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A DIVERGENCE-FREE ANTISYMMETRIC TENSOR

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Consider a V_n with metric tensor $g_{\alpha\beta}$. It is well known that the only tensor containing derivatives of $g_{\alpha\beta}$ no higher than the second order, linear in the second order derivatives, and having vanishing covariant divergence is the Einstein tensor

(1) $$G_{\alpha\beta} \equiv R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R.$$

Suppose that a system of ennuple vectors $h^i{}_\alpha$ and their inverses $h_i{}^\alpha$ is set up at each point of the V_n. (Here the latin suffixes are ennuple suffixes and greek suffixes are tensor suffixes.) These vectors satisfy the relations

(2) $$h^i{}_\alpha h_{i\beta} = g_{\alpha\beta}$$

(3) $$h_{i\alpha} h_j{}^\alpha = \eta_{ij}$$

where $\eta_{ij} = \operatorname{diag}(e_1, \ldots, e_n)$; $e_i = \pm 1$ $(i=1, \ldots, n)$. Using (2) to replace the metric tensor and its derivatives in (1) it is seen that the Einstein tensor is of the second order and first degree in the $h^i{}_\alpha$. The problem to be considered here is whether or not there are any other tensors with vanishing covariant divergence which also can be expressed in terms of the ennuple vectors and their derivatives and which also are of the second order and first degree in the $h^i{}_\alpha$.

The Ricci rotation coefficients are defined by

(4) $$\gamma_{ijk} = h_{i\alpha;\beta} h_j{}^\alpha h_k{}^\beta$$

where the semi-colon denotes covariant differentiation with respect to the Christoffel bracket connection formed from the metric tensor. The corresponding tensor quantity is

(5) $$\gamma_{\alpha\beta\gamma} = h^i{}_\alpha h^j{}_\beta h^k{}_\gamma \gamma_{ijk}.$$

This tensor is antisymmetric in the first two suffixes and from (4) may be written in the form

(6) $$\gamma_{\alpha\beta\gamma} = h_{i\alpha} h^i{}_{\beta;\gamma}.$$

A vector γ_β may be defined by

(7) $$\gamma_\beta = \gamma^\alpha{}_{\beta\alpha} = h_i{}^\alpha h^i{}_{\beta;\alpha}.$$

Consider now the following tensor

(8) $$S^{\alpha\beta} = (\gamma^{\alpha\beta\gamma} + \gamma^{\beta\gamma\alpha} + \gamma^{\gamma\alpha\beta})_{;\gamma}.$$

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From the antisymmetry property of $\gamma^{\alpha\beta\gamma}$ it follows that $S^{\alpha\beta}$ is antisymmetric and from (6) it is clear that $S^{\alpha\beta}$ is of the second order and first degree in the $h^i_{\ \alpha}$. It remains to show that it has vanishing divergence.

Let $P^{\alpha\beta\gamma}=\gamma^{\alpha\beta\gamma}+\gamma^{\beta\gamma\alpha}+\gamma^{\gamma\alpha\beta}$. Then $P^{\alpha\beta\gamma}$ is antisymmetric in all its suffixes and it follows that

$$\begin{aligned} S^{\alpha\beta}{}_{;\beta} &= P^{\alpha\beta\gamma}{}_{;\gamma\beta} \\ &= \tfrac{1}{2}(P^{\alpha\beta\gamma}{}_{;\gamma\beta}-P^{\alpha\beta\gamma}{}_{;\beta\gamma}) \\ &= \tfrac{1}{2}(R_{\varepsilon\ \beta\gamma}^{\ \alpha}P^{\varepsilon\beta\gamma}+R_{\varepsilon\ \beta\gamma}^{\ \beta}P^{\alpha\varepsilon\gamma}+R_{\varepsilon\ \beta\gamma}^{\ \gamma}P^{\alpha\beta\varepsilon}). \end{aligned}$$

The last two terms cancel so

$$S^{\alpha\beta}{}_{;\beta} = \tfrac{1}{2}R_{\varepsilon\ \beta\gamma}^{\ \alpha}(\gamma^{\varepsilon\beta\gamma}+\gamma^{\beta\gamma\varepsilon}+\gamma^{\gamma\varepsilon\beta}).$$

By suitably permuting suffixes this becomes

$$S^{\alpha\beta}{}_{;\beta} = \tfrac{1}{2}\gamma^{\varepsilon\beta\gamma}(R_{\varepsilon\ \beta\gamma}^{\ \alpha}+R_{\gamma\ \varepsilon\beta}^{\ \alpha}+R_{\beta\ \gamma\varepsilon}^{\ \alpha}).$$

The term in the bracket is zero from the properties of the curvature tensor. Thus in addition to the symmetric tensor $G_{\alpha\beta}$ we have the antisymmetric tensor $S_{\alpha\beta}$ which has similar properties and any tensor of the form $aG_{\alpha\beta}+bS_{\alpha\beta}$, where a, b are constants, will also have these properties. It seems likely that these are the only two tensors with these properties but no proof of this has yet been found.

Note that $G_{\alpha\beta}$ also can be written in terms of $\gamma_{\alpha\beta\gamma}$ and its derivatives. By considering the relation

$$h^i_{\ \beta;\gamma\delta}-h^i_{\ \beta;\delta\gamma} = h^i_{\ \alpha}R^{\alpha}_{\ \beta\gamma\delta}$$

it is found that

$$R^{\alpha}_{\ \beta\gamma\delta} = \gamma^{\alpha}_{\ \beta\gamma;\delta}-\gamma^{\alpha}_{\ \beta\delta;\gamma}+\gamma^{\alpha}_{\ \varepsilon\delta}\gamma^{\varepsilon}_{\ \beta\gamma}-\gamma^{\alpha}_{\ \varepsilon\gamma}\gamma^{\varepsilon}_{\ \beta\delta}.$$

This leads to the expression

$$G_{\alpha\beta} = \gamma^{\varepsilon}_{\ \alpha\beta;\varepsilon}-\gamma_{\alpha;\beta}+g_{\alpha\beta}\gamma^{\varepsilon}{}_{;\varepsilon}+\gamma^{\varepsilon}\gamma_{\varepsilon\alpha\beta}-\gamma^{\delta}_{\ \varepsilon\beta}\gamma^{\varepsilon}_{\ \alpha\delta}+\tfrac{1}{2}g_{\alpha\beta}\gamma_{\varepsilon}\gamma^{\varepsilon}-\tfrac{1}{2}g_{\alpha\beta}\gamma^{\varepsilon\gamma\delta}\gamma_{\varepsilon\delta\gamma}. \tag{9}$$

The physical significance, if any, of $S_{\alpha\beta}$ in a four-dimensional space-time is not clear. In view of the recent attempts [1], [2], [3] to formulate field equations in general relativity in which the ennuple vectors, rather than the metric tensor, are regarded as the quantities with the basic physical meaning, this tensor may prove to be useful.

Added in Proof. The suggestion made in this note that $S^{\alpha\beta}$ is the only antisymmetric tensor with the properties described is unfounded. At least one other such tensor exists in a V_4, namely

$$W^{\alpha\beta} = \eta^{\alpha\beta\mu\nu}\gamma_{\mu;\nu},$$

where γ_μ is defined by equation (7) and $\eta^{\alpha\beta\mu\nu}$ is the permutation tensor. Unlike $S^{\alpha\beta}$ the tensor $W^{\alpha\beta}$ is dimensionally dependent in that it is a second-rank tensor only because the permutation tensor has four suffixes in a V_n. In a general V_n $(n \geq 4)$

the corresponding completely antisymmetric divergence-free tensor has rank $(n-2)$.

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