\le \boldsymbol{m}$ implies having naive slope $\le \boldsymbol{m}$. This fact can also be seen as a particular case of the following:

Proposition 3.4. *An element $F \in \mathcal{A}^+$ has slope $\le \boldsymbol{m}$ if and only if*

$$\Delta(F) = (\text{anything}) \otimes (\text{naive slope} \le \boldsymbol{m}). \tag{3.10}$$

Similarly, an element $G \in \mathcal{A}^-$ has slope $\le \boldsymbol{m}$ if and only if

$$\Delta(G) = (\text{naive slope} \le \boldsymbol{m}) \otimes (\text{anything}). \tag{3.11}$$

The meaning of the right-hand side of these equations is that $\Delta(F)$ (resp., $\Delta(G)$) is an infinite sum of tensors, all of whose second (resp., first) factors have naive slope $\le \boldsymbol{m}$. Moreover, these statements would remain true if we replaced 'naive slope' with 'slope'.

Proof. Let us prove the statements pertaining to F, and leave the analogous case of G as an exercise for the interested reader. We will write

$$F = \sum_{\{d_{ia}\}_{1\leq a\leq n_i}^{i\in I}} \text{coefficient} \cdot \prod_{1\leq a\leq n_i}^{i\in I} z_{ia}^{d_{ia}}$$

for various coefficients. Note that

$$\zeta_{ji}\left(\frac{1}{x}\right)^{-1} \in x^{\#\overrightarrow{ji}}\mathbb{F}[[x]]^{\times}.$$

Then, as a consequence of equation (2.35), we have

$$\Delta(F) = \sum_{\mathbf{0}\leq\mathbf{k}\leq\mathbf{n}} \sum_{\{d_{ia}\}_{1\leq a\leq n_i}^{i\in I}} \sum_{\substack{\{p_{jb}\geq 0\}_{k_j<b\leq n_j}^{j\in I} \\ \left\{e_{jb}^{ia}\geq\#\overrightarrow{ji}\right\}_{1\leq a\leq k_i, k_j<b\leq n_j}^{i,j\in I}}} \text{coefficient}\cdot$$

$$\prod_{k_j<b\leq n_j}^{j\in I} h_{j,+p_{jb}} \prod_{1\leq a\leq k_i}^{i\in I} z_{ia}^{d_{ia}+\sum_{k_j<b\leq n_j}^{j\in I} e_{jb}^{ia}} \otimes \underbrace{\prod_{k_j<b\leq n_j}^{j\in I} z_{jb}^{d_{jb}-p_{jb}-\sum_{1\leq a\leq k_i}^{i\in I} e_{jb}^{ia}}}_{\text{call this } F_2}. \tag{3.12}$$

The homogeneous degree of any second tensor factor in this formula satisfies

$$\text{vdeg} F_2 \leq \sum_{k_j<b\leq n_j}^{j\in I}\left(d_{jb} - \sum_{1\leq a\leq k_i}^{i\in I} \#\overrightarrow{ji}\right) = \sum_{k_j<b\leq n_j}^{j\in I} d_{jb} - \langle \mathbf{n}-\mathbf{k}, \mathbf{k}\rangle.$$

By the assumption (3.6), the right-hand side of this expression is

$$\leq \mathbf{m}\cdot(\mathbf{n}-\mathbf{k}) = \mathbf{m}\cdot\text{hdeg} F_2,$$

which implies that F_2 has naive slope $\leq \mathbf{m}$. Conversely, the terms F_2 in equation (3.12) with maximal naive slope are the ones corresponding to $p_{jb}=0$ and $e_{jb}^{ia}=\#\overrightarrow{ji}$. If these F_2s have naive slope $\leq \mathbf{m}$, then the chain of inequalities implies precisely

$$\sum_{k_j<b\leq n_j}^{j\in I} d_{jb} \leq \mathbf{m}\cdot(\mathbf{n}-\mathbf{k}) + \langle\mathbf{n}-\mathbf{k}, \mathbf{k}\rangle.$$

Since this holds for all $\mathbf{0}\leq\mathbf{k}\leq\mathbf{n}$, we obtain precisely expression (3.6).

It remains to show that we can replace the weaker notion of 'naive slope' with the stronger 'slope' in equation (3.10). To this end, we will explicitly show that if we write

$$\Delta(F) = \sum_{s\in S} F_{1,s}\otimes F_{2,s},$$

where $\{F_{1,s}\}_{s\in S}$ is an arbitrary linear basis of $\mathcal{A}^{\geq}$, then every $F_{2,s}$ has slope $\leq \mathbf{m}$. The key to proving this fact is the coassociativity of the coproduct

$$(\text{Id}\otimes\Delta)\circ\Delta(F) = (\Delta\otimes\text{Id})\circ\Delta(F).$$

The left-hand side of this expression is precisely

$$\sum_{s \in S} F_{1,s} \otimes \Delta(F_{2,s}),$$

and the right-hand side is of the form

$$(\text{anything}) \otimes (\text{anything}) \otimes (\text{naive slope} \leq \boldsymbol{m})$$

by equation (3.10). For any given $s \in S$, identifying the coefficients of $F_{1,s} \otimes - \otimes -$ in the two expressions here implies that $\Delta(F_{2,s}) = (\text{anything}) \otimes (\text{naive slope} \leq \boldsymbol{m})$. By equation (3.10), this precisely means that $F_{2,s}$ has slope $\leq \boldsymbol{m}$, as we needed to prove. $\qquad\square$

Let us denote the subspaces of shuffle elements of slope $\leq \boldsymbol{m}$ by

$$\mathcal{A}^{\pm}_{\leq \boldsymbol{m}} \subset \mathcal{A}^{\pm}. \tag{3.13}$$

Proposition 3.4 and the multiplicativity of Δ show that $\mathcal{A}^{\pm}_{\leq \boldsymbol{m}}$ are algebras.

3.5. It is easy to see that the graded pieces of $\mathcal{A}^{\pm}_{\leq \boldsymbol{m}}$, namely

$$\mathcal{A}_{\leq \boldsymbol{m}|\pm \boldsymbol{n}, \pm d} = \mathcal{A}_{\pm \boldsymbol{n}, \pm d} \cap \mathcal{A}^{\pm}_{\leq \boldsymbol{m}},$$

are finite-dimensional for any $(\boldsymbol{n}, d) \in \mathbb{N}^I \times \mathbb{Z}$. This is because expression (3.6) (resp., (3.7)) imposes upper (resp., lower) bounds on the exponents of the variables that make up the Laurent polynomials F (resp., G). If we also fix the total homogeneous degree of such a polynomial, then there are finitely many choices for the monomials which make up F (resp., G).

Definition 3.6. For any $\boldsymbol{m} \in \mathbb{Q}^I$, we will write

$$\mathcal{B}^{\pm}_{\boldsymbol{m}} \subset \mathcal{A}^{\pm} \tag{3.14}$$

for the subalgebras consisting of elements of slope $\leq \boldsymbol{m}$ and naive slope $= \boldsymbol{m}$.[7]

We will denote the graded pieces of $\mathcal{B}^{\pm}_{\boldsymbol{m}}$ by

$$\mathcal{B}^{\pm}_{\boldsymbol{m}} = \bigoplus_{\boldsymbol{n} \in \mathbb{N}^I} \mathcal{B}_{\boldsymbol{m}|\pm \boldsymbol{n}}, \tag{3.15}$$

where $\mathcal{B}_{\boldsymbol{m}|\pm \boldsymbol{n}} = \mathcal{A}_{\leq \boldsymbol{m}|\pm \boldsymbol{n}, \pm \boldsymbol{m} \cdot \boldsymbol{n}}$. If $\boldsymbol{m} \cdot \boldsymbol{n} \notin \mathbb{Z}$ for some $\boldsymbol{n} \in \mathbb{N}^I$, the respective direct summand in equation (3.15) is zero. As for the nonzero direct summands, they are all finite-dimensional, as was explained in the beginning of this subsection.

3.7. We can make the algebras $\mathcal{B}^{\pm}_{\boldsymbol{m}}$ into Hopf algebras if we first extend them:

$$\mathcal{B}^{\geq}_{\boldsymbol{m}} = \mathcal{B}^{+}_{\boldsymbol{m}} \otimes \mathbb{F}\left[h^{\pm 1}_{i,+0}\right]_{i \in I} \Big/ \text{relation (2.33)} \tag{3.16}$$

$$\mathcal{B}^{\leq}_{\boldsymbol{m}} = \mathcal{B}^{-}_{\boldsymbol{m}} \otimes \mathbb{F}\left[h^{\pm 1}_{i,-0}\right]_{i \in I} \Big/ \text{relation (2.34)}. \tag{3.17}$$

---

[7]Having naive slope $= \boldsymbol{m}$ means having equality in formulas (3.8) and (3.9). This terminology is slightly ambiguous, as there may be infinitely many values of $\boldsymbol{m}$ for which equality holds, but in what follows this ambiguity will be clarified by the context.

There is a coproduct $\Delta_{\boldsymbol{m}}$ on the subalgebras (3.16) and (3.17), determined by

$$\Delta_{\boldsymbol{m}}\left(h_{i,\pm 0}\right) = h_{i,\pm 0} \otimes h_{i,\pm 0}$$

and the following formulas for any $F \in \mathcal{B}_{\boldsymbol{m}|\boldsymbol{n}}$ and $G \in \mathcal{B}_{\boldsymbol{m}|-\boldsymbol{n}}$:

$$\Delta_{\boldsymbol{m}}(F) = \sum_{\boldsymbol{0} \leq \boldsymbol{k} \leq \boldsymbol{n}} \lim_{\xi \to \infty} \frac{h_{\boldsymbol{n}-\boldsymbol{k}} F\left(\ldots, z_{i1}, \ldots, z_{ik_i} \otimes \xi z_{i,k_i+1}, \ldots, \xi z_{in_i}, \ldots\right)}{\xi^{\boldsymbol{m}\cdot(\boldsymbol{n}-\boldsymbol{k})} \cdot \text{lead}\left[\prod_{1 \leq a \leq k_i}^{i \in I} \prod_{k_j < b \leq n_j}^{j \in I} \zeta_{ji}\left(\frac{\xi z_{jb}}{z_{ia}}\right)\right]} \tag{3.18}$$

$$\Delta_{\boldsymbol{m}}(G) = \sum_{\boldsymbol{0} \leq \boldsymbol{k} \leq \boldsymbol{n}} \lim_{\xi \to 0} \frac{G\left(\ldots, \xi z_{i1}, \ldots, \xi z_{ik_i} \otimes z_{i,k_i+1}, \ldots, z_{in_i}, \ldots\right) h_{-\boldsymbol{k}}}{\xi^{-\boldsymbol{m}\cdot\boldsymbol{k}} \cdot \text{lead}\left[\prod_{1 \leq a \leq k_i}^{i \in I} \prod_{k_j < b \leq n_j}^{j \in I} \zeta_{ij}\left(\frac{\xi z_{ia}}{z_{jb}}\right)\right]}, \tag{3.19}$$

where 'lead $[\ldots]$' refers to the leading-order term in ξ of the expression marked by the ellipsis (expanded as $\xi \to \infty$ or as $\xi \to 0$, depending on the situation) and

$$h_{\pm \boldsymbol{n}} = \prod_{i \in I} h_{i,\pm 0}^{n_i} \tag{3.20}$$

for all $\boldsymbol{n} \in \mathbb{N}^I$. By its very definition, $\Delta_{\boldsymbol{m}}$ consists of the leading naive slope terms in formulas (2.35) and (2.36), in the sense that

$$\Delta_{\boldsymbol{m}}(F) = \text{component of } \Delta(F) \text{ in } \bigoplus_{\boldsymbol{n}=\boldsymbol{n}_1+\boldsymbol{n}_2} h_{\boldsymbol{n}_2} \mathcal{A}_{\boldsymbol{n}_1, \boldsymbol{m}\cdot\boldsymbol{n}_1} \otimes \mathcal{A}_{\boldsymbol{n}_2, \boldsymbol{m}\cdot\boldsymbol{n}_2}$$

$$\Delta_{\boldsymbol{m}}(G) = \text{component of } \Delta(G) \text{ in } \bigoplus_{\boldsymbol{n}=\boldsymbol{n}_1+\boldsymbol{n}_2} \mathcal{A}_{-\boldsymbol{n}_1, -\boldsymbol{m}\cdot\boldsymbol{n}_1} \otimes \mathcal{A}_{-\boldsymbol{n}_2, -\boldsymbol{m}\cdot\boldsymbol{n}_2} h_{-\boldsymbol{n}_1}$$

for all $F \in \mathcal{B}_{\boldsymbol{m}}^+, G \in \mathcal{B}_{\boldsymbol{m}}^-$. Thus, the fact that $\Delta_{\boldsymbol{m}}$ makes $\mathcal{B}_{\boldsymbol{m}}^{\geq}$ and $\mathcal{B}_{\boldsymbol{m}}^{\leq}$ into bialgebras is induced by the fact that Δ makes $\mathcal{A}^{\geq}$ and $\mathcal{A}^{\leq}$ into bialgebras.

Proposition 3.8. *The restriction of the pairing (2.21) to*

$$\langle \cdot, \cdot \rangle : \mathcal{B}_{\boldsymbol{m}}^{\geq} \otimes \mathcal{B}_{\boldsymbol{m}}^{\leq} \longrightarrow \mathbb{F} \tag{3.21}$$

satisfies properties (2.24) and (2.25) with respect to the coproduct $\Delta_{\boldsymbol{m}}$.

Proof. Let us check equation (2.24) and leave the analogous formula (2.25) as an exercise for the interested reader. Moreover, we will consider only the case when $a \in \mathcal{B}_{\boldsymbol{m}}^+$ and $b_1, b_2 \in \mathcal{B}_{\boldsymbol{m}}^-$, as the situation when one or more of a, b_1, b_2 is of the form (3.20) is quite easy, and so left as an exercise for the interested reader. Thus, let us write

$$\Delta(a) = \sum_{s \in S} a_{1,s} \otimes a_{2,s},$$

where S is some indexing set. Formula (3.10) gives us

$$\text{vdeg}a_{2,s} \leq \boldsymbol{m} \cdot (\text{hdeg}a_{2,s}) \Leftrightarrow \text{vdeg}a_{1,s} \geq \boldsymbol{m} \cdot (\text{hdeg}a_{1,s}) \tag{3.22}$$

(the equivalence is due to the fact that $\text{vdeg}a = \boldsymbol{m} \cdot (\text{hdeg}a)$, on account of the very definition of $\mathcal{B}_{\boldsymbol{m}}^{+} \ni a$). The definition of $\Delta_{\boldsymbol{m}}$ implies that

$$\Delta_{\boldsymbol{m}}(a) = \sum_{s \in S'} a_{1,s} \otimes a_{2,s},$$

where the indexing set $S' \subset S$ consists of those $s \in S$ for which equality holds in formula (3.22). The fact that equation (2.24) holds with respect to Δ means that

$$\langle a, b_1 b_2 \rangle = \sum_{s \in S} \langle a_{1,s}, b_1 \rangle \langle a_{2,s}, b_2 \rangle.$$

However, because $\text{vdeg}b_{1,2} = \boldsymbol{m} \cdot (\text{hdeg}b_{1,2})$, the pairings in this formula are nonzero only if we have equality in formula (3.22) – that is, only if $s \in S'$. Therefore,

$$\langle a, b_1 b_2 \rangle = \sum_{s \in S'} \langle a_{1,s}, b_1 \rangle \langle a_{2,s}, b_2 \rangle,$$

which precisely states that equation (2.24) also holds with respect to the coproduct $\Delta_{\boldsymbol{m}}$. □

3.9. The pairing (3.21) is nondegenerate,[8] as we will show in Proposition 3.18. This will allow us to define the Drinfeld double

$$\mathcal{B}_{\boldsymbol{m}} = \mathcal{B}_{\boldsymbol{m}}^{\geq} \otimes \mathcal{B}_{\boldsymbol{m}}^{\leq} \tag{3.23}$$

as in §2.7, which has a universal R-matrix as in §2.8:

$$\mathcal{R}_{\boldsymbol{m}} \in \mathcal{B}_{\boldsymbol{m}}^{\geq} \widehat{\otimes} \mathcal{B}_{\boldsymbol{m}}^{\leq} \subset \mathcal{B}_{\boldsymbol{m}} \widehat{\otimes} \mathcal{B}_{\boldsymbol{m}}.$$

Explicitly, $\mathcal{R}_{\boldsymbol{m}}$ is the canonical tensor of the pairing (3.21). As in equation (2.28), we have

$$\mathcal{R}_{\boldsymbol{m}} = \mathcal{R}_{\boldsymbol{m}}' \cdot [\text{a sum of products involving the } h_{i,\pm 0}], \tag{3.24}$$

where $\mathcal{R}_{\boldsymbol{m}}'$ is the canonical tensor of the restriction of the pairing (3.21) to

$$\langle \cdot, \cdot \rangle : \mathcal{B}_{\boldsymbol{m}}^{+} \otimes \mathcal{B}_{\boldsymbol{m}}^{-} \longrightarrow \mathbb{F}. \tag{3.25}$$

Although they look similar, we emphasize the fact that the Drinfeld doubles $\mathcal{A}$ and $\mathcal{B}_{\boldsymbol{m}}$ are defined with respect to the different coproducts Δ and $\Delta_{\boldsymbol{m}}$, respectively. Since the product in a Drinfeld double (namely relation (2.27)) is controlled by the coproduct that is used to define the double, the following result is nontrivial:

Proposition 3.10. *The inclusion map $\mathcal{B}_{\boldsymbol{m}} \subset \mathcal{A}$ (obtained by tensoring together the natural inclusion maps $\mathcal{B}_{\boldsymbol{m}}^{\geq} \subset \mathcal{A}^{\geq}$ and $\mathcal{B}_{\boldsymbol{m}}^{\leq} \subset \mathcal{A}^{\leq}$) is an algebra homomorphism.*

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[8]As usual in the theory of quantum groups, this statement is true as stated for the restricted pairing (3.25) to the $\pm$ subalgebras. To have the statement hold for the $\geq, \leq$ subalgebras, one needs to work instead over the power series ring in $\log(q), \log(t_e)$ instead of over $\mathbb{F} = \mathbb{Q}(q, t_e)_{e \in E}$.

Proof. Consider any $a \in \mathcal{B}_{\boldsymbol{m}}^{\geq}$ and $b \in \mathcal{B}_{\boldsymbol{m}}^{\leq}$, and let us write

$$\Delta(a) = \sum_{s \in S} a_{1,s} \otimes a_{2,s} \qquad \text{and} \qquad \Delta(b) = \sum_{t \in T} b_{1,t} \otimes b_{2,t}$$

for some sets S and T. By the definition of $\Delta_{\boldsymbol{m}}$, we have

$$\Delta_{\boldsymbol{m}}(a) = \sum_{s \in S'} a_{1,s} \otimes a_{2,s} \qquad \text{and} \qquad \Delta_{\boldsymbol{m}}(b) = \sum_{t \in T'} b_{1,t} \otimes b_{2,t},$$

where the indexing sets $S' \subset S, T' \subset T$ consist of those $s \in S, t \in T$ such that

$$\text{vdeg}a_{1,s} = \boldsymbol{m} \cdot (\text{hdeg}a_{1,s}) \Leftrightarrow \text{vdeg}a_{2,s} = \boldsymbol{m} \cdot (\text{hdeg}a_{2,s})$$
$$\text{vdeg}b_{1,t} = \boldsymbol{m} \cdot (\text{hdeg}b_{1,t}) \Leftrightarrow \text{vdeg}b_{2,t} = \boldsymbol{m} \cdot (\text{hdeg}b_{2,t}).$$

Formula (2.27) implies that the following relation holds in $\mathcal{A}$:

$$\sum_{s \in S, t \in T} a_{1,s} b_{1,t} \langle a_{2,s}, b_{2,t} \rangle = \sum_{s \in S, t \in T} b_{2,t} a_{2,s} \langle a_{1,s}, b_{1,t} \rangle. \tag{3.26}$$

However, formulas (3.10) and (3.11) imply that $a_{2,s}$ and $b_{1,t}$ have naive slope $\leq \boldsymbol{m}$, for all $s \in S$ and $t \in T$. This implies that

$$\text{vdeg}a_{2,s} \leq \boldsymbol{m} \cdot (\text{hdeg}a_{2,s}) \Rightarrow \text{vdeg}a_{1,s} \geq \boldsymbol{m} \cdot (\text{hdeg}a_{1,s})$$
$$\text{vdeg}b_{1,t} \geq \boldsymbol{m} \cdot (\text{hdeg}b_{1,t}) \Rightarrow \text{vdeg}b_{2,t} \leq \boldsymbol{m} \cdot (\text{hdeg}b_{2,t}),$$

where in both cases, the implication is due to our assumption that $\text{vdeg}a = \boldsymbol{m} \cdot (\text{hdeg}a)$ and $\text{vdeg}b = \boldsymbol{m} \cdot (\text{hdeg}b)$. Therefore, the only way for the pairings in the left- and right-hand sides of equation (3.26) to be nonzero is to have equality in all these inequalities, which would imply $s \in S'$ and $t \in T'$. We therefore have

$$\sum_{s \in S', t \in T'} a_{1,s} b_{1,t} \langle a_{2,s}, b_{2,t} \rangle = \sum_{s \in S', t \in T'} b_{2,t} a_{2,s} \langle a_{1,s}, b_{1,t} \rangle.$$

However, this is simply equation (2.27) in the double $\mathcal{B}_{\boldsymbol{m}}$, which implies that the same multiplicative relations hold in $\mathcal{A}$ as in $\mathcal{B}_{\boldsymbol{m}}$. $\qquad \square$

3.11. Let us now fix $\boldsymbol{m} \in \mathbb{Q}^I$ and $\boldsymbol{\theta} \in \mathbb{Q}_+^I$, and consider the subalgebras $\{\mathcal{B}_{\boldsymbol{m}+r\boldsymbol{\theta}}\}_{r \in \mathbb{Q}}$.

Proposition 3.12. *For any $\boldsymbol{m} \in \mathbb{Q}^I$ and $\boldsymbol{\theta} \in \mathbb{Q}_+^I$, we have*

$$\left\langle \overrightarrow{\prod_{r \in \mathbb{Q}}} a_r, \overrightarrow{\prod_{r \in \mathbb{Q}}} b_r \right\rangle = \overrightarrow{\prod_{r \in \mathbb{Q}}} \langle a_r, b_r \rangle \tag{3.27}$$

for all elements $\left\{ a_r \in \mathcal{B}_{\boldsymbol{m}+r\boldsymbol{\theta}}^+, b_r \in \mathcal{B}_{\boldsymbol{m}+r\boldsymbol{\theta}}^- \right\}_{r \in \mathbb{Q}}$, almost all of which are equal to 1.

Proof. Let $r \in \mathbb{Q}$ be maximal such that $a_r \neq 1$ or $b_r \neq 1$, and let us assume that $|\text{hdeg}a_r| \geq -|\text{hdeg}b_r|$ (the opposite case is treated analogously, so we leave it as an exercise for the

interested reader). Then formula (2.25) implies

$$\left\langle \prod_{r'<r} a_{r'} \cdot a_r, \prod_{r'\leq r} b_{r'} \right\rangle = \left\langle \prod_{r'<r} a_{r'}, \prod_{r'\leq r} b_{r',2} \right\rangle \left\langle a_r, \prod_{r'\leq r} b_{r',1} \right\rangle, \tag{3.28}$$

where we use the Sweedler notation $\Delta(b_r) = b_{r,1} \otimes b_{r,2}$. Because of equation (3.11), all of $b_{r',1}$ with $r' < r$ have slope strictly smaller than $\boldsymbol{m} + r\boldsymbol{\theta}$. Therefore, the only term in the right-hand side of this expression which could pair nontrivially with $a_r \in \mathcal{B}^+_{\boldsymbol{m}+r\boldsymbol{\theta}}$ is $b_{r,1}$. However, because of the assumption that $|\mathrm{hdeg}a_r| \geq -|\mathrm{hdeg}b_r|$, such a nontrivial pairing is possible only if all three of the following properties hold:

- $\mathrm{hdeg}a_r = -\mathrm{hdeg}\, b_r$;
- $b_{r,1}$ is the first tensor factor in the first summand of

$$\Delta(b_r) = b_r \otimes h_{-\mathrm{hdeg}b_r} + \cdots;$$

- $b_{r',1}$ is the first tensor factor in the first summand of

$$\Delta(b_{r'}) = 1 \otimes b_{r'} + \cdots$$

for all $r' < r$.

Therefore, equation (3.28) implies

$$\left\langle \prod_{r'\leq r} a_{r'}, \prod_{r'\leq r} b_{r'} \right\rangle = \left\langle \prod_{r'<r} a_{r'}, \prod_{r'<r} b_{r'} \cdot h_{-\mathrm{hdeg}b_r} \right\rangle \langle a_r, b_r \rangle. \tag{3.29}$$

Since equation (2.24) implies the identity

$$\langle a, b \cdot h_{-\boldsymbol{n}} \rangle = \langle a, b \rangle \langle 1, h_{-\boldsymbol{n}} \rangle = \langle a, b \rangle$$

for any $a \in \mathcal{A}^+, b \in \mathcal{A}^-$ and any $\boldsymbol{n} \in \mathbb{N}^I$, the right-hand side of equation (3.29) is unchanged if we remove $h_{-\mathrm{hdeg}b_r}$. Iterating identity (3.29) for the various r' for which $a_{r'} \neq 1$ or $b_{r'} \neq 1$, one obtains identity (3.27). $\qquad\square$

3.13. Still fixing $\boldsymbol{m} \in \mathbb{Q}^I$ and $\boldsymbol{\theta} \in \mathbb{Q}^I_+$ as before, our main goal in the next subsections (see Corollary 3.20) is to prove that multiplication yields isomorphisms

$$\overset{\rightarrow}{\bigotimes_{r\in\mathbb{Q}}} \mathcal{B}^\pm_{\boldsymbol{m}+r\boldsymbol{\theta}} \xrightarrow{\sim} \mathcal{A}^\pm. \tag{3.30}$$

If we write $\mathcal{B}^\pm_{\boldsymbol{m}+\infty\boldsymbol{\theta}} = \mathbb{F}\left[h^{\pm 1}_{i,\pm 0}, h_{i,\pm 1}, \dots \right]_{i\in I}$, then formula (3.30) leads to isomorphisms

$$\overset{\rightarrow}{\bigotimes_{r\in\mathbb{Q}\sqcup\infty}} \mathcal{B}^+_{\boldsymbol{m}+r\boldsymbol{\theta}} \xrightarrow{\sim} \mathcal{A}^\geq \quad \text{and} \quad \overset{\rightarrow}{\bigotimes_{r\in\mathbb{Q}\sqcup\infty}} \mathcal{B}^-_{\boldsymbol{m}+r\boldsymbol{\theta}} \xrightarrow{\sim} \mathcal{A}^\leq. \tag{3.31}$$

Thus, the entire $\mathcal{A} = \mathcal{A}^\geq \otimes \mathcal{A}^\leq$ factors as the tensor product of the $\left\{ \mathcal{B}^\pm_{\boldsymbol{m}+r\boldsymbol{\theta}} \right\}_{r\in\mathbb{Q}\sqcup\infty}$.

Proposition 3.14. *For any $\boldsymbol{m} \in \mathbb{Q}^I$, $\boldsymbol{\theta} \in \mathbb{Q}_+^I$, and $p \in \mathbb{Q}$, the multiplication map*

$$\overset{\rightarrow}{\bigotimes_{r \in \mathbb{Q}_{\leq p}}} \mathcal{B}_{\boldsymbol{m}+r\boldsymbol{\theta}}^{\pm} \twoheadrightarrow \mathcal{A}_{\leq \boldsymbol{m}+p\boldsymbol{\theta}}^{\pm} \tag{3.32}$$

is surjective (here, $\mathbb{Q}_{\leq p}$ denotes the set of rational numbers $\leq p$).

Since any element of $\mathcal{A}^+$ has slope $\leq \boldsymbol{m}+r\boldsymbol{\theta}$ for $r \in \mathbb{Q}$ large enough (this is because $\boldsymbol{\theta} \in \mathbb{Q}_+^I$), the surjectivity of formula (3.32) implies that the multiplication map

$$\overset{\rightarrow}{\bigotimes_{r \in \mathbb{Q}}} \mathcal{B}_{\boldsymbol{m}+r\boldsymbol{\theta}}^{\pm} \twoheadrightarrow \mathcal{A}^{\pm} \tag{3.33}$$

is also surjective.

Proof of Proposition 3.14. Let us consider the restriction of the multiplication map (3.32) to the subspaces of given degree $(\boldsymbol{n}, d) \in \mathbb{N}^I \times \mathbb{Z}$:

$$\bigoplus_{\substack{\sum_{r \in \mathbb{Q}_{\leq p}} \boldsymbol{n}_r = \boldsymbol{n} \\ \sum_{r \in \mathbb{Q}_{\leq p}} (\boldsymbol{m}+r\boldsymbol{\theta}) \cdot \boldsymbol{n}_r = d}} \left[\overset{\rightarrow}{\bigotimes_{r \in \mathbb{Q}_{\leq p}}} \mathcal{B}_{\boldsymbol{m}+r\boldsymbol{\theta}|\pm\boldsymbol{n}_r} \right] \xrightarrow{\phi_{\pm\boldsymbol{n}, \pm d}} \mathcal{A}_{\leq \boldsymbol{m}+p\boldsymbol{\theta}|\pm\boldsymbol{n}, \pm d} \tag{3.34}$$

(the indexing set goes over all sequences $(\boldsymbol{n}_r)_{r \in \mathbb{Q}_{\leq p}}$ of elements of $\mathbb{N}^I$, almost all of which are 0). We will prove that $\phi_{\pm\boldsymbol{n}, \pm d}$ is surjective by induction on $\boldsymbol{n}$, with respect to the ordering (3.4) (the base case, when $\boldsymbol{n} = \boldsymbol{\varsigma}^i$ for some $i \in I$, is trivial). To streamline the subsequent explanation, if a certain element of the shuffle algebra has slope (or naive slope) $\boldsymbol{m}+r\boldsymbol{\theta}$, we will refer to the number $r \in \mathbb{Q}$ as its slope (or naive slope). This also has the added benefit of making the notion 'naive slope $= r$' unambiguous, as the fact that $\boldsymbol{\theta} \in \mathbb{Q}_+^I$ means that for any $(\boldsymbol{n}, d) \in \mathbb{N}^I \times \mathbb{Z}$, there exists exactly one rational number r for which $(\boldsymbol{m}+r\boldsymbol{\theta}) \cdot \boldsymbol{n} = d$.

So let us show that any element $F \in \mathcal{A}_{\leq \boldsymbol{m}+p\boldsymbol{\theta}|\boldsymbol{n}, d}$ lies in the image of the map $\phi_{\boldsymbol{n}, d}$ (we will discuss only the case $\pm = +$, as the $\pm = -$ case is analogous). Let $r \leq p$ denote the naive slope of F, and let us call *hinges* those

$$(\boldsymbol{k}, e) \in \mathbb{N}^I \times \mathbb{Z} \tag{3.35}$$

such that $\Delta(F)$ has a nonzero component in

$$\mathcal{A}_{\boldsymbol{n}-\boldsymbol{k}, d-e} \otimes \mathcal{A}_{\boldsymbol{k}, e}. \tag{3.36}$$

Clearly, a hinge would need to satisfy formula (3.4), and by equation (3.10) also the inequality

$$e \leq (\boldsymbol{m}+p\boldsymbol{\theta}) \cdot \boldsymbol{k}. \tag{3.37}$$

We will call a hinge *bad* if

$$e > (\boldsymbol{m}+r\boldsymbol{\theta}) \cdot \boldsymbol{k}. \tag{3.38}$$

It is easy to see that F has finitely many bad hinges, as there are only finitely many values of $\boldsymbol{k}$ satisfying formula (3.4), and for any such $\boldsymbol{k}$, finitely many integers e that satisfy the inequalities (3.37) and (3.38). If F has no bad hinges, then by equation (3.10), it lies in $\mathcal{B}_{\boldsymbol{m}+r\boldsymbol{\theta}}^{+}$ and we are done. Thus our strategy will be to successively subtract from F elements in the image of $\phi_{\boldsymbol{n},d}$ so as to 'kill' all its bad hinges.

The slope of a bad hinge (3.35) is that rational number $\rho \in (r,p)$ such that

$$e = (\boldsymbol{m} + \rho\boldsymbol{\theta}) \cdot \boldsymbol{k}. \tag{3.39}$$

Let us consider the partial order on the set of bad hinges, given primarily by slope, and then by $|\boldsymbol{k}|$ break ties between hinges of the same slope. Let us write $(\boldsymbol{k},e)$ for a maximal bad hinge of F. The number ρ from equation (3.39) is minimal such that

$$F \in \mathcal{A}_{\leq \boldsymbol{m}+\rho\boldsymbol{\theta}}^{+}. \tag{3.40}$$

By the maximality of $(\boldsymbol{k},e)$, the component of $\Delta(F)$ in degree (3.36) is given by

$$\Delta_{(\boldsymbol{n}-\boldsymbol{k},d-e),(\boldsymbol{k},e)}(F) =$$
$$\text{top}\left[\frac{h_{\boldsymbol{k}}F\left(\ldots,z_{i1},\ldots,z_{i,n_i-k_i}\otimes\xi z_{i,n_i-k_i+1},\ldots,\xi z_{in_i},\ldots\right)}{\gamma \cdot \prod_{1\leq a\leq n_i-k_i}^{i\in I}\prod_{n_j-k_j<b\leq n_j}^{j\in I}\left(\frac{\xi z_{jb}}{z_{ia}}\right)^{\#_{\overrightarrow{ji}}}}\right], \tag{3.41}$$

where 'top $[\ldots]$' refers to the top coefficient in ξ of the expression marked by the ellipsis, and $\gamma \in \mathbb{F}^{\times}$. The reason for this formula is that the maximality of $(\boldsymbol{k},e)$ implies that only the leading term of the h power series in the numerator (resp., the ζ rational functions in the denominator) of equation (2.35) can contribute (see equation (3.12)). By writing F as a linear combination of monomials, we have

$$\Delta_{(\boldsymbol{n}-\boldsymbol{k},d-e),(\boldsymbol{k},e)}(F) = \sum_{s\in S} h_{\boldsymbol{k}}F_{1,s}\otimes F_{2,s}, \tag{3.42}$$

where S is some indexing set and $F_{1,s},F_{2,s}$ denote various elements in $\mathcal{A}^+$ that one obtains by summing up the various top coefficients in ξ of equation (3.41).

Claim 3.15. *The element*

$$G = \sum_{s\in S} F_{1,s}F_{2,s} \tag{3.43}$$

lies in the image of $\phi_{\boldsymbol{n},d}$. All its bad hinges are less than or equal to $(\boldsymbol{k},e)$, and

$$\Delta_{(\boldsymbol{n}-\boldsymbol{k},d-e),(\boldsymbol{k},e)}(F) = \gamma' \cdot \Delta_{(\boldsymbol{n}-\boldsymbol{k},d-e),(\boldsymbol{k},e)}(G) \tag{3.44}$$

for some $\gamma' \in \mathbb{F}^{\times}$.

Let us complete the induction step of the surjectivity of $\phi_{\boldsymbol{n},d}$. Claim 3.15 allows us to reduce the fact that F lies in the image of $\phi_{\boldsymbol{n},d}$ to the analogous fact for $F - \gamma'G$, where γ' is the constant that features in equation (3.44). Moreover, the claim implies that $F - \gamma'G$ does not in fact have a bad hinge at $(\boldsymbol{k},e)$. Thus, repeating this argument finitely many

times allows us to reduce F to an element without any bad hinges, which as we have seen must lie in $\mathcal{B}_{\boldsymbol{m}+r\boldsymbol{\theta}|\boldsymbol{n}}$. This concludes the induction step.

Proof of Claim 3.15. To eliminate redundancy in the sum (3.42), we will assume the various $F_{1,s}$ which appear are part of a fixed linear basis of $\mathcal{A}_{\boldsymbol{n}-\boldsymbol{k},d-e}$. Then expression (3.40) together with the last sentence of Proposition 3.4 imply that $F_{2,s}$ has slope $\leq \rho$ for all $s \in S$. Because $F_{2,s}$ has naive slope $= \rho$ by equation (3.39), we conclude that

$$F_{2,s} \in \mathcal{B}_{\boldsymbol{m}+\rho\boldsymbol{\theta}|\boldsymbol{k}} \tag{3.45}$$

for all $s \in S$. Let us now express every $F_{2,s}$ in terms of a fixed linear basis

$$\{F_{2,t}\}_{t \in T} \quad \text{of} \quad \mathcal{B}_{\boldsymbol{m}+\rho\boldsymbol{\theta}|\boldsymbol{k}}$$

and then re-express equation (3.42) in this new basis:

$$\Delta_{(\boldsymbol{n}-\boldsymbol{k},d-e),(\boldsymbol{k},e)}(F) = \sum_{t \in T} h_{\boldsymbol{k}} F_{1,t} \otimes F_{2,t}. \tag{3.46}$$

Claim 3.16. *Every $F_{1,t}$ which appears in equation (3.46) has slope $< \rho$.*

By Claim 3.16 and the induction hypothesis of the surjectivity of the map (3.34),

$$G = \sum_{t \in T} F_{1,t} F_{2,t}$$

is a sum of products of elements of $\{\mathcal{B}_{\boldsymbol{m}+r\boldsymbol{\theta}}\}_{r \leq \rho}$ in increasing order of r. This implies that $G \in \operatorname{Im}\phi_{\boldsymbol{n},d}$. To compute the bad hinges of G, we note that

$$\Delta(G) = \sum_{t \in T} \underbrace{\Delta(F_{1,t})}_{X_1 \otimes X_2} \underbrace{\Delta(F_{2,t})}_{Y_1 \otimes Y_2}.$$

Since every $F_{1,t}$ has slope $< \rho$ and every $F_{2,t}$ has slope $= \rho$, then every X_2 that appears in this formula has naive slope $< \rho$ and every Y_2 has naive slope $\leq \rho$. But unless $X_2 = 1$ and $Y_2 = F_{2,t}$, either the product $X_2 Y_2$ has naive slope $< \rho$ or it has naive slope $= \rho$ but smaller |hdeg| than $|\boldsymbol{k}|$, and thus cannot contribute to the component of $\Delta(G)$ in degree (3.36). Thus, we have

$$\Delta_{(\boldsymbol{n}-\boldsymbol{k},d-e),(\boldsymbol{k},e)}(G) = \sum_{t \in T} (F_{1,t} \otimes 1)(h_{\boldsymbol{k}} \otimes F_{2,t}).$$

This matches equation (3.42) up to an overall constant that one obtains when commuting $h_{\boldsymbol{k}}$ past the various $F_{1,t}$ (this constant can be read off from equation (2.33), and depends only on $\boldsymbol{k}$ and the horizontal degree of the $F_{1,t}$s, which is equal to $\boldsymbol{n} - \boldsymbol{k}$ for all $t \in T$).

Proof of Claim 3.16. By expression (3.6) and the fact that F has slope $\leq \rho$ (see expression (3.40)), we have

$$\text{total degree of } F \text{ in } \{z_{i1}, \ldots, z_{il_i}\}_{i \in I} \leq (\boldsymbol{m} + \rho\boldsymbol{\theta}) \cdot \boldsymbol{l} + \langle \boldsymbol{l}, \boldsymbol{n} - \boldsymbol{l} \rangle \tag{3.47}$$

for all $\boldsymbol{0} \leq \boldsymbol{l} \leq \boldsymbol{n}$. The inequality is strict if $|\boldsymbol{k}| < |\boldsymbol{l}|$, on account of the maximality of the hinge $(\boldsymbol{k}, e)$. By the symmetry of the Laurent polynomial F, the same inequality holds if we

replace the set of variables $z_{i1}, \ldots, z_{il_i}$ by any other subset of l_i of the variables $z_{i1}, \ldots, z_{in_i}$. Let us zoom in on a certain monomial μ that appears in the Laurent polynomial F, and write for all $\mathbf{0} < \boldsymbol{k}' \leq \boldsymbol{n} - \boldsymbol{k}$

$$\alpha = \text{total degree of } \mu \text{ in } \left\{z_{i1}, \ldots, z_{ik_i'}\right\}_{i \in I}$$

$$\beta = \text{total degree of } \mu \text{ in } \left\{z_{i, n_i - k_i + 1}, \ldots, z_{in_i}\right\}_{i \in I}.$$

By applying formula (3.47) for $\boldsymbol{l} = \boldsymbol{k} + \boldsymbol{k}'$, we conclude that

$$\alpha + \beta < (\boldsymbol{m} + \rho\boldsymbol{\theta}) \cdot (\boldsymbol{k} + \boldsymbol{k}') + \langle \boldsymbol{k} + \boldsymbol{k}', \boldsymbol{n} - \boldsymbol{k} - \boldsymbol{k}' \rangle. \tag{3.48}$$

On the other hand, if the monomial μ survives in the limit (3.41), this implies

$$\beta = (\boldsymbol{m} + \rho\boldsymbol{\theta}) \cdot \boldsymbol{k} + \langle \boldsymbol{k}, \boldsymbol{n} - \boldsymbol{k} \rangle. \tag{3.49}$$

Subtracting equation (3.49) from formula (3.48) yields

$$\alpha < (\boldsymbol{m} + \rho\boldsymbol{\theta}) \cdot \boldsymbol{k}' + \langle \boldsymbol{k}', \boldsymbol{n} - \boldsymbol{k} - \boldsymbol{k}' \rangle - \langle \boldsymbol{k}, \boldsymbol{k}' \rangle. \tag{3.50}$$

However, the homogeneous degree of the first tensor factor of equation (3.41) in the variables $\left\{z_{i1}, \ldots, z_{ik_i'}\right\}_{i \in I}$ is equal to $\alpha + \langle \boldsymbol{k}, \boldsymbol{k}' \rangle$. By inequality (3.50), this quantity is

$$< (\boldsymbol{m} + \rho\boldsymbol{\theta}) \cdot \boldsymbol{k}' + \langle \boldsymbol{k}', \boldsymbol{n} - \boldsymbol{k} - \boldsymbol{k}' \rangle.$$

According to expression (3.6), this precisely means that $F_{1,t}$ has slope $< \rho$ for all $t \in T$. $\quad\square$

3.17. We are now ready to prove that the pairing (3.25) is nondegenerate, which is a necessary hypothesis when constructing the Drinfeld double (3.23).

Proposition 3.18. *For any $\boldsymbol{m} \in \mathbb{Q}^I$, the pairing (3.25) is nondegenerate.*

Proof. Because the pairing (2.29) is nondegenerate, we have

$$\langle F, \mathcal{A}^- \rangle = 0 \Rightarrow F = 0.$$

However, the surjectivity of the map (3.33) allows us to write

$$\left\langle F, \overset{\rightarrow}{\bigotimes_{r \in \mathbb{Q}}} \mathcal{B}_{\boldsymbol{m} + r\boldsymbol{\theta}}^- \right\rangle = 0 \Rightarrow F = 0$$

if $F \in \mathcal{B}_{\boldsymbol{m}}^+$, then Proposition 3.12 implies that F pairs trivially with all ordered products of elements from different $\mathcal{B}_{\boldsymbol{m} + r\boldsymbol{\theta}}^-$s, and pairs nontrivially only with $\mathcal{B}_{\boldsymbol{m}}^-$ itself. We conclude that

$$\left\langle F, \mathcal{B}_{\boldsymbol{m}}^- \right\rangle = 0 \Rightarrow F = 0,$$

which is precisely the nondegeneracy of formula (3.25) in the first factor. The case of nondegeneracy in the second factor is completely analogous. $\quad\square$

Proposition 3.19. *For any $\boldsymbol{m} \in \mathbb{Q}^I$ and $\boldsymbol{\theta} \in \mathbb{Q}_+^I$, the map (3.33) is injective.*

In particular, the proposition implies that the maps (3.32) are injective for all $p \in \mathbb{Q}$.

Proof. By Proposition 3.18, we may fix dual linear bases

$$\{a_{r,s}\}_{s \in \mathbb{N}} \subset \mathcal{B}^+_{\boldsymbol{m}+r\boldsymbol{\theta}} \qquad \text{and} \qquad \{b_{r,s}\}_{s \in \mathbb{N}} \subset \mathcal{B}^-_{\boldsymbol{m}+r\boldsymbol{\theta}} \tag{3.51}$$

for all $r \in \mathbb{Q}$. We will assume $a_{r,0} = b_{r,0} = 1$ is an element in our bases. If the map (3.33) (we assume $\pm = +$, as the case of $\pm = -$ is analogous) failed to be injective, then there would exist a nontrivial linear relation

$$\sum_{\{s_r\}_{r \in \mathbb{Q}}} \gamma_{\{s_r\}} \overrightarrow{\prod_{r \in \mathbb{Q}}} a_{r,s_r} = 0 \in \mathcal{A}^+ \tag{3.52}$$

(the sum goes over all collections of indices s_r, almost all of which are equal to 0). For any fixed collection of indices $\{t_r\}_{r \in \mathbb{Q}}$, almost all of which are equal to 0, this relation implies

$$\sum_{\{s_r\}_{r \in \mathbb{Q}}} \gamma_{\{s_r\}} \left\langle \overrightarrow{\prod_{r \in \mathbb{Q}}} a_{r,s_r}, \overrightarrow{\prod_{r \in \mathbb{Q}}} b_{r,t_r} \right\rangle = 0 \in \mathcal{A}^+.$$

By Proposition 3.12, the only pairing which survives in this formula is the one for $s_r = t_r, \forall r \in \mathbb{Q}$, thus implying that $\gamma_{\{t_r\}} = 0$. Since this holds for all collections of indices $\{t_r\}_{r \in \mathbb{Q}}$, this precludes the existence of a nontrivial relation (3.52) and establishes the injectivity of the map (3.33). $\qquad\square$

Corollary 3.20. *For any $\boldsymbol{m} \in \mathbb{Q}^I$, $\boldsymbol{\theta} \in \mathbb{Q}^I_+$, and $p \in \mathbb{Q}$, the maps (3.32) and (3.33) are isomorphisms.*

We have completed the construction of the isomorphisms (3.30). As these isomorphisms preserve the pairing in the sense of Proposition 3.12, we have the following:

Corollary 3.21. *For any $\boldsymbol{m} \in \mathbb{Q}^I$ and $\boldsymbol{\theta} \in \mathbb{Q}^I_+$, we have*

$$\mathcal{R}' = \overrightarrow{\prod_{r \in \mathbb{Q}}} \mathcal{R}'_{\boldsymbol{m}+r\boldsymbol{\theta}}, \tag{3.53}$$

where $\mathcal{R}'$ is defined in equation (2.28), and $\mathcal{R}'_{\boldsymbol{m}}$ is defined in equation (3.24).

Proof. Let us consider dual bases (3.51). The canonical tensor of the pairing (3.25) (for $\boldsymbol{m}$ replaced by $\boldsymbol{m}+r\boldsymbol{\theta}$) is

$$\mathcal{R}'_{\boldsymbol{m}+r\boldsymbol{\theta}} = \sum_{s \in \mathbb{N}} a_{r,s} \otimes b_{r,s}. \tag{3.54}$$

Meanwhile, by formulas (3.27) and (3.30), we have

$$\left\{ \overrightarrow{\prod_{r \in \mathbb{Q}}} a_{r,s_r} \right\} \subset \mathcal{A}^+ \qquad \text{and} \qquad \left\{ \overrightarrow{\prod_{r \in \mathbb{Q}}} b_{r,s_r} \right\} \subset \mathcal{A}^-$$

(as $\{s_r\}_{r\in\mathbb{Q}}$ goes over all collections of natural numbers, almost all of which are 0) are dual bases. Therefore, the canonical tensor of the pairing (2.29) is

$$\mathcal{R}' = \sum_{\{s_r\}} \overrightarrow{\prod_{r\in\mathbb{Q}}} a_{r,s_r} \otimes \overrightarrow{\prod_{r\in\mathbb{Q}}} b_{r,s_r}. \tag{3.55}$$

Comparing formulas (3.54) and (3.55) yields equation (3.53). □

3.22. The universal R-matrix intertwines the coproduct with its opposite

$$\Delta^{\mathrm{op}}(a) = \mathcal{R} \cdot \Delta(a) \cdot \mathcal{R}^{-1}$$

for all $a \in \mathcal{A}$. However, we now have the factorization

$$\mathcal{R} = \overrightarrow{\prod_{r\in\mathbb{Q}\sqcup\{\infty\}}} \mathcal{R}'_{\boldsymbol{m}+r\boldsymbol{\theta}},$$

where $\mathcal{R}'_{\boldsymbol{m}+\infty\boldsymbol{\theta}}$ denotes the factor in square brackets in equation (2.28). Therefore,

$$\Delta_{(\boldsymbol{m})}(a) = \left[\overrightarrow{\prod_{r\in\mathbb{Q}_{>0}\sqcup\{\infty\}}} \mathcal{R}'_{\boldsymbol{m}+r\boldsymbol{\theta}}\right] \cdot \Delta(a) \cdot \left[\overrightarrow{\prod_{r\in\mathbb{Q}_{>0}\sqcup\{\infty\}}} \mathcal{R}'_{\boldsymbol{m}+r\boldsymbol{\theta}}\right]^{-1} \tag{3.56}$$

defines another coproduct on the algebra $\mathcal{A}$, for all $\boldsymbol{m} \in \mathbb{Q}^I$. We expect the restriction of $\Delta_{(\boldsymbol{m})}$ to the subalgebra $\mathcal{B}_{\boldsymbol{m}}$ to match the coproduct $\Delta_{\boldsymbol{m}}$ of §3.7.

The existence of many coproducts on quantum groups is a well-known phenomenon in representation theory. For example, when Q is of finite type and $\mathcal{A}$ is the corresponding quantum affine algebra, Δ is the Drinfeld new coproduct and $\Delta_{(\boldsymbol{0})}$ is the Drinfeld–Jimbo coproduct. When Q is the cyclic quiver and $\mathcal{A}$ is the corresponding quantum toroidal algebra, we expect $\Delta_{(\boldsymbol{0})}$ to match the coproduct defined in [29]. For general quivers, Conjecture 4.22 suggests that the coproducts (3.56) match the ones defined by [1, 18, 32, 33, 34] using the theory of stable bases.

Remark 3.23. All the results in this paper would continue to hold if the equivariant parameters $\{q,t_e\}_{e\in E}$ were not generic but specialized in any way which satisfies [31, Assumption Ь] – for example,

$$t_e = q^{\frac{1}{2}}, \quad \forall e \in E.$$

Indeed, as explained in §5 there, this assumption allows us to define $\mathcal{A}$ as a Drinfeld double, and then all the notions of the current section would carry through. The main caveat is that the wheel conditions (2.4) are no longer enough to define $\mathcal{S} \subset \mathcal{V}$; one needs to impose the stronger conditions [31, formula (5.2)] instead.

4. Connections to geometry

4.1. We think of formula (3.30) as a PBW theorem for the shuffle algebras $\mathcal{A}^{\pm}$: it says that a linear basis of $\mathcal{A}^{\pm}$ is given by ordered products of linear bases of the subalgebras $\{\mathcal{B}_{\boldsymbol{m}+r\boldsymbol{\theta}}\}_{r\in\mathbb{Q}}$, for any fixed $\boldsymbol{m} \in \mathbb{Q}^I$ and $\boldsymbol{\theta} \in \mathbb{Q}_+^I$. Moreover, by equation (3.27), these linear

bases can be chosen to be dual to each other under the pairing (2.29). Formula (3.53) also emphasizes the role the subalgebras $\mathcal{B}_{\boldsymbol{m}}^{\pm}$ play in understanding the universal R-matrix of $\mathcal{A}$. This motivates our interest in understanding the subalgebras $\mathcal{B}_{\boldsymbol{m}}$. For starters, it is easy to see that the automorphisms (2.17) send

$$\tau_{\boldsymbol{k}} : \mathcal{B}_{\boldsymbol{m}} \xrightarrow{\sim} \mathcal{B}_{\boldsymbol{m}+\boldsymbol{k}} \tag{4.1}$$

for any $\boldsymbol{k} \in \mathbb{Z}^I$. Therefore, the classification of the algebras $\mathcal{B}_{\boldsymbol{m}}$ depends only on $\boldsymbol{m} \in (\mathbb{Q}/\mathbb{Z})^I$. A more substantial reduction would be the following:

Problem 4.2. Show that $\mathcal{B}_{\boldsymbol{m}}$ for the quiver Q is isomorphic to the algebra $\mathcal{B}_{\boldsymbol{0}}$ for some other quiver $Q_{\boldsymbol{m}}$, and understand the dependence of the latter on $\boldsymbol{m} \in (\mathbb{Q}/\mathbb{Z})^I$.

For example, in [27], when Q is the cyclic quiver of length n, we showed that

$$\mathcal{B}_{(m_1,\ldots,m_n)} = U_q\left(\widehat{\mathfrak{gl}}_{n_1}\right) \otimes \cdots \otimes U_q\left(\widehat{\mathfrak{gl}}_{n_d}\right),$$

where the natural numbers d and $n_1 + \cdots + n_d = n$ are defined by an explicit procedure from the rational numbers $m_1, \ldots, m_n \in \mathbb{Q}/\mathbb{Z}$. In particular, we have

$$\mathcal{B}_{(0,\ldots,0)} = U_q\left(\widehat{\mathfrak{gl}}_n\right).$$

Thus we encounter a particular instance of Problem 4.2: when Q is the cyclic quiver of length n, the statement of the problem holds with $Q_{(m_1,\ldots,m_n)}$ being a disjoint union of cyclic quivers of lengths $n_1, \ldots, n_d$.

4.3. With Problem 4.2 in mind, we will now focus on the algebra $\mathcal{B}_{\boldsymbol{0}}$.

Definition 4.4. The Kac polynomial of Q in dimension $\boldsymbol{n} \in \mathbb{N}^I$, denoted by

$$A_{Q,\boldsymbol{n}}(t), \tag{4.2}$$

counts the number of isomorphism classes of $\boldsymbol{n}$-dimensional absolutely indecomposable representations of the quiver Q over a finite field with t elements.

It was shown in [12] that the number of absolutely indecomposable representations is a polynomial in t, and so expression (4.2) lies in $\mathbb{Z}[t]$. This was further shown in [11] to lie in $\mathbb{N}[t]$, thus opening the door to the notion that $A_{Q,\boldsymbol{n}}(1)$ counts 'something'. Before we make a conjecture as to what this something is, let us assemble all the Kac polynomials into a power series:

$$A_Q(t,\boldsymbol{z}) = \sum_{\boldsymbol{n} \in \mathbb{N}^I \setminus \boldsymbol{0}} A_{Q,\boldsymbol{n}}(t) \boldsymbol{z}^{\boldsymbol{n}},$$

where $\boldsymbol{z}^{\boldsymbol{n}} = \prod_{i \in I} z_i^{n_i}$. Similarly, define

$$\chi_{\mathcal{B}_{\boldsymbol{0}}}(\boldsymbol{z}) = \sum_{\boldsymbol{n} \in \mathbb{N}^I} \dim \mathcal{B}_{\boldsymbol{0}|\boldsymbol{n}} \boldsymbol{z}^{\boldsymbol{n}} \tag{4.3}$$

and consider the plethystic exponential

$$\mathrm{Exp}\left[\sum_{\boldsymbol{n}\in\mathbb{N}^I\setminus\boldsymbol{0}} d_{\boldsymbol{n}}\boldsymbol{z}^{\boldsymbol{n}}\right] := \prod_{\boldsymbol{n}\in\mathbb{N}^I\setminus\boldsymbol{0}} \frac{1}{(1-\boldsymbol{z}^{\boldsymbol{n}})^{d_{\boldsymbol{n}}}}$$

for any collection of natural numbers $\{d_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{N}^I\setminus\boldsymbol{0}}$.

Conjecture 4.5. *For any quiver Q, we have*

$$\chi_{\mathcal{B}_{\boldsymbol{0}}}(\boldsymbol{z}) = \mathrm{Exp}\left[A_Q(1,\boldsymbol{z})\right]. \tag{4.4}$$

In other words, $\mathcal{B}_{\boldsymbol{0}}$ is isomorphic (as a graded vector space) to the symmetric algebra of a graded vector space of graded dimension $A_Q(1,\boldsymbol{z})$.

Conjecture 4.5 gives an elementary combinatorial formula for the Kac polynomial at $t = 1$, since $\dim \mathcal{B}_{\boldsymbol{0}|\boldsymbol{n}}$ is the dimension of the vector space of Laurent polynomials satisfying the wheel conditions (2.4) and the growth conditions (3.6) for $\boldsymbol{m} = \boldsymbol{0}$. We computed these dimensions using mathematical software and verified Conjecture 4.5 in the following cases:

- Q is the quiver with one vertex and $g \in \{1,2,3\}$ loops, up to dimension $n = 5$;
- Q is the quiver with two vertices and $d \in \{1,2,3,4\}$ edges between them, up to dimension vector $(n_1, n_2) = (3,3)$.

These two types of quivers are relevant because they control the various wheel conditions (2.4). Moreover, in the particular instances $g = 1$, $d = 1$, and $d = 2$ of these two cases, we have

$$\mathcal{B}_{\boldsymbol{0}} = U_q\left(\widehat{\mathfrak{gl}}_1\right), \qquad \mathcal{B}_{\boldsymbol{0}} = U_q(\mathfrak{sl}_3), \qquad \text{and} \qquad \mathcal{B}_{\boldsymbol{0}} = U_q\left(\widehat{\mathfrak{sl}}_2\right),$$

respectively. In these cases, Conjecture 4.5 is easily verified.

Remark 4.6. We may consider the subspace $\mathcal{B}_{\boldsymbol{0}}^{\mathrm{prim}} \subset \mathcal{B}_{\boldsymbol{0}}$ of primitive elements – that is, those for which the coproduct $\Delta_{\boldsymbol{0}}$ has no intermediate terms:

$$\Delta_{\boldsymbol{0}}(P) = P \otimes 1 + h_{\mathrm{hdeg}P} \otimes P. \tag{4.5}$$

We expect $\dim \mathcal{B}_{\boldsymbol{0}|\boldsymbol{n}}^{\mathrm{prim}}$ to be given by the number $C_{Q,\boldsymbol{n}}(1)$ of [2], for all $\boldsymbol{n} \in \mathbb{N}^I$.

In particular, if Q is the quiver with one vertex and $g \geq 2$ loops (such a vertex is called hyperbolic in the language of [2]), then the $q \to 1$ limit of $\mathcal{B}_{\boldsymbol{0}}$ should be the free Lie algebra on the vector space which is the $q \to 1$ limit of $\mathcal{B}_{\boldsymbol{0}}^{\mathrm{prim}}$. We thank Andrei Okounkov and Olivier Schiffmann for pointing out this expectation.

4.7. We will now present two more frameworks which are conjecturally related to our constructions. The first of these is the K-theoretic Hall algebra of the quiver Q. To define it, let us consider the stack of $\boldsymbol{n}$-dimensional representations of Q, for any $\boldsymbol{n} \in \mathbb{N}^I$:

$$\mathfrak{Z}_{\boldsymbol{n}} = \bigoplus_{\overrightarrow{ij}=e\in E} \mathrm{Hom}\left(V_i, V_j\right) \Big/ \prod_{i\in I} GL(V_i), \tag{4.6}$$

where V_i denotes a vector space dimension n_i, for every $i \in I$. The action of the product of general linear groups in this equation is by conjugating homomorphisms $V_i \to V_j$.

The Kac polynomial of Definition 4.4 counts the number of (certain) points of $\mathfrak{Z}_{\boldsymbol{n}}$ over the field with t elements. There are also other fruitful ways to count points of the stack (4.6), but one can obtain similarly beautiful constructions by looking at other enumerative invariants of $\mathfrak{Z}_{\boldsymbol{n}}$. For example, Schiffmann and Vasserot consider the equivariant algebraic K-theory groups of the cotangent bundle of the stack $\mathfrak{Z}_{\boldsymbol{n}}$,

$$K = \bigoplus_{\boldsymbol{n} \in \mathbb{N}^I} K_T(T^* \mathfrak{Z}_{\boldsymbol{n}}), \tag{4.7}$$

where the torus $T = \mathbb{C}^* \times \prod_{e \in E} \mathbb{C}^*$ acts on $T^* \mathfrak{Z}_{\boldsymbol{n}}$ as follows: the first factor of $\mathbb{C}^*$ scales the cotangent fibers, and the eth $\mathbb{C}^*$ in the product scales the homomorphism corresponding to the same-named edge e in equation (4.6). As K is a module over the ring

$$K_T(\text{point}) = \mathbb{Z}\left[q^{\pm 1}, t_e^{\pm 1}\right]_{e \in E}, \tag{4.8}$$

we may consider its localization with respect to the fraction field $\mathbb{F} = \mathbb{Q}(q, t_e)_{e \in E}$:

$$K_{\text{loc}} = K \bigotimes_{\mathbb{Z}\left[q^{\pm 1}, t_e^{\pm 1}\right]_{e \in E}} \mathbb{Q}(q, t_e)_{e \in E}. \tag{4.9}$$

We refer to [38] for a survey of K-theoretic Hall algebras, to [31, §2] for a quick overview in notation similar to ours, and to Remark 4.11 for an explicit presentation of $T^* \mathfrak{Z}_{\boldsymbol{n}}$. In particular, the reason for summing over all $\boldsymbol{n}$ in equation (4.7) is to make K into an $\mathbb{N}^I$-graded $\mathbb{F}$-algebra. Moreover, we have an $\mathbb{F}$-algebra homomorphism

$$K_{\text{loc}} \xrightarrow{\iota} \mathcal{V},$$

where $\mathcal{V}$ is the algebra (2.1). It was shown in [42] that ι is injective, in [44] that $\text{Im}\,\iota \subseteq \mathcal{S}$, and in [31] that $\text{Im}\,\iota = \mathcal{S}$. We therefore have an $\mathbb{F}$-algebra isomorphism

$$K_{\text{loc}} \xrightarrow{\sim} \mathcal{S}. \tag{4.10}$$

Problem 4.8. What is the geometric meaning of the slope subalgebras $\mathcal{B}_{\boldsymbol{m}}^+ \subset \mathcal{A}^+ = \mathcal{S}$, for various $\boldsymbol{m} \in \mathbb{Q}^I$, when pulled back to K_{loc} via the isomorphism (4.10)?

4.9. A bridge between Conjecture 4.5 and Problem 4.8 is provided by the work of Davison and Meinhardt [4, 5], who studied the version of equation (4.7) when equivariant K-theory is replaced by Borel–Moore homology. The resulting object H is called the cohomological Hall algebra of the quiver Q, and its study goes back to Kontsevich and Soibelman in [15]. The algebra H is related to the algebra K as Yangians are related to quantum loop groups. For a general quiver Q, Davison and Meinhardt constructed in [4, 5] an $\mathbb{N}^I$-graded Lie algebra $\mathfrak{g}_{\text{BPS}}$ with an algebra embedding

$$U(\mathfrak{g}_{\text{BPS}}) \subset H.$$

The Lie algebra $\mathfrak{g}_{\text{BPS}}$ has the following graded dimension (see equation (4.3)):

$$\chi_{\mathfrak{g}_{\text{BPS}}}(\boldsymbol{z}) = A_Q(t, \boldsymbol{z}) \Rightarrow \chi_{U(\mathfrak{g}_{\text{BPS}})}(\boldsymbol{z}) = \text{Exp}\left[A_Q(t, \boldsymbol{z})\right],$$

where t keeps track of the homological degree on H. Therefore, it is natural to conjecture that the degeneration map $K \rightsquigarrow H$ sends $\mathcal{B}_0 \rightsquigarrow U(\mathfrak{g}_{\mathrm{BPS}})$. The homological grading, ubiquitous on the H-side, is not readily seen on the K-side. This is why we expect equation (4.4) to only see the value at $t = 1$ of the Kac polynomial $A_Q(t, \boldsymbol{z}) \in \mathbb{N}[t][[\boldsymbol{z}]]$.

4.10. We will now recall the construction of Nakajima quiver varieties associated to the quiver Q [20]. To define these, consider for any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{N}^I$ the affine space

$$N_{\boldsymbol{v},\boldsymbol{w}} = \bigoplus_{\overrightarrow{ij}=e\in E} [\mathrm{Hom}\,(V_i, V_j) \oplus \mathrm{Hom}\,(V_j, V_i)] \bigoplus_{i \in I} [\mathrm{Hom}(W_i, V_i) \oplus \mathrm{Hom}(V_i, W_i)],$$

where V_i (resp., W_i) are vector spaces of dimension v_i (resp., w_i) for all $i \in I$. Points of this affine space will be denoted by quadruples

$$(X_e, Y_e, A_i, B_i)_{e \in E, i \in I}, \tag{4.11}$$

where X_e, Y_e, A_i, B_i denote homomorphisms in the four types of Hom spaces that enter the definition of $N_{\boldsymbol{v},\boldsymbol{w}}$. Consider the action of

$$G_{\boldsymbol{v}} = \prod_{i \in I} GL(V_i) \tag{4.12}$$

on $N_{\boldsymbol{v},\boldsymbol{w}}$ by conjugating X_e, Y_e, left-multiplying A_i, and right-multiplying B_i. It is easy to see that $G_{\boldsymbol{v}}$ acts freely on the open locus of *stable*[9] points

$$N^s_{\boldsymbol{v},\boldsymbol{w}} \subset N_{\boldsymbol{v},\boldsymbol{w}},$$

that is, those (4.11) such that there does not exist a collection of subspaces $\{V_i' \subseteq V_i\}_{i \in I}$ (other than $V_i' = V_i$ for all $i \in I$) which is preserved by the maps X_e and Y_e, and contains $\mathrm{Im} A_i$ for all $i \in I$. Let us consider the quadratic moment map

$$N_{\boldsymbol{v},\boldsymbol{w}} \xrightarrow{\mu} \mathrm{Lie}\, G_{\boldsymbol{v}} = \bigoplus_{i \in I} \mathrm{Hom}(V_i, V_i) \tag{4.13}$$

$$\mu\left((X_e, Y_e, A_i, B_i)_{e \in E, i \in I}\right) = \sum_{e \in E} (X_e Y_e - Y_e X_e) + \sum_{i \in I} A_i B_i.$$

If we write $\mu_{\boldsymbol{v},\boldsymbol{w}}^{-1}(0)^s = \mu_{\boldsymbol{v},\boldsymbol{w}}^{-1}(0) \cap N^s_{\boldsymbol{v},\boldsymbol{w}}$, then there is a geometric quotient

$$\mathcal{N}_{\boldsymbol{v},\boldsymbol{w}} = \mu_{\boldsymbol{v},\boldsymbol{w}}^{-1}(0)^s / GL_{\boldsymbol{v}}, \tag{4.14}$$

which is called the *Nakajima quiver variety* for the quiver Q, associated to $\boldsymbol{v}, \boldsymbol{w}$.

Remark 4.11. The $\boldsymbol{v} = \boldsymbol{n}, \boldsymbol{w} = \boldsymbol{0}$ version of this construction, where instead of taking the geometric quotient (4.14) one takes the stack quotient, is simply $T^* \mathfrak{Z}_{\boldsymbol{n}}$.

4.12. The algebraic group

$$T_{\boldsymbol{w}} = \mathbb{C}^* \times \prod_{e \in E} \mathbb{C}^* \times \prod_{i \in I} GL(W_i) \tag{4.15}$$

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[9]In general, stability conditions for quiver varieties are indexed by $\boldsymbol{\theta} \in \mathbb{R}^I$; the one studied herein corresponds to $\boldsymbol{\theta} = (1, \ldots, 1)$.

acts on Nakajima quiver varieties as follows:

$$\left(\bar{q}, \bar{t}_e, \bar{U}_i\right)_{e \in E, i \in I} \cdot (X_e, Y_e, A_i, B_i)_{e \in E, i \in I} = \left(\frac{X_e}{\bar{t}_e}, \frac{\bar{t}_e Y_e}{\bar{q}}, A_i \bar{U}_i^{-1}, \frac{\bar{U}_i B_i}{\bar{q}} \right)_{e \in E, i \in I}.$$

With respect to this action, the $T_{\boldsymbol{w}}$-equivariant algebraic K-theory groups of Nakajima quiver varieties are modules over the ring

$$K_{T_{\boldsymbol{w}}}(\text{point}) = \mathbb{Z}\left[q^{\pm 1}, t_e^{\pm 1}, u_{ia}^{\pm 1}\right]^{\text{sym}}_{e \in E, i \in I, 1 \leq a \leq w_i}$$

(where 'sym' means symmetric in the equivariant parameters $u_{i1}, \ldots, u_{iw_i}$ for each $i \in I$ separately). We will localize our K-theory groups by analogy with equation (4.9):

$$K_{\boldsymbol{v}, \boldsymbol{w}} = K_{T_{\boldsymbol{w}}}\left(\mathcal{N}_{\boldsymbol{v}, \boldsymbol{w}}\right) \bigotimes_{\mathbb{Z}\left[q^{\pm 1}, t_e^{\pm 1}, u_{ia}^{\pm 1}\right]^{\text{sym}}_{e \in E, i \in I, 1 \leq a \leq w_i}} \mathbb{Q}(q, t_e, u_{ia})^{\text{sym}}_{e \in E, i \in I, 1 \leq a \leq w_i}. \tag{4.16}$$

As with the K-theoretic Hall algebra, it makes sense to consider the direct sum

$$K_{\boldsymbol{w}} = \bigoplus_{\boldsymbol{v} \in \mathbb{N}^I} K_{\boldsymbol{v}, \boldsymbol{w}}. \tag{4.17}$$

For every $i \in I$, consider the *tautological bundle* V_i of rank v_i, whose fiber over a point (4.11) is the vector space V_i itself; this is a nontrivial vector bundle, because Nakajima quiver varieties arise as quotients by the group (4.12). We formally write

$$[V_i] = x_{i1} + \cdots + x_{iv_i} \in K_{\boldsymbol{v}, \boldsymbol{w}}.$$

The symbols x_{ia} are not elements of $K_{\boldsymbol{v}, \boldsymbol{w}}$, but any symmetric Laurent polynomial in them is (specifically, it is obtained by taking the K-theory class of an appropriate Schur functor of the tautological vector bundle V_i). We will abbreviate

$$\boldsymbol{X_v} = \{\ldots, x_{i1}, \ldots, x_{iv_i}, \ldots\}_{i \in I}.$$

By the foregoing discussion, any Laurent polynomial $p(\ldots, x_{i1}, \ldots, x_{iv_i}, \ldots)$ which is symmetric in the x_{ia}s (for each $i \in I$ separately) yields an element of K-theory

$$p(\boldsymbol{X_v}) \in K_{\boldsymbol{v}, \boldsymbol{w}} \tag{4.18}$$

called a *tautological class*. By [19, Theorem 1.2], tautological classes (as p runs over all symmetric Laurent polynomials) linearly span $K_{\boldsymbol{v}, \boldsymbol{w}}$ for any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{N}^I$.

Example 4.13. Recall the function $\zeta_{ij}(x)$ (2.3), and consider its close cousin

$$\widetilde{\zeta}_{ij}(x) = \frac{\zeta_{ij}(x)}{\left(1 - \frac{x}{q}\right)^{\delta^i_j} \left(1 - \frac{1}{qx}\right)^{\delta^i_j}} = \frac{\prod_{e = \overrightarrow{ij} \in E} \left(\frac{1}{t_e} - x\right) \prod_{e = \overrightarrow{ji} \in E} \left(1 - \frac{t_e}{qx}\right)}{(1 - x)^{\delta^i_j} \left(1 - \frac{1}{qx}\right)^{\delta^i_j}}. \tag{4.19}$$
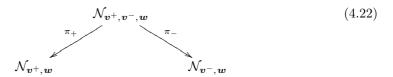
Then for any $\boldsymbol{n} \in \mathbb{N}^I$, let us define

$$\widetilde{\zeta}\left(\frac{\boldsymbol{Z_n}}{\boldsymbol{X_v}}\right) = \prod_{1 \leq a \leq n_i}^{i \in I} \prod_{1 \leq b \leq v_j}^{j \in I} \frac{\prod_{e = \overrightarrow{ij} \in E} \left(\frac{1}{t_e} - \frac{z_{ia}}{x_{jb}}\right) \prod_{e = \overrightarrow{ji} \in E} \left(1 - \frac{t_e x_{jb}}{q z_{ia}}\right)}{\left(1 - \frac{z_{ia}}{x_{jb}}\right)^{\delta^i_j} \left(1 - \frac{x_{jb}}{q z_{ia}}\right)^{\delta^i_j}} \tag{4.20}$$

$$\widetilde{\zeta}\left(\frac{\boldsymbol{X_v}}{\boldsymbol{Z_n}}\right) = \prod_{1\le a\le n_i}^{i\in I}\prod_{1\le b\le v_j}^{j\in I} \frac{\prod_{e=\overrightarrow{ji}\in E}\left(\frac{1}{t_e}-\frac{x_{jb}}{z_{ia}}\right)\prod_{e=\overrightarrow{ij}\in E}\left(1-\frac{t_e z_{ia}}{q x_{jb}}\right)}{\left(1-\frac{x_{jb}}{z_{ia}}\right)^{\delta_j^i}\left(1-\frac{z_{ia}}{q x_{jb}}\right)^{\delta_j^i}} \tag{4.21}$$

as elements of $K_{\boldsymbol{v},\boldsymbol{w}}\left[\left[\ldots,z_{i1}^{\pm1},\ldots,z_{in_i}^{\pm1},\ldots\right]\right]$.

4.14. One important reason for considering all $\boldsymbol{v}$s together in equation (4.17) is given by the following correspondences of Nakajima, which give operators $K_{\boldsymbol{v}^+,\boldsymbol{w}}\leftrightarrows K_{\boldsymbol{v}^-,\boldsymbol{w}}$ whenever $\boldsymbol{v}^+=\boldsymbol{v}^-+\boldsymbol{\varsigma}^i$ for some $i\in I$. Explicitly, for any such $\boldsymbol{v}^\pm$, let the diagram

$$\tag{4.22}$$

be the Hecke correspondences of [21, §5]. There is a tautological line bundle

$$\mathcal{L}_i\in\mathrm{Pic}\left(\mathcal{N}_{\boldsymbol{v}^+,\boldsymbol{v}^-,\boldsymbol{w}}\right)\rightsquigarrow l_i=[\mathcal{L}_i]\in K_{T_{\boldsymbol{w}}}\left(\mathcal{N}_{\boldsymbol{v}^+,\boldsymbol{v}^-,\boldsymbol{w}}\right).$$

With this notation in mind, let us consider the endomorphisms of $K_{\boldsymbol{w}}$

(for all $\boldsymbol{v}^+=\boldsymbol{v}^-+\boldsymbol{\varsigma}^i$ and $\boldsymbol{v}$ in $\mathbb{N}^I$) given by the formulas

$$E_{i,d}(\alpha)=\pi_{+*}\left(l_i^d\cdot\frac{\prod_{e=\overrightarrow{ij}}^{j\in I}t_e^{-v_j^+}\prod_{e=\overrightarrow{ji}}^{j\in I}\left(\det V_j^+\right)\left(\frac{-t_e}{l_i q}\right)^{v_j^+}}{\left(\det V_i^+\right)\left(\frac{-1}{l_i q}\right)^{v_i^+}}\cdot\pi_-^*(\alpha)\right) \tag{4.23}$$

$$F_{i,d}(\alpha)=\pi_{-*}\left(l_i^d\cdot\frac{\prod_{e=\overrightarrow{ij}}^{j\in I}\left(\det V_j^-\right)\left(\frac{-q}{l_i t_e}\right)^{v_j^-}\prod_{e=\overrightarrow{ji}}^{j\in I}t_e^{-v_j^-}}{\left(\det W_i\right)^{-1}\left(-l_i\right)^{r_i}\left(\det V_i^-\right)\left(\frac{-q}{l_i}\right)^{v_i^-}}\cdot\pi_+^*(\alpha)\right) \tag{4.24}$$

$$H_i^\pm(z_{i1})(\alpha)=\frac{\widetilde{\zeta}\left(\frac{\boldsymbol{Z_{\varsigma^i}}}{\boldsymbol{X_v}}\right)}{\widetilde{\zeta}\left(\frac{\boldsymbol{X_v}}{\boldsymbol{Z_{\varsigma^i}}}\right)}\cdot\frac{\wedge^\bullet\left(\frac{z_{i1}q}{W_i}\right)}{\wedge^\bullet\left(\frac{z_{i1}}{W_i}\right)}\cdot\alpha \tag{4.25}$$

for all $i \in I$ and $d \in \mathbb{Z}$.[10] In formulas (4.23) and (4.24), the fractions are the K-theory classes of certain line bundles on $\mathcal{N}_{\boldsymbol{v}^+,\boldsymbol{v}^-,\boldsymbol{w}}$ (built out of the determinants of the vector bundles $\{V_j^\pm, W_j\}_{j \in I}$, as well as the tautological line bundle) times equivariant constants. It was shown in [21] that the operators (4.23)–(4.25) induce an action of the quantum loop group associated to the quiver Q on $K_{\boldsymbol{w}}$, in the case when the quiver has no edge loops (see also [10, 41] for earlier work on the cyclic quiver case).

4.15. It is natural to expect the operators (4.23) to extend to an action $K_{\mathrm{loc}} \curvearrowright K_{\boldsymbol{w}}, \forall \boldsymbol{w} \in \mathbb{N}^I$ (more specifically, the operators $E_{i,d}$ should correspond to the action of the $\boldsymbol{n} = \boldsymbol{\varsigma}^i$ summand of equation (4.7)). This was proved in complete generality in [43]. As [31] showed that the localized K-theoretic Hall algebra is isomorphic to the shuffle algebra $\mathcal{S}$, it becomes natural to ask for a 'shuffle' version of formula (4.23), and analogously for equation (4.24). As shown in [24], the shuffle algebra naturally arises when we present formulas in terms of tautological classes (4.18). To this end, we have the following generalization of [24, Theorem 4.7] (which treated the Jordan quiver case):

Theorem 4.16. *The formulas*

$$p(\boldsymbol{X_v}) \xrightarrow{F} \frac{1}{\boldsymbol{n}!} \int^+ \frac{F(\boldsymbol{Z_n})}{\widetilde{\zeta}\left(\frac{\boldsymbol{Z_n}}{\boldsymbol{Z_n}}\right)} p(\boldsymbol{X_{v+n}} - \boldsymbol{Z_n}) \widetilde{\zeta}\left(\frac{\boldsymbol{Z_n}}{\boldsymbol{X_{v+n}}}\right) \wedge^\bullet \left(\frac{\boldsymbol{Z_n}q}{\boldsymbol{W}}\right) \qquad (4.26)$$

$$p(\boldsymbol{X_v}) \xrightarrow{G} \frac{1}{\boldsymbol{n}!} \int^- \frac{G(\boldsymbol{Z_n})}{\widetilde{\zeta}\left(\frac{\boldsymbol{Z_n}}{\boldsymbol{Z_n}}\right)} p(\boldsymbol{X_{v-n}} + \boldsymbol{Z_n}) \widetilde{\zeta}\left(\frac{\boldsymbol{X_{v-n}}}{\boldsymbol{Z_n}}\right)^{-1} \wedge^\bullet \left(\frac{\boldsymbol{Z_n}}{\boldsymbol{W}}\right)^{-1} \qquad (4.27)$$

give actions $\mathcal{A}^\pm \curvearrowright K_{\boldsymbol{w}}$ for all $F \in \mathcal{A}_{\boldsymbol{n}}, G \in \mathcal{A}_{-\boldsymbol{n}}$ (the notation will be explained after the statement of the theorem). Together with equation (4.25), these formulas glue to an action $\mathcal{A} \curvearrowright K_{\boldsymbol{w}}$.

Let us now explain the notation in formulas (4.26) and (4.27), except for the definition of the integrals $\int^\pm$, which will be given in §4.17. We write

$$(F \text{ or } G)(\boldsymbol{Z_n}) = (F \text{ or } G)(\dots, z_{i1}, \dots, z_{in_i}, \dots)_{i \in I}$$

$$\widetilde{\zeta}\left(\frac{\boldsymbol{Z_n}}{\boldsymbol{Z_n}}\right) = \prod_{\substack{1 \le a \le n_i, 1 \le b \le n_j \\ (i,a) \ne (j,b)}}^{i,j \in I} \frac{\zeta_{ij}\left(\frac{z_{ia}}{z_{jb}}\right)}{\left(1 - \frac{z_{ia}}{z_{jb}q}\right)^{\delta_j^i}}$$

$$\wedge^\bullet \left(\frac{\boldsymbol{Z_n}(1 \text{ or } q)}{\boldsymbol{W}}\right) = \prod_{1 \le a \le n_i}^{i \in I} \wedge^\bullet \left(\frac{z_{ia}(1 \text{ or } q)}{W_i}\right).$$

---

[10] The notation in equation (4.25) is such that for any variable z and any vector space S, we set

$$\wedge^\bullet \left(\frac{z}{S}\right) = \sum_{k=0}^{\dim S} (-z)^k \left[\wedge^k(S^\vee)\right] \qquad \text{and} \qquad \wedge^\bullet \left(\frac{S}{z}\right) = \sum_{k=0}^{\dim S} (-z)^{-k} \left[\wedge^k(S)\right].$$

The operation

$$p(\boldsymbol{X_v}) \mapsto p(\boldsymbol{X_{v-n}} + \boldsymbol{Z_n}) \tag{4.28}$$

is called a *plethysm*, and it is defined by evaluating the Laurent polynomial p at the collection of variables $\{\dots, x_{i1}, \dots, x_{i,v_i-n_i}, z_{i1}, \dots, z_{in_i}, \dots\}_{i \in I}$. The notation

$$p(\boldsymbol{X_v}) \mapsto p(\boldsymbol{X_{v+n}} - \boldsymbol{Z_n}) \tag{4.29}$$

would like to refer to the 'inverse' operation of formula (4.28), but the problem is that it is not uniquely defined. Indeed, by the fundamental theorem of symmetric polynomials, the Laurent polynomial p can be written as

$$p(\boldsymbol{X_v}) = \frac{\text{polynomial in } \left\{ x_{i1}^s + \cdots + x_{iv_i}^s \right\}_{i \in I, s \in \mathbb{N}^*}}{\prod_{i \in I} \left(x_{i1} \cdots x_{iv_i} \right)^N} \tag{4.30}$$

in infinitely many ways, for various polynomials in the numerator and various natural numbers N in the denominator. We define formula (4.29) by

$$p(\boldsymbol{X_{v+n}} - \boldsymbol{Z_n}) = \frac{\text{polynomial in } \left\{ x_{i1}^s + \cdots + x_{i,v_i+n_i}^s - z_{i1}^s - \cdots - z_{in_i}^s \right\}_{i \in I, s \in \mathbb{N}^*}}{\prod_{i \in I} \left(\frac{x_{i1} \cdots x_{i,v_i+n_i}}{z_{i1} \cdots z_{in_i}} \right)^N}. \tag{4.31}$$

Of course, the right-hand side of these expressions depends on the particular polynomial and the number N in equation (4.30), but we will show in the proof of Theorem 4.16 that the right-hand side of formula (4.26) does not.

4.17. The integrals (4.26)–(4.27) are defined [28, Definition 3.15] by

$$\int^+ T(\dots, z_{ia}, \dots) = \sum_{\sigma : \{(i,a)\} \to \{\pm 1\}}^{\text{functions}} \int_{|z_{ia}| = r^{\sigma(i,a)}}^{|q/t_e|^{\pm 1}, |t_e|^{\pm 1} > 1} T(\dots, z_{ia}, \dots) \prod_{(i,a)} \frac{\sigma(i,a) dz_{ia}}{2\pi\sqrt{-1} z_{ia}} \tag{4.32}$$

$$\int^- T(\dots, z_{ia}, \dots) = \sum_{\sigma : \{(i,a)\} \to \{\pm 1\}}^{\text{functions}} \int_{|z_{ia}| = r^{\sigma(i,a)}}^{|q/t_e|^{\pm 1}, |t_e|^{\pm 1} < 1} T(\dots, z_{ia}, \dots) \prod_{(i,a)} \frac{\sigma(i,a) dz_{ia}}{2\pi\sqrt{-1} z_{ia}} \tag{4.33}$$

for some positive real number $r \ll 1$. In each summand, each variable z_{ia} is integrated over either a very small circle of radius r or a very large circle of radius r^{-1}. The meaning of the superscripts $|q/t_e|^{\pm 1}, |t_e|^{\pm 1} > 1$ that adorn the integral (4.32) is the following: in the summand corresponding to a particular σ, if

$$\sigma(i,a) = \sigma(j,b) = 1 \qquad (\text{resp., } \sigma(i,a) = \sigma(j,b) = -1),$$

then the variables z_{ia} and z_{jb} are both integrated over the small (resp., large) circle. The corresponding integral is computed via residues under the assumption $|q/t_e|, |t_e| > 1$ (resp., $|q/t_e|^{-1}, |t_e|^{-1} > 1$). If $\sigma(i,a) \neq \sigma(j,b)$, then we do not need to assume anything about the sizes of q and t_e. One defines equation (4.33) analogously.

The following proposition is precisely the motivation behind our definition of $\int^\pm$, and its proof closely follows the analogous computation in [28, Theorem 3.17]:

Proposition 4.18. *The right-hand side of formula (4.26) (resp., (4.27)) for*

$$F = e_{i_1,d_1} * \cdots * e_{i_n,d_n} \qquad (resp., \ G = f_{i_1,d_1} * \cdots * f_{i_n,d_n}) \qquad (4.34)$$

is equal to the composition of the right-hand sides of formula (4.26) for $F = e_{i_1,d_1}, \ldots, F = e_{i_n,d_n}$ (resp., the right-hand sides of formula (4.27) for $G = f_{i_1,d_1}, \ldots, G = f_{i_n,d_n}$).

Indeed, it is easy to see that the composition of the right-hand sides of formulas (4.26) and (4.27) for $F = e_{i_1,d_1}, \ldots, F = e_{i_n,d_n}$ and $G = f_{i_1,d_1}, \ldots, G = f_{i_n,d_n}$ is

$$\int_{\{0,\infty\} \succ z_1 \succ \cdots \succ z_n} \frac{z_1^{d_1} \cdots z_n^{d_n}}{\prod_{1 \le a < b \le n} \widetilde{\zeta}_{i_b i_a}\left(\frac{z_b}{z_a}\right)}$$
$$p(\boldsymbol{X_{v+n}} - \boldsymbol{Z_n})\widetilde{\zeta}\left(\frac{\boldsymbol{Z_n}}{\boldsymbol{X_{v+n}}}\right) \wedge^{\bullet}\left(\frac{\boldsymbol{Z_n}q}{\boldsymbol{W}}\right) \prod_{a=1}^{n} \frac{dz_a}{2\pi\sqrt{-1}z_a} \qquad (4.35)$$

$$\int_{\{0,\infty\} \succ z_1 \succ \cdots \succ z_n} \frac{z_1^{d_1} \cdots z_n^{d_n}}{\prod_{1 \le a < b \le n} \widetilde{\zeta}_{i_a i_b}\left(\frac{z_a}{z_b}\right)}$$
$$p(\boldsymbol{X_{v-n}} + \boldsymbol{Z_n})\widetilde{\zeta}\left(\frac{\boldsymbol{X_{v-n}}}{\boldsymbol{Z_n}}\right)^{-1} \wedge^{\bullet}\left(\frac{\boldsymbol{Z_n}}{\boldsymbol{W}}\right)^{-1} \prod_{a=1}^{n} \frac{dz_a}{2\pi\sqrt{-1}z_a}. \qquad (4.36)$$

The notation $\int_{\{0,\infty\} \succ z_1 \succ \cdots \succ z_n}$ means that the variable z_1 is integrated over a contour in the complex plane which surrounds 0 and ∞, the variable z_2 is integrated over a contour which surrounds the previous contour, and so on, and the contours are also far away from each other compared to the size of the equivariant parameters q, t_e.

Moreover, in formulas (4.35) and (4.36), we implicitly identify the variables

$$\{z_1, \ldots, z_n\} \leftrightarrow \{\ldots, z_{i1}, \ldots, z_{in_i}, \ldots\}_{i \in I}$$

by mapping z_a in a one-to-one way to some $z_{i_a \bullet}$ (the specific choice of $\bullet \in \mathbb{N}$ does not matter, due to the symmetry of all expressions involved in the variables which make up $\boldsymbol{Z_n}$). Note that we need $n = |\boldsymbol{n}|$ in order for this notation to be consistent. We leave the equivalence of formulas (4.26) and (4.27) for the shuffle elements (4.34) with formulas (4.35) and (4.36) as an exercise for the interested reader (it closely follows the analogous computation in [28, proof of Theorem 3.17], which dealt with a close relative of our construction in the particular case when Q is the cyclic quiver).

Proof of Theorem 4.16. Our main task will be to establish the following claim:

Claim 4.19. *The case $F = e_{i,d}$ of formula (4.26) yields the same formula as equation (4.23), $\forall i \in I, d \in \mathbb{Z}$. Similarly, the case $G = f_{i,d}$ of formula (4.27) yields the same formula as equation (4.24).*

This claim establishes the fact that formulas (4.26) and (4.27) yield well-defined operators on $K_{\boldsymbol{w}}$ when $F = e_{i,d}$ and $G = f_{i,d}$, respectively. The meaning of the phrase 'well-defined' here is that

if $p(\boldsymbol{X_v}) = 0$, then the right-hand side of formulas (4.26) and (4.27) is also 0

(in particular, the right-hand side of formula (4.26) does not depend on the choices we made in defining formula (4.29)). By Proposition 4.18, formulas (4.26) and (4.27) also yield well-defined operators on $K_{\boldsymbol{w}}$ for any F and G of the form (4.34). Since by Theorem 2.3, any element of $\mathcal{S}$ and $\mathcal{S}^{\mathrm{op}}$ is a linear combination of such Fs and Gs, this proves that formulas (4.26) and (4.27) are well defined for any $F \in \mathcal{A}^+$ and any $G \in \mathcal{A}^-$. Similarly, the fact that the aforementioned formulas are multiplicative in F and G (thus implying that formulas (4.26) and (4.27) yield actions $\mathcal{A}^{\pm} \curvearrowright K_{\boldsymbol{w}}$) is an immediate consequence of Proposition 4.18. To prove that the actions $\mathcal{A}^{\pm} \curvearrowright K_{\boldsymbol{w}}$ glue to an action $\mathcal{A} \curvearrowright K_{\boldsymbol{w}}$, one only needs to check relations (2.13)–(2.15); this closely follows the $Q =$ cyclic quiver case treated in [27, Theorem II.9.].

Proof of Claim 4.19. Consider the complex of vector bundles on $\mathcal{N}_{\boldsymbol{v},\boldsymbol{w}}$

$$U_i = \left[V_i \cdot q \xrightarrow{(B_i, -X_e, Y_{e'})} W_i \oplus \bigoplus_{e=\overrightarrow{ij}} V_j \cdot \frac{q}{t_e} \oplus \bigoplus_{e'=\overrightarrow{ji}} V_j \cdot t_e \xrightarrow{(A_i, Y_e, X_{e'})} V_i \right] \tag{4.37}$$

(which originated in [20]) with the middle term in homological degree 0. By the stability condition, the second arrow is point-wise surjective, and thus its kernel K_i is a vector bundle; thus U_i is quasi-isomorphic to a complex $[V_i \cdot q \to K_i]$ of vector bundles. For such a complex, we may construct the projectivization

$$\mathbb{P}_{\mathcal{N}_{\boldsymbol{v},\boldsymbol{w}}}(U_i) = \mathrm{Proj}_{\mathcal{N}_{\boldsymbol{v},\boldsymbol{w}}}(\mathrm{Sym}^\bullet(U_i))$$

as a dg-scheme over $\mathcal{N}_{\boldsymbol{v},\boldsymbol{w}}$ (see [25, §5.18] for our notational conventions), and analogously for the dual complex $U_i^\vee[1] \cdot q = [K_i^\vee \cdot q \to V_i^\vee]$. With this in mind, it is well known that we have isomorphisms:

$$\mathcal{N}_{\boldsymbol{v}^+, \boldsymbol{v}^-, \boldsymbol{w}} \cong \mathbb{P}_{\mathcal{N}_{\boldsymbol{v}^-, \boldsymbol{w}}}(U_i) \tag{4.38}$$

$$\mathcal{N}_{\boldsymbol{v}^+, \boldsymbol{v}^-, \boldsymbol{w}} \cong \mathbb{P}_{\mathcal{N}_{\boldsymbol{v}^+, \boldsymbol{w}}}(U_i^\vee[1] \cdot q) \tag{4.39}$$

with respect to which the line bundle $\mathcal{L}_i$ is isomorphic to $\mathcal{O}(1)$ and $\mathcal{O}(-1)$, respectively. A straightforward computation, which follows directly from the well-known formulas in [25, Proposition 5.19], yields for any tautological class p as in expression (4.18)

$$E_{i,d}(p(\boldsymbol{X}_{\boldsymbol{v}^-})) = \int^+ z_{i1}^d p(\boldsymbol{X}_{\boldsymbol{v}^+} - \boldsymbol{Z}_{\boldsymbol{\varsigma}^i}) \frac{\prod_{e=\overrightarrow{ij}}^{j \in I} t_e^{-v_j} \prod_{e=\overrightarrow{ji}}^{j \in I} (\det V_j) \left(\frac{-t_e}{z_{i1}q}\right)^{v_j}}{(\det V_i) \left(\frac{-1}{z_{i1}q}\right)^{v_i}} \wedge^\bullet \left(\frac{z_{i1}q}{U_i}\right)$$

$$F_{i,d}(p(\boldsymbol{X}_{\boldsymbol{v}^+})) = \int^- z_{i1}^d p(\boldsymbol{X}_{\boldsymbol{v}^-} + \boldsymbol{Z}_{\boldsymbol{\varsigma}^i}) \frac{\prod_{e=\overrightarrow{ij}}^{j \in I} (\det V_j) \left(\frac{-q}{z_{i1}t_e}\right)^{v_j} \prod_{e=\overrightarrow{ji}}^{j \in I} t_e^{-v_j}}{(\det W_i)^{-1} (-z_{i1})^{r_i} (\det V_i) \left(\frac{-q}{z_{i1}}\right)^{v_i}} \wedge^\bullet \left(-\frac{U_i}{z_{i1}}\right).$$

One can express U_i in terms of the vector bundles V_i and the trivial vector bundles W_i using equation (4.37), and one notices that the right-hand sides of these formulas are precisely the right-hand sides of formulas (4.26) and (4.27) when $F = e_{i,d}$ and $G = f_{i,d}$. $\qquad\square$

4.20. As a consequence of Theorem 4.16, we obtain an algebra homomorphism

$$\mathcal{A} \xrightarrow{\Psi} \prod_{\boldsymbol{w} \in \mathbb{N}^I} \operatorname{End}(K_{\boldsymbol{w}}). \tag{4.40}$$

To make the map Ψ into a bialgebra homomorphism, one needs to place a coproduct on the right-hand side, which interweaves the modules $K_{\boldsymbol{w}}$ as $\boldsymbol{w}$ varies over $\mathbb{N}$. Consider any $\boldsymbol{w}^1, \boldsymbol{w}^2 \in \mathbb{N}^I$, let $\boldsymbol{w} = \boldsymbol{w}^1 + \boldsymbol{w}^2$, and take the one-parameter subgroup

$$\mathbb{C}^* \ni t \xrightarrow{\tau} \prod_{i \in I} \operatorname{diag}(\underbrace{1,\ldots,1}_{w_i^1 \text{ times}}, \underbrace{t,\ldots,t}_{w_i^2 \text{ times}}) \in \prod_{i \in I} GL(W_i). \tag{4.41}$$

The fixed locus of τ acting on $\mathcal{N}_{\boldsymbol{v},\boldsymbol{w}}$ is

$$\mathcal{N}_{\boldsymbol{v},\boldsymbol{w}}^{\mathbb{C}^*} \cong \bigsqcup_{\boldsymbol{v}^1 + \boldsymbol{v}^2 = \boldsymbol{v}} \mathcal{N}_{\boldsymbol{v}^1,\boldsymbol{w}^1} \times \mathcal{N}_{\boldsymbol{v}^2,\boldsymbol{w}^2} \xhookrightarrow{\iota} \mathcal{N}_{\boldsymbol{v},\boldsymbol{w}} \tag{4.42}$$

consisting of quadruples (4.11) which respect fixed direct sum decompositions $V_i = V_i^1 \oplus V_i^2$ and $W_i = W_i^1 \oplus W_i^2$. We have a decomposition of the normal bundle

$$T_{\mathcal{N}_{\boldsymbol{v},\boldsymbol{w}}/\mathcal{N}_{\boldsymbol{v}^1,\boldsymbol{w}^1} \times \mathcal{N}_{\boldsymbol{v}^2,\boldsymbol{w}^2}} = T^+ \oplus T^-,$$

where T^+ (resp., T^-) consists of the attracting (resp., repelling) sub-bundles with respect to the action of the one-parameter subgroup τ of formula (4.41). Because τ preserves the holomorphic symplectic form on $\mathcal{N}_{\boldsymbol{v},\boldsymbol{w}}$ (which we have not defined), the sub-bundles T^+ and T^- are dual to each other, and so have the same rank. This allows us to think of T^+ as 'half' of the normal bundle, and set
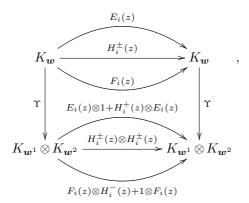
$$\Upsilon : K_{\boldsymbol{v},\boldsymbol{w}} \xrightarrow{\wedge^\bullet(-T^{+\vee}) \cdot \iota^*} \bigoplus_{\boldsymbol{v} = \boldsymbol{v}^1 + \boldsymbol{v}^2} K_{\boldsymbol{v}^1,\boldsymbol{w}^1} \otimes K_{\boldsymbol{v}^2,\boldsymbol{w}^2}. \tag{4.43}$$

Conjugation with the (product over all $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{N}^I$ of the) map Υ yields a coproduct

$$\prod_{\boldsymbol{w} \in \mathbb{N}^I} \operatorname{End}(K_{\boldsymbol{w}}) \longrightarrow \prod_{\boldsymbol{w}^1 \in \mathbb{N}^I} \operatorname{End}(K_{\boldsymbol{w}^1}) \widehat{\otimes} \prod_{\boldsymbol{w}^2 \in \mathbb{N}^I} \operatorname{End}(K_{\boldsymbol{w}^2}). \tag{4.44}$$

It is straightforward to check that the map Ψ of formula (4.40) intertwines the coproduct on $\mathcal{A}$ of formulas (2.18)–(2.20) with the coproduct (4.44). Explicitly, this boils down to the commutativity of the following squares, which we leave as exercises for the interested

reader:

$$
\begin{array}{ccc}
K_{\boldsymbol{w}} & \xrightarrow{\;\;H_i^{\pm}(z)\;\;} & K_{\boldsymbol{w}} \\
\end{array}
$$

with upper arc $E_i(z)$, lower arc $F_i(z)$; vertical maps Υ on both sides; middle label $E_i(z)\otimes 1 + H_i^+(z)\otimes E_i(z)$

$$
\begin{array}{ccc}
K_{\boldsymbol{w}^1}\otimes K_{\boldsymbol{w}^2} & \xrightarrow{\;\;H_i^{\pm}(z)\otimes H_i^{\pm}(z)\;\;} & K_{\boldsymbol{w}^1}\otimes K_{\boldsymbol{w}^2}
\end{array}
$$

lower arc $F_i(z)\otimes H_i^-(z) + 1\otimes F_i(z)$

(with $E_i(z) = \sum_{d\in\mathbb{Z}} \frac{E_{i,d}}{z^d}$ and $F_i(z) = \sum_{d\in\mathbb{Z}} \frac{F_{i,d}}{z^d}$) for any $\boldsymbol{w} = \boldsymbol{w}^1 + \boldsymbol{w}^2$ in $\mathbb{N}^I$.

4.21. The construction of the previous subsection (which is by now folklore among experts) is quite straightforward, but unfortunately has a number of drawbacks. The first is that it heavily uses equivariant localization. The second is that it produces only the topological coproduct Δ, instead of the more desirable coproducts discussed in §3.22 (chief among which is the Drinfeld–Jimbo coproduct).

To remedy these issues, [22] suggested considering the attracting subvariety of the fixed-point locus $\mathcal{N}_{\boldsymbol{v},\boldsymbol{w}}^{\mathbb{C}^*} \hookrightarrow \mathcal{N}_{\boldsymbol{v},\boldsymbol{w}}$, as a replacement for the class $\wedge^\bullet(-T^{+\vee})$ in formula (4.43). Through a wide-reaching framework that pertains to conical symplectic resolutions, [18] defined a specific class on the disjoint union of all attracting subvarieties, called the *stable basis*, which gives a better analogue of the map (4.43). The K-theoretic version of this construction was developed in [1, 32, 33, 34], yielding a map

$$
\Upsilon_{\boldsymbol{m}} : K_{\boldsymbol{v},\boldsymbol{w}} \longrightarrow \bigoplus_{\boldsymbol{v}=\boldsymbol{v}^1+\boldsymbol{v}^2} K_{\boldsymbol{v}^1,\boldsymbol{w}^1}\otimes K_{\boldsymbol{v}^2,\boldsymbol{w}^2} \tag{4.45}
$$

for any decomposition $\boldsymbol{w} = \boldsymbol{w}^1 + \boldsymbol{w}^2$ in $\mathbb{N}^I$ and any $\boldsymbol{m} \in \mathbb{Q}^I$. As explained in [18], applying the FRT formalism to the maps (4.45) gives rise to a Hopf algebra

$$
U_q\big(\widehat{\mathfrak{g}}^Q\big) \subset \prod_{\boldsymbol{w}\in\mathbb{N}^I} \mathrm{End}(K_{\boldsymbol{w}}). \tag{4.46}
$$

A well-known conjecture in the field (see [36, Conjecture 1.2] or [40, Conjecture] for various incarnations) posits that the integral version of the Hopf algebra (4.46) is isomorphic to the double K-theoretic Hall algebra (4.7). As the localization (4.9) was shown to be isomorphic to the shuffle algebra $\mathcal{S}$ in [31], we propose the following:

Conjecture 4.22. *The map (4.40) yields an isomorphism* $\mathcal{A} \cong U_q\big(\widehat{\mathfrak{g}}^Q\big)$.

We remark that the definition of $U_q\big(\widehat{\mathfrak{g}}^Q\big)$ relies on many choices that we do not recall here: chambers, alcoves, polarization (see [34] for an overview). An essential precondition to proving Conjecture 4.22 is to properly make these choices such that the map (4.40) indeed maps $\mathcal{A}$ into $U_q\big(\widehat{\mathfrak{g}}^Q\big)$, although this is straightforward.

4.23. In comparing the maps (4.45) for $\boldsymbol{m} - \boldsymbol{\varepsilon}$ and $\boldsymbol{m} + \boldsymbol{\varepsilon}$ (for some $\boldsymbol{\varepsilon} \in \mathbb{Q}_+^I$ very close to $\boldsymbol{0}$), the application of the FRT formalism in [34] yields subalgebras

$$U_q\left(\mathfrak{g}_{\boldsymbol{m}}^Q\right) \subset U_q\left(\widehat{\mathfrak{g}}^Q\right) \tag{4.47}$$

for any $\boldsymbol{m} \in \mathbb{Q}^I$. We expect Conjecture 4.22 to match these subalgebras to the slope subalgebras of §3.9 – that is, there should be a commutative diagram

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\ \sim\ } & U_q\left(\widehat{\mathfrak{g}}^Q\right) \\
\uparrow & & \uparrow \\
\mathcal{B}_{\boldsymbol{m}} & \xrightarrow{\ \sim\ } & U_q\left(\mathfrak{g}_{\boldsymbol{m}}^Q\right),
\end{array}
$$

where the leftmost vertical map is prescribed by Proposition 3.10. Moreover, the factorization (3.53) should match the analogous factorization of the universal R-matrix of $U_q\left(\widehat{\mathfrak{g}}^Q\right)$ into the universal R-matrices of the subalgebras $U_q\left(\mathfrak{g}_{\boldsymbol{m}+r\boldsymbol{\theta}}^Q\right)$, which is quite tautological in the construction of [34].

Of particular interest is the case $\boldsymbol{m} = \boldsymbol{0}$ of the subalgebra (4.47), which is a q-deformation of the universal enveloping algebra of the Lie algebra $\mathfrak{g}^Q$ defined by [18]. Okounkov conjectured that the graded dimension of the latter Lie algebra should be equal to the value of the Kac polynomial of the quiver Q. If we knew that $\mathcal{B}_{\boldsymbol{0}} \cong U_q\left(\mathfrak{g}_{\boldsymbol{0}}^Q\right)$, then this conjecture would be equivalent to Conjecture 4.5.

Competing Interests. None.

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