PROBLEMS FOR SOLUTION

P 43. (Corrected.) Let G be a group generated by P and Q, and let H be the cyclic subgroup generated by P. If P and Q satisfy the relations $P^2QP = Q^2$ and $Q^2PQ^{-4} = P^k$ for some k, then the index of H in G is 1 or 7.

N. S. Mendelsohn

P 44. Show that

$$\pi^2 = 10 - \Sigma_{n=1}^{\infty} \frac{1}{n^3(n+1)^3}.$$

E. L. Whitney

P 45. Show that

$$\Sigma_{i=0}^{n} \binom{n+1}{i} \int_0^1 \binom{t}{i+2} dt = 0$$

for $n = 1, 3, 5, \ldots$, where $\binom{t}{k} = t(t-1)(t-2)\ldots(t-k+1)/k!$.

B. Wolk

P 46. Given infinitely many points in the plane such that

(a) the distance between any two of them is greater than 1,

(b) for infinitely many n, there are more than cn^2 points in the circle $|z| < n$.

Show that for any $\varepsilon > 0$ there is a line which comes closer than ε to infinitely many of the points.

P. Erdős

SOLUTIONS

P 10. (a) Prove that every set of six points in the plane can be colored in three colors in such a way that no two points unit distance apart have the same color.

(b) Show that in (a) six cannot be replaced by seven.

L. Moser and W. Moser

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Solution by the proposers. Two points which are unit distance apart we call friends, otherwise enemies. Obviously 4 points cannot all be friends of each other; and 2 points cannot have 3 common friends. If a finite set of points can be colored in 3 colors so that no pair of friends have the same color, we say this set can be 3-colored.

Any set of 4 points P_1, P_2, P_3, P_4 can be 3-colored. For at least one pair, say P_1 and P_2, are enemies; color these alike and use the two remaining colors for P_3 and P_4.

Let P_1, P_2, P_3, P_4, P_5 be a set of 5 points. Not all of them have precisely 3 friends each. For, in this case, if P_1 and P_2 are enemies, then they would have P_3, P_4, P_5 as common friends, and this is impossible. Now, if P_1 has ≤ 2 friends 3-color the four points P_2, P_3, P_4, P_5 and use for P_1 the available color different from P_1's friends. If P_1 has 4 friends, they lie on a unit circle (whose center is P_1) and can obviously be colored in 2 colors, leaving the third color for P_1. Thus every set of 5 points can be 3-colored.

Let P_1, P_2, P_3, P_4, P_5, P_6 be a set of 6 points. If P_1, say, has ≤ 2 friends, 3-color the set P_2, P_3, P_4, P_5, P_6 and use for P_1 the color different from those used for P_1's friends. If P_1 has 5 friends, they lie on a unit circle and can obviously be colored in 2 colors, leaving the third color for P_1. If P_1 has 4 friends, say P_2, P_3, P_4, P_5, and has enemy P_6, use 2 colors for P_2, P_3, P_4, P_5 (they lie on a unit circle) and use the third color for P_1 and P_6. Thus we may restrict our attention to the situation where each of the 6 points has precisely 3 friends.

Let P_1 and P_2 be enemies. Since each has 3 friends in the set P_3, P_4, P_5, P_6 they must have 2 common friends say P_3 and P_4. Let P_5 be P_1's third friend (and hence P_1 and P_6 are enemies). P_5 and P_2 are enemies; for otherwise P_1 and P_2 would have 3 common friends (namely P_3, P_4, P_5). Furthermore P_5 cannot be friends with both P_3 and P_4; for otherwise P_3 and P_4 would have 3 common friends (namely P_1, P_2, P_5). It follows that P_5 is friendly with P_6 and either P_3 or P_4, say P_3. Finally, since P_1 and P_3 each have 3 friends different from P_6 it follows that P_6 must have P_2 and P_4 as friends. Hence the set can be 3-colored by using color A for P_1 and P_6, color B for P_3 and P_4, color C

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for P_5, P_2, e.g. fig. 1, where two points are joined by a unit line if they are friends.

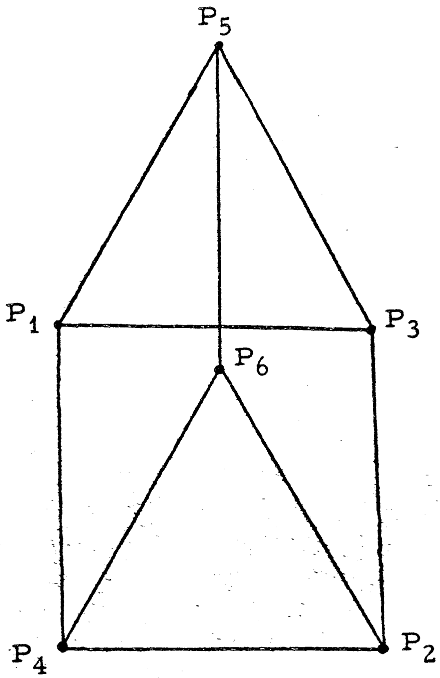


Fig. 1

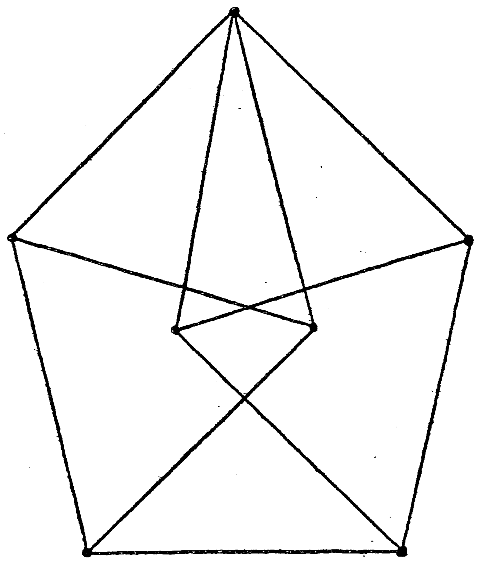


Fig. 2

Fig. 2 exhibits a configuration of 7 points which cannot be 3-colored.

P 30. Show that every triangle can be dissected into n isosceles triangles for every $n \geq 4$ but that some triangles cannot be dissected into 3 isosceles triangles.

L. Sauvé

Solution by the proposer. For $n = 2$, the dissecting line must pass through a vertex, and an investigation of the possible cases shows that the following triangles, and only those, can be dissected into 2 isosceles triangles:

(1) all right-angled triangles,

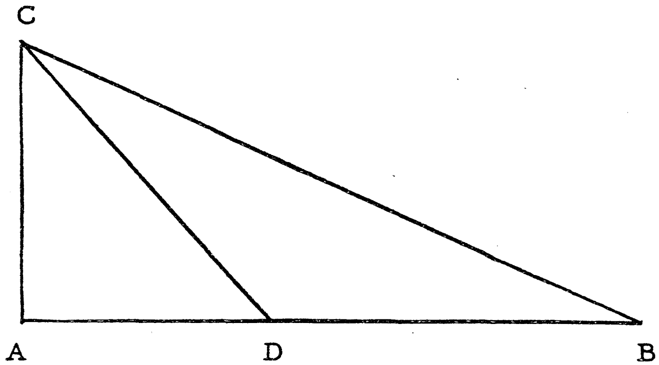
(2) all triangles in which one angle is twice another,

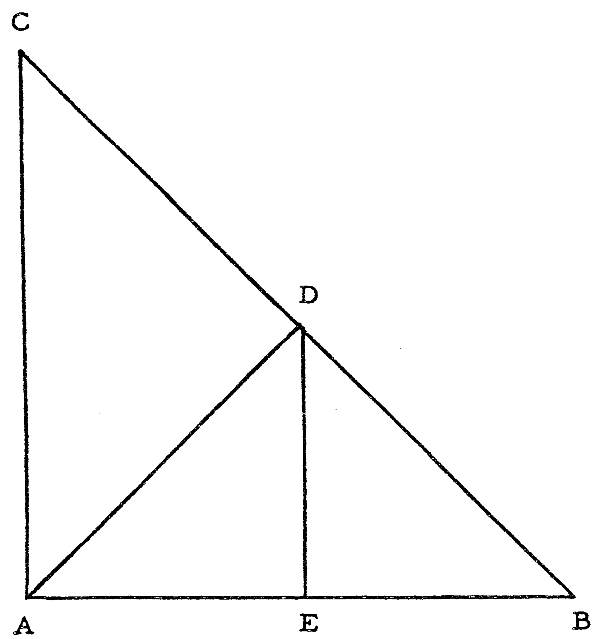
(3) all triangles in which one angle is three times another.

For $n = 3$, the dissection is always possible in the following cases:

(1) all acute-angled triangles; simply join the circumcentre to the vertices,

(2) all right-angled triangles. Let ABC be a right-angled triangle with the right angle at A.





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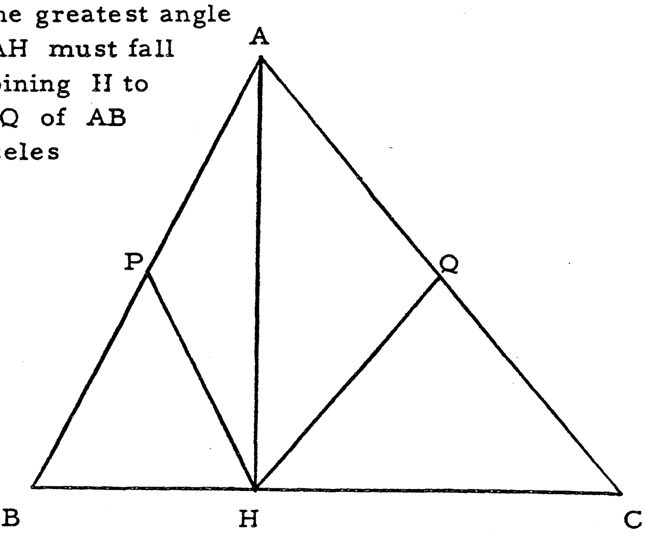
If $\angle B \neq \angle C$, say $\angle B < \angle C$, draw $\angle BCD = \angle B$. Then BCD is isosceles and ACD can be dissected into 2 isosceles triangles as seen in case $n = 2$. If $\angle B = \angle C$, draw $AD \perp BC$ and $DE \perp AB$.

The triangle with angles 1^o, 8^o, 171^o serves as an example to show that not every obtuse angled triangle can be dissected into 3 isosceles triangles. For one of the dissecting lines must pass through a vertex; but no such line can be found which yields an isosceles triangle and a triangle of the types for which $n = 2$ is possible.

For $n = 4$, the theorem holds for every triangle. For, given ABC in which the greatest angle is at A, the altitude AH must fall within the triangle. Joining H to the midpoints P and Q of AB and AC yields 4 isosceles triangles.

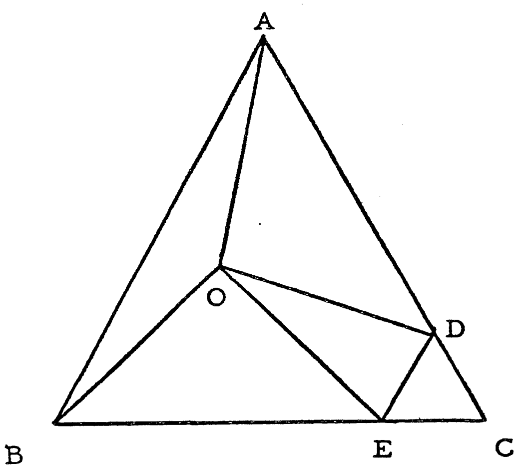


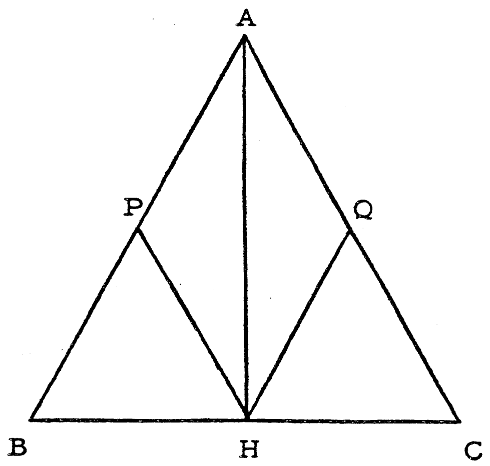
Assume that the theorem holds for $n = m$ and let ABC be a given triangle. We distinguish two cases:

(1) ABC is not equilateral. Then ABC can be dissected into an isosceles triangle and another triangle; the latter can be dissected into m isosceles triangles and thus ABC can be dissected into $m + 1$ isosceles triangles.

(2) ABC is equilateral. Then the theorem holds for $n = 3, 4, 5$. The cases $n = 3, 4$ have been proved above. For $n = 5$, select a point D on AC such that $CD < AD$ and draw $DE \parallel AD$. Then ABED is a cyclic quadrilateral and the centre O of its circumcircle lies within it. Join O to A, B, D, E and we have ABC dissected into 5 isosceles triangles. Now, if P, Q, H are the mid points of AB, AC, BC then triangles PBH,

APH, AQH are isosceles, and triangle QHC (which is equilateral) can be dissected into m isosceles triangles by the induction hypothesis; hence ABC can be dissected into m + 3 isosceles triangles. Thus the truth of the theorem for n = 3, 4, 5 implies its truth for n > 5.





COROLLARY. Every convex m-gon can be dissected into n isosceles triangles for every $n \geq 4(m - 2)$, and this inequality is the best possible.

Also solved by the proposer, R.J. Wisner, and L. Moser.

P 31. Prove that if $p > 3$ is a prime $\equiv 3 \pmod 4$ and $\zeta = e^{2\pi i/p}$, then

$$\prod_r (1 + \zeta^r) = \left(\frac{2}{p}\right)$$

where r runs through the quadratic residues of p, and $\left(\frac{2}{p}\right)$ is the Legendre symbol of quadratic residuacity.

L. J. Mordell

Solution by Emma Lehmer and P. Chowla. Suppose first that $p \equiv 7 \pmod 8$. Then $\left(\frac{2}{p}\right) = 1$ and hence the quadratic residues may be denoted by either r or 2r. Thus

$$\prod_r (1 - \zeta^{2r}) = \prod_r (1 - \zeta^r)$$

and then

$$\prod_r (1 + \zeta^r) = 1.$$

Suppose next that $p \equiv 3 \pmod 8$. Then $(\frac{2}{p}) = -1$, $(\frac{-1}{p}) = -1$ and so the quadratic residues may be denoted by either r or $-2r$. Hence

$$\prod_r (1 - \zeta^{-2r}) = \prod_r (1 - \zeta^r),$$

$$(-1)^{(p-1)/2} \prod_r (1 + \zeta^r) = \prod_r \zeta^{2r},$$

and then if $p > 3$, $\prod_r (1 + \zeta^r) = 1$, since $\Sigma r \equiv 0 \pmod p$.

Also solved by the proposer, L. Carlitz, and R. Ayoub.

P 32. The equation

$$(1 + 2\cos\frac{\pi}{p})(1 + 2\cos\frac{\pi}{q}) = 1$$

is obviously satisfied by $p = q = 2$. Are there any other rational solutions with $p \geq q \geq 1$?

N. W. Johnson

Solution by H. Schwerdtfeger. Let us write the given equation in the form

$$(1 + 2\cos\frac{2\pi a}{n})(1 + 2\cos\frac{2\pi b}{n}) = 1$$

and ask for integral solutions (a, b, n). Such may be obtained in the following way. Let

$$z = e^{2\pi i/n}$$

(or any other primitive n-th root of 1). Then the equation becomes

$$(1 + z^a + z^{-a})(1 + z^b + z^{-b}) = 1,$$

and after multiplication by $z^a z^b$,

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$$f(a,b;z) = z^{2(a+b)} + z^{a+2b} + z^{2a+b} + z^{2b} + z^{2a} + z^{b} + z^{a} + 1 = 0.$$

Now we put the question in the following way. For which positive integral a, b, n is a primitive n-th root of unity a root of this equation?

It may be noted that automatically with x also x^{-1} is a root of the equation.

Series of solutions are obtained as follows. Let $z^a = y$. Then

$$f(a,a;z) = y^4 + 2y^3 + 2y^2 + 2y + 1 = (y+1)^2(y^2+1).$$

Hence

$$z = e^{2\pi i/2a} \quad \text{or} \quad e^{2\pi i/4a}$$

which for any positive integer a yields the solutions $b = a$, $n = 2a$ and $b = a$, $n = 4a$.

Further we examine

$$f(a,2a;z) = y^6 + y^5 + 2y^4 + 2y^2 + y + 1.$$

By testing divisibility with the cyclotomic polynomials of degree < 6 it is found that no root of unity is root of this polynomial.

Finally,

$$\begin{aligned} f(a,3a;z) &= y^8 + y^7 + y^6 + y^5 + y^3 + y^2 + y + 1 \\ &= (y+1)^2(y^2+1)(y^4 - y^3 + y^2 - y + 1) \end{aligned}$$

whence the following solutions are derived: for $b = 3a$; $n = 2a$, $n = 4a$, and $n = 10a$.

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