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THE SEXTIC PERIOD POLYNOMIAL

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In this paper we show that the method of calculating the Gaussian period polynomial which originated with Gauss can be replaced by a more general method based on formulas for Lagrange resolvants. The period polynomial of cyclic sextic fields of arbitrary conductor is determined by way of example.

1. INTRODUCTION

Suppose $p = ef + 1$ is prime. Define the e cyclotomic classes

$$\mathcal{C}_j = \{g^{ek+j} \bmod p, \quad j = 0, \dots, e-1, \quad k = 0, \dots, f-1\},$$

where g is any primitive root modulo p. The Gaussian periods η_j are defined by

$$\eta_j = \sum_{\nu \in \mathcal{C}_j} \zeta_p^{\nu}, \quad \zeta_p = \exp(2\pi i/p). \tag{1.1}$$

The principal class $\mathcal{C}_0$ contains the e-th power residues and the other classes are its cosets. The η_j are Galois conjugates and the *period polynomial* $\Psi_e(X)$ is their common minimal polynomial over $\mathbb{Q}$. Gauss introduced the *cyclotomic numbers* (h,k) determined, for a given g, by

$$(h,k) = \#\{\nu \in (\mathbb{Z}/p\mathbb{Z})^* : \nu \in \mathcal{C}_h, \ \nu + 1 \in \mathcal{C}_k\}.$$

It follows that

$$\eta_0 \eta_h = \sum_{k=0}^{e-1} (h,k)\eta_k + f\delta(h,\ell) \tag{1.2}$$

where δ is Kronecker's delta and $\ell = 0$ or $e/2$ according as f is even or odd. The coefficients of $\Psi_3(X)$ in terms of p and the coefficients of the quadratic form $4p = A^2 + 27B^2$ were determined by Gauss in *Disquisitiones Arithmeticae*: enough relations exist to determine all (h,k) in terms of p, A, and B. The period polynomial's

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coefficients are then calculated as the symmetric functions of the ηs. (See Gauss, Bachmann [1], or Mathews [13].) The same general method and (1.2), with the appropriate quadratic form, was used to solve the cases $e = 4$ (Gauss [5], 1825), $e = 5$ (Emma Lehmer [10], 1951), $e = 6$ (D. H. and Emma Lehmer [9], 1984), and $e = 8$ (Evans [4], 1983).

Throughout ζ_n is the root of unity $\exp(2\pi i/n)$ and $\mu(\cdot)$ is the Möbius function. Let χ be a primitive character of degree e and modulus p. Recall the Lagrange resolvant (sometimes called a Gauss sum) defined by $\tau(\chi) = \sum_{j=0}^{p-1} \chi(j)\zeta_p^j$. Provided that the periods are defined with primitive root g such that $\chi(g) = \zeta_e$,

$$\tau(\chi^j) = \sum_{k=0}^{e-1} \chi^j(k)\eta_k. \tag{1.3}$$

The inverse of (1.3) is

$$\eta_j = e^{-1} \sum_{h=0}^{e-1} \zeta_p^{-hj}\tau(\chi^h). \tag{1.4}$$

Since (1.4) does not depend on the existence of a primitive root, it defines η_j for any character χ of *arbitrary* modulus.

Field-theoretically, embed an abelian field K of degree $[K : \mathbb{Q}] = e$, Galois group $\mathcal{G} = \mathrm{Gal}(K/\mathbb{Q})$, and conductor m in $\mathbb{Q}[\zeta_m]$. Let $\widehat{\mathcal{G}}$ be the group of Dirichlet characters modulo m which annihilate $\mathrm{Gal}(\mathbb{Q}[\zeta_m]/K) \subset \mathrm{Gal}(\mathbb{Q}[\zeta_m]/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^*$. We say that K *belongs to* $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{G}}$ is *associated to* K. Then $\widehat{\mathcal{G}}$ is the dual of $\mathcal{G}$ and $\widehat{\mathcal{G}} \cong \mathcal{G}$ [15, Chapter 3]. The Gaussian period is defined in this most general case by

$$\eta_j = e^{-1} \sum_{\chi\in\widehat{\mathcal{G}}} \zeta_p^{-hj}\tau(\chi). \tag{1.5}$$

This reduces to (1.4) when $\widehat{\mathcal{G}}$ is cyclic and to (1.1) when m is prime. It is easy to see that $\eta_0 = \mathrm{Tr}_K^{\mathbb{Q}[\zeta_m]} \zeta_m$. The class $\mathcal{C}_0$ becomes the kernel of $\widehat{\mathcal{G}}$ in $(\mathbb{Z}/m\mathbb{Z})^*$. For all $j \in \mathcal{C}_0$, the map $\zeta_m \mapsto \zeta_m^j$ is an automorphism of $\mathbb{Q}[\zeta_m]$ which fixes K. The period polynomial in this general case was determined in an ad hoc way for cyclic cubic fields by Châtelet [3], and for cyclic quartic fields by Nakahara [14] and (independently) by the author [8]. The computation of period polynomials can be made systematic through well-known formulas for arithmetic of Lagrange resolvants.

Lemma 1.1. (A) *For χ_m and χ_n of conductors m, n respectively with* $\gcd(m,n) = 1$,

$$\tau(\chi_m\chi_n) = \chi_n(m)\chi_m(n)\tau(\chi_m)\tau(\chi_n).$$

(B) *If the conductor of χ is m and c is odd, then*

$$\sum_{j=0}^{cm-1} \chi(j)\zeta_{cm} = \mu(c)\chi(c)\tau(\chi).$$

If c is even the sum vanishes.

(C) *If $\mathcal{X}$ is the set of characters of prime conductor p, $\ell \mid p-1$, and $\chi^\ell \neq 1$, then*

$$\prod_{\substack{\psi\in\mathcal{X}\\ \psi^\ell=1}} \tau(\chi\psi) = \overline{\chi}^\ell(\ell)\tau(\chi^\ell) \prod_{\substack{\psi\in\mathcal{X}\\ \psi^\ell=1}} \tau(\psi).$$

PROOF: The first two formulas are routine; the last is the theorem of Hasse and Davenport [**6**, 20.4.IX]. □

REMARKS. (1) The Hasse-Davenport Theorem has been used extensively in cyclotomy of prime modulus and composite *degree*; see, for example, Buck, Smith, Spearman and Williams [**2**].

(2) The Gaussian periods appear as summands of the 'Basiszahl' of an abelian number field in the elegant paper of Lettl [**11**].

2. Sextic Dirichlet Characters

NOTATION. From now on ψ, ξ, and χ will be primitive quadratic, cubic, and sextic characters, respectively. A subscript, for example χ_m, will indicate the conductor. Powers such as χ_m^2 denote a possibly imprimitive character.

We may associate to every cubic character ξ_m an integer $\mathfrak{m}$ in $\mathbb{Q}[\zeta_3]$ as follows. There is a prime-power decomposition of

$$\xi_m = \xi_{p_1} \cdots \xi_{p_\nu}$$

where each p_j is either 9 or a prime congruent to 1 modulo 3. We may assume that if $3 \mid m$, the divisors are ordered such that $p_1 = 9$. ξ_{p_j} is a complex cubic residue symbol modulo $\mathfrak{p}_j$, where $\mathfrak{p}_j$ is a prime of $\mathbb{Q}[\zeta_3]$ lying over p_j (over 3, if $p_1 = 9$). Set

$$\mathfrak{m} = \begin{cases} \prod_{j=1}^{\nu} \mathfrak{p}_j & 3 \nmid m \\ 3\mathfrak{p}_1 \prod_{j=2}^{\nu} \mathfrak{p}_j & \text{otherwise.} \end{cases} \tag{2.1}$$

Since there are two conjugate primes lying over each p_j, it is clear that there are 2^ν different cubic characters, where m is divisible by ν distinct primes, but ξ and $\overline{\xi}$ generate the same group and are associated to the same field. Therefore there are $2^{\nu-1}$ distinct cubic fields of conductor m.

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Lemma 2.1. *If ξ_m is a cubic character and ψ_m is a quadratic character, both primitive with conductor m, not necessarily prime, then*

$$\tau(\xi_m\psi_m) = \xi_m(2)\tau(\psi_m)\frac{\tau(\overline{\xi}_m)^2}{m}.$$

Proof: The conductor of a primitive quadratic (respectively cubic) character is square-free except for powers of 2 (respectively 3), so m is square-free. If m is prime, the lemma is just the Hasse-Davenport Theorem (Lemma 1.1C). Proceeding by induction on the number of primes dividing m, write $m = pq$, p prime, and factor $\xi_m = \xi_p\xi_q$ (similarly ψ_m).

$$\begin{aligned}\tau(\xi_m\psi_m) &= \xi_q(p)\psi_q(p)\xi_p(q)\psi_p(q)\tau(\xi_p\psi_p)\tau(\xi_q\psi_q), \quad \text{by Lemma 1.1A and } \gcd(p,q)=1\\ &= \frac{\xi_p(2)}{p}\frac{\xi_q(2)}{q}\xi_q(p)\xi_p(q)\tau(\overline{\xi}_p)^2\tau(\overline{\xi}_q)^2\psi_q(p)\psi_p(q)\tau(\psi_p)\tau(\psi_q),\\ &\qquad\qquad \text{by inductive hypothesis}\\ &= \xi_m(2)\tau(\psi_m)\frac{\tau(\overline{\xi}_m)^2}{m}.\end{aligned}$$

□

Lemma 2.2. *The complex integer $\mathfrak{m}$ defined by (2.1) is equal to the Jacobi sum $J(\xi_m,\xi_m) = \tau(\xi_m)^2/\tau(\overline{\xi}_m)$. The sextic resolvant $\tau(\xi_m\psi_m) = \xi_m(2)\tau(\psi_m)\tau(\xi_m)\overline{\mathfrak{m}}/m$.*

Proof: The first clause is well-known, for example, Hasse [7, Section 2(1)]. The second now follows using the previous lemma. □

3. The period polynomial

The sextic period polynomial will be determined from these lemmas. Let K be a cyclic sextic field of conductor m. K is the compositum of a quadratic field K_2 of conductor m_2 and a cyclic (hence real) cubic field K_3 of conductor m_3. Set $g = \gcd(m_2, m_3)$. In this section we shall assume that $3 \nmid g$; we shall treat the other case afterwards. Let $n_i = m_i/g$, $i = 2,3$. Then $m = gn_2n_3$ and $3 \nmid g$ implies g, n_2, and n_3 are pairwise co-prime.The sextic character χ_m can be factored into a product of cubic and quadratic characters:

$$\chi_m = \xi_{m_3}\psi_{m_2} = \xi_g\xi_{n_3}\psi_g\psi_{n_2}.$$

We define $\mathfrak{g}$ and $\mathfrak{m}$ as the Jacobi sums of ξ_g and ξ_{m_3}. (If $g = 1$, we define $\mathfrak{g} = 1$.) Let $\mathfrak{n} = \mathfrak{m}/\mathfrak{g}$, which is $J(\xi_{n_3},\xi_{n_3})$ if $n_3 > 1$.

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We are now equipped to determine the resolvants necessary to use (1.4). Clearly $\tau(\chi^6) = \mu(m)$. Since K_3 is real, $\psi_{m_2}(-1) = \chi_m(-1)$. From Lemma 1.1B and the result of Gauss on $\tau(\psi)$,

$$\tau(\chi_m^3) = \mu(n_3)\psi_{m_2}(n_3)\sqrt{m_2^*}, \quad m_2^* = \chi_m(-1)m_2.$$

From Lemmas 1.1B and 2.2,

$$\tau(\chi_m^2) = \mu(n_2)\xi_{m_3}(n_2)\tau(\overline{\xi}_{m_3}).$$

The sextic resolvant can be found in terms of the quadratic and cubic ones by Lemmas 1.1 and 2.2.

(3.1)

$$\begin{aligned}\tau(\chi_m) &= \xi_{n_3}\psi_{n_2}(g)\,\xi_g\psi_g(n_3n_2)\,\tau(\xi_{n_3}\psi_{n_2})\,\tau(\xi_g\psi_g)\\ &= \xi_{n_3}\psi_{n_2}(g)\,\xi_g\psi_g(n_3n_2)\,\xi_{n_3}(n_2)\,\psi_{n_2}(n_3)\,\tau(\xi_{n_3})\,\tau(\psi_{n_2})\,\xi_g(2)\,\tau(\psi_g)\,\tau(\xi_g)\overline{\mathfrak{g}}/g\\ &= \xi_{m_3}(n_2)\psi_{m_2}(n_3)\xi_g(2)\tau(\psi_{m_2})\tau(\xi_{m_3})\overline{\mathfrak{g}}/g\end{aligned}$$

It is easy to see that $\tau(\chi_m^4) = \overline{\tau(\chi_m^2)}$ and $\tau(\chi_m^5) = \chi_m(-1)\overline{\tau(\chi_m)}$.

The symbolic coefficients are simpler if we work with the *reduced Gaussian period* $\lambda_j = e\eta_j - \mu(m)$. The reduced period polynomial $\Lambda(X)$ is given by

$$\Lambda(X) = \operatorname{Irr}_{\mathbb{Q}}^{K}\lambda_0 = \operatorname{Irr}_{K_2}(\lambda_0)\operatorname{Irr}_{K_2}(\lambda_1).$$

The normal and reduced period polynomials of degree e are related by $\Psi_e(X) = e^{-e}\Lambda_e(eX - \mu(m))$.

PROPOSITION 3.1. *The minimal polynomial* $\operatorname{Irr}_{K_2}(\lambda_0)$ *of* λ_0 *over* K_2 *is*

$$(X - \lambda_0)(X - \lambda_2)(X - \lambda_4) = X^3 + c_2X^2 + c_1X + c_0$$

where a calculation shows

$$\begin{aligned}c_2 &= -3\,\mu(n_3)\,\psi_{m_2}(n_3)\,\sqrt{m_2^*}\\ c_1 &= -3\,\mu(n_2)^2m_3 - 3\,\chi_m(-1)\,\left(m - \mu(n_3)^2m_2\right)\\ &\quad - 6\,\mu(n_2)\,\psi_{m_2}(n_3)\,n_3\,\operatorname{Re}(\xi_g(2)\,\overline{\mathfrak{g}})\sqrt{m_2^*}\\ c_0 &= -2\,\mu(n_2)\,m_3\,\operatorname{Re}(\mathrm{m}) - 6\,\mu(n_2)\,m\chi_m(-1)\,\operatorname{Re}(\xi_g(2)[\overline{\mathrm{n}}\mathrm{g} - \mu(n_3)\,\overline{\mathfrak{g}}])\\ &\quad \{-3\,\mu(n_2)^2m_3\,\left(2\,\operatorname{Re}(\xi_g(2)\,\mathrm{n}) - \mu(n_3)\right)\\ &\quad -n_2\,\chi_m(-1)\,\left(2\,n_3\,\operatorname{Re}(\overline{\mathrm{n}}\,\mathrm{g}^2) + \mu(n_3)\,(g - 3\,m_3)\right)\}\psi_{m_2}(n_3)\,\sqrt{m_2^*}.\end{aligned}$$

PROOF: The coefficients were computed from (1.4) and the values of $\tau(\chi^j)$ using the Maple symbolic algebra system, and then simplified by Lemma 2.2. ∎

To rewrite Proposition 3.1 with rational integer coefficients, create new variables with the assignments

(3.2)
$$\xi_g(2) = \frac{z_0 + \sqrt{-3}z_1}{2}, \quad \mathfrak{g} = \frac{A + 3\sqrt{-3}B}{2}, \quad \mathfrak{m} = \frac{L + 3\sqrt{-3}M}{2}, \quad \mathfrak{n} = \frac{R + 3\sqrt{-3}S}{2}$$

so that

$$R = \frac{AL + 27BM}{2g}, \qquad S = \frac{AM - BL}{2g} \tag{3.3}$$

Maple's Gröbner basis reduction package normalised c_0 and c_1 with respect to (3.3) and the obvious relations on the conductors. We have that

$$\begin{aligned}
c_1 = &-3\,\mu(n_2)^2 m_3 - 3\,\chi_m(-1)\left(m - \mu(n_3)^2 m_2\right)\\
&- \frac{3}{2}\,\mu(n_2)\,\psi_{m_2}(n_3)\,n_3\,(z_0\,A + 9\,z_1\,B)\sqrt{m_2^*}\\
c_0 = &-\mu(n_2)\,m_3\,L\\
&- \frac{3}{4}\,\mu(n_2)\,m\chi_m(-1)\,[z_0\,(AR + 27\,BS) + 9z_1\,(AS - BR) - 2\,\mu(n_3)\,(z_0\,A + 9z_1\,B)]\\
&- \left\{\frac{3}{2}\,\mu(n_2)^2 m_3\,(z_0R - 9z_1S - 2\,\mu(n_3)) + \frac{1}{2}\,n_2\,\chi_m(-1)\,[n_3\,(2\,gR\right.\\
&\left.+27\,ABS - 27\,B^2R) + 2\,\mu(n_3)\,(g - 3\,m_3)]\right\}\psi_{m_2}(n_3)\sqrt{m_2^*}.
\end{aligned}$$

Define rational numbers c_0' and c_0'' by $c_0 = c_0' + c_0''\sqrt{m_2^*}$; similarly c_2'', c_1' and c_1''. The conjugate polynomial $\mathrm{Irr}_{K_2}(\lambda_1)$ is

$$X^3 - c_2X^2 + \left(c_1' - c_1''\sqrt{m_2^*}\right)X + \left(c_0' - c_0''\sqrt{m_2^*}\right).$$

Writing the reduced period polynomial

$$\Lambda(X) = X^6 + 0\,X^5 + d_4X^4 + d_3X^3 + d_2X^2 + d_1X + d_0$$

we have our main result.

Theorem 1. *The coefficients d_ν are given by*

$$\begin{aligned}
d_4 &= -6\,\mu(n_2)^2 m_3 - 3\left(2\,m + \mu(n_3)^2 m_2\right)\chi_m(-1)\\
d_3 &= \mu(n_2)\Big\{-2\,m_3\,L\\
&\quad + \left(-6\,\mu(n_3)\left(z_0A + 9\,z_1B\right) - \frac{3}{2}\,[z_0\,(AR + 27\,BS) + 9\,z_1\,(AS - BR)]\right)\chi_m(-1)m\Big\}\\
d_2 &= {c_1'}^2 - \left({c_1''}^2 + 2c_0''c_2''\right)\chi_m(-1)m_2\\
d_1 &= 2(c_0'c_1' - c_0''c_1''\chi_m(-1)m_2)\\
d_0 &= {c_0'}^2 - {c_0''}^2\chi_m(-1)m_2.
\end{aligned}$$

The expanded expressions for the three trailing coefficients are very large.

REMARKS. (1) d_4 depends on m_3 but not on m, that is, the choice of K_3 is irrelevant.
(2) The coefficients are all divisible by g. This is to be expected, since λ_j is a sum of terms divisible by a prime over g.
(3) When $4 \mid m_2$ the polynomial is even. (This can also be seen directly from the definition.)
(4) The factor $\psi_{m_2}(n_3)$ does not appear explicitly.

COROLLARY 3.2. *If* $m = m_2 = m_3 = g$, *then*

$$
\begin{aligned}
c_2 &= -3\sqrt{\chi_m(-1)m} \\
c_1 &= -3m - \frac{9}{2}(z_0 L + 3 z_1 M)\sqrt{\chi_m(-1)m} \\
c_0 &= -mL + 27\chi_m(-1) z_1 Mm + \left([3 - 3z_0 + 4\chi_m(-1)]m - \chi_m(-1)L^2\right)\sqrt{\chi_m(-1)m} \\
d_4 &= -3m(3\chi_m(-1) + 2) \\
d_3 &= [-9(z_0 L + 3 z_1 M)\chi_m(-1) - 2L]m \\
d_2 &= \left[9\left(2 - 2z_0 - 3z_1^2\right)\chi_m(-1) + 33\right]m^2 \\
&\quad - \frac{3}{4}\left[3\left(z_0^2 L^2 + 18 z_0 z_1 LM - 3 z_1^2 L^2\right)\chi_m(-1) + 8L^2\right]m \\
d_1 &= -9\left(9 z_1 M + z_0^2 L + 9 z_0 z_1 M - z_0 L\right)\chi_m(-1)m^2 \\
&\quad + 3\left(2Lm + 4 z_0 Lm + 36 z_1 Mm - 9 z_1 L^2 M - z_0 L^3\right)m \\
d_0 &= (27\chi_m(-1) z_1 M - L)^2 m^2 - \left([3 - 3z_0 + 4\chi_m(-1)]m - \chi_m(-1)L^2\right)^2 \chi_m(-1)m.
\end{aligned}
$$

REMARK. If moreover m is prime we recover formulas (8)–(12) of [9].

EXAMPLE. To illustrate the theorem, we show in Table 1 all ten sextic period polynomials of conductor 91. The two sextic subfields of $\mathbb{Q}[\zeta_{91}]$ of smaller conductor are omitted. Although the reduced polynomial is simpler symbolically, the period polynomial in the table has smaller coefficients.

TABLE 1. Sextic period polynomials of conductor 91

m_2	m_3	g	$\mathfrak{m}$	C_0	$\xi_g(2)$	$\Psi_6(X)$
7	13	1	$\frac{-5+3\sqrt{-3}}{2}$	$\{1,8,18,25,44,51,53,57,60,64,79,86\}$	1	$X^3 - X^5 + 14X^4 + 13X^3 + 58X^2 + 16X + 8$
7	91	$\frac{1+3\sqrt{-3}}{2}$	$-8-3\sqrt{-3}$	$\{1,2,4,8,16,23,32,37,46,57,64,74\}$	ζ_3	$X^6 - X^5 + X^4 + 13X^3 + 162X^2 - 400X + 736$
7	91	$\frac{1+3\sqrt{-3}}{2}$	$\frac{11+9\sqrt{-3}}{2}$	$\{1,8,9,11,30,57,58,64,67,72,81,88\}$	$\bar{\zeta}_3$	$X^6 - X^5 + X^4 + 13X^3 + 71X^2 + 419X + 827$
13	7	1	$\frac{1+3\sqrt{-3}}{2}$	$\{1,22,27,29,36,43,48,55,62,64,69,90\}$	1	$X^6 - X^5 - 17X^4 + 4X^3 + 57X^2 - 18X - 27$
13	91	$\frac{-5+3\sqrt{-3}}{2}$	$-8-3\sqrt{-3}$	$\{1,4,16,17,23,27,64,68,74,75,87,90\}$	$\bar{\zeta}_3$	$X^6 - X^5 - 31X^4 + 4X^3 + 253X^2 + 101X - 391$
13	91	$\frac{-5+3\sqrt{-3}}{2}$	$\frac{11+9\sqrt{-3}}{2}$	$\{1,3,9,10,27,30,61,64,81,82,88,90\}$	$\bar{\zeta}_3$	$X^6 - X^5 - 31X^4 + 4x^3 + 162X^2 - 81X - 27$
91	7	$\frac{1+3\sqrt{-3}}{2}$	$\frac{1+3\sqrt{-3}}{2}$	$\{1,6,20,22,29,34,36,41,43,64,76,83\}$	ζ_3	$X^6 - X^5 + 22X^4 - 22X^3 + 148X^2 - 148X + 337$
91	13	$\frac{-5+3\sqrt{-3}}{2}$	$\frac{-5+3\sqrt{-3}}{2}$	$\{1,5,25,31,34,47,51,53,64,73,79,83\}$	$\bar{\zeta}_3$	$X^6 - X^5 + 21X^4 - 22X^3 + 58X^2 + 23X + 155$
91	91	$-8-3\sqrt{-3}$	$-8-3\sqrt{-3}$	$\{1,4,16,23,34,45,54,59,64,74,83,89\}$	1	$X^6 - X^5 + 8X^4 - 113X^3 + 435X^2 - 666X + 428$
91	91	$\frac{11+9\sqrt{-3}}{2}$	$\frac{11+9\sqrt{-3}}{2}$	$\{1,9,19,24,30,33,34,64,80,81,83,88\}$	ζ_3	$X^6 - X^5 + 8X^4 - 22X^3 - 20X^2 + 426X + 1611$

REMARK. For units in sextic fields see Mäki [12].

4. THE SPECIAL CASE $3 \mid \gcd(m_2, m_3)$

In this case, we write $g = \gcd(m_2, m_3)/3$, $n_3 = m_3/9g$, $n_2 = m_2/3g$, so that 9,

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g, n_2, and n_3 are pairwise co-prime. We have

$$\chi_m = \xi_9 \xi_g \xi_{n_3} \psi_3 \psi_g \psi_{n_2}.$$

We can relate the Gauss sums for χ_m to the character $\chi'_m = \psi_3 \chi_m$.

LEMMA 4.1. *If ξ_9 is the cubic residue symbol $\left(\frac{\zeta_3}{\cdot}\right)$, then $\tau(\xi_9\psi_3) = \tau(\xi_9)$.*

Since the character belonging to a field is determined only up to complex conjugation, we may assume the condition of the lemma without loss of generality.

LEMMA 4.2. *With χ_9 normalised, $\tau(\chi_m^6) = \tau\left({\chi'_m}^6\right) = \tau(\chi_m^3) = \tau\left({\chi'_m}^3\right) = 0$, $\tau(\chi_m^2) = \tau\left({\chi'_m}^2\right)$, $\tau\left({\chi'_m}^4\right) = \tau\left({\chi'_m}^4\right) = \overline{\tau(\chi_m^2)}$, $\tau(\chi_m) = \psi_3(m/9)\tau(\chi'_m)$ and $\tau(\chi_m^5) = -\psi_3(m/9)\tau\left({\chi'_m}^5\right)$.*

PROOF: We have $\tau(\chi_m^6) = \tau\left({\chi'_m}^6\right) = \mu(9(m/9)) = 0$. For the cube, use Lemma 1.1B. From the definition $\chi_m^2 = {\chi'_m}^2$. Since exactly one of χ_m, χ'_m is an even character, we need prove only the relation on $\tau(\chi_m)$. By Lemma 1.1,

$$\begin{aligned}\tau(\chi_m) &= \chi_9(gn_2n_3)\chi_g\psi_{n_2}\xi_{n_3}(9)\tau(\chi_9)\tau(\chi_9\psi_{n_2}\xi_{n_3})\\ &= \chi_9(gn_2n_3)\chi_g\xi_{n_3}(9)\tau(\xi_9)\xi_{gn_3}(n_2)\psi_{gn_2}(n_3)\xi_g(2)\tau(\psi_{gn_2})\tau(\xi_{gn_3})\mathfrak{g}/g\end{aligned}$$

using Lemmas 2.2, 4.1, and $\psi_g(9) = 1$. Expanding $\chi_9(gn_2n_3)$ and recombining terms,

$$\tau(\chi_m) = \xi_{m_3}(n_2)\psi_3(gn_2n_3)\psi_{gn_2}(n_3)\xi_g(2)\tau(\xi_{m_3})\tau(\psi_{gn_2})\mathfrak{g}/g$$

Since gn_2 is the conductor of the quadratic field associated to χ', comparing this expression to (3.1) gives the lemma. □

The coefficients can now be computed.

PROPOSITION 4.3. *The polynomial* $\mathrm{Irr}_{K_2}(\lambda_0)$ *is given by* $X^3 + 0X^2 + c_1X + c_0$ *where*

$$\begin{aligned}c_1 &= -3\left(\mu(n_2)^2 m_3 + \chi_m(-1)m\right)\\ &\quad - 6\sqrt{-1}\mu(n_2)\psi_{m_2}(n_3)\psi_3(m/9)n_3\,\mathrm{Im}\,(\xi_g(2)\bar{\mathfrak{g}})\sqrt{\chi'_m(-1)gn_2}\\ c_0 &= -2\mu(n_2)m_3\,\mathrm{Re}\,(\mathfrak{m}) + 6\,\mu(n_2)\chi_m(-1)m\,\mathrm{Re}\,(\xi_g(2)\bar{\mathfrak{n}}\mathfrak{g})\\ &\quad - 6\sqrt{-1}\mu(n_2)^2\psi_{m_2}(n_3)\psi_3(m/9)m_3\,\mathrm{Im}\,(\xi_g(2)\mathfrak{n})\sqrt{\chi'_m(-1)gn_2}\\ &\quad - 18\sqrt{-1}\psi_{m_2}(n_3)\psi_3(m/9)\chi_m(-1)n_2n_3\,\mathrm{Im}\,(\bar{\mathfrak{n}}\mathfrak{g}^2)\sqrt{\chi'_m(-1)gn_2}.\end{aligned}$$

The radicand $\chi'_m(-1)gn_2 = -m_2/3$ appears because $\sqrt{3}$ is hidden in the imaginary parts. We have $\sqrt{m_2^*} = -\chi_m(-1)\sqrt{-3}\sqrt{\chi'_m(-1)gn_2}$. Making the assignments (3.2) and (3.3) we obtain:

Theorem 2. *For $3 \mid \gcd(m_2, m_3)$ the reduced period polynomial $\Lambda(X) = X^6 + \sum_{\nu=0}^{4} d_\nu X^\nu$ is given by*

$$\begin{aligned}
c_1 &= -3\left[\left(\mu(n_2)^2 m_3 + \chi_m(-1)m\right)\right] \\
&\quad - \frac{3}{2}\mu(n_2)\psi_{m_2}(n_3)\psi_3(m/9)\chi_m(-1)n_3(3\,z_0 B - z_1 A)\sqrt{m_2^*} \\
c_0 &= -\mu(n_2)m_3 L + \frac{3}{4}\,\mu(n_2)\chi_m(-1)m[z_0(AR + 27\,BS) + 9\,z_1(AS - BR)] \\
&\quad + \frac{27}{2}\,\psi_{m_2}(n_3)\psi_3(m/9)\left(2\,gS + ABR - A^2 S\right)\sqrt{m_2^*} \\
&\quad + \frac{3}{2}\,\mu(n_2)^2\psi_{m_2}(n_3)\psi_3(m/9)\chi_m(-1)m_3(3\,z_0 S + z_1 R)\sqrt{m_2^*} \\
d_4 &= -6\left[\left(\mu(n_2)^2 m_3 + \chi_m(-1)m\right)\right] \\
d_3 &= -2\mu(n_2)m_3 L + \frac{3}{2}\,\mu(n_2)\chi_m(-1)m[z_0(AR + 27\,BS) + 9\,z_1(AS - BR)] \\
d_2 &= {c_1'}^2 - {c_1''}^2\chi_m(-1)m_2 \\
d_1 &= 2\,(c_0' c_1' - c_0'' c_1''\chi_m(-1)m_2) \\
d_0 &= {c_0'}^2 - {c_0''}^2\chi_m(-1)m_2
\end{aligned}$$

where $c_j = c_j' + c_j''\sqrt{m_2^*}$.

As in the general case we suppress writing the trailing terms in full.

Remark. The remarks after Theorem 1 hold. In addition, if $n_2 = 1$ and $m_2 \equiv 3 \bmod 4$, d_4 vanishes and d_2 is independent of K_3. d_2 is also independent of K_3 whenever g is prime or trivial, since we can fix $\mathfrak{g}$.

Example. In Table 2 we give the period polynomials of the six sextic fields of conductor 63. When $m_2 = 7$, Theorem 1 holds; for $m_2 = 21$, we use Theorem 2.

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TABLE 1. Sextic period polynomials of conductor 63

m_2	m_3	g	C_0	$\mathfrak{m}$	$\xi_g(2)$	$\Psi_6(X)$
7	9	1	$\{1,8,37,44,46,53\}$	$\frac{-3+3\sqrt{-3}}{2}$	1	$X^6+9X^4+5X^3+36X^2+12X+8$
7	63	$\frac{1+3\sqrt{-3}}{2}$	$\{1,8,11,23,25,58\}$	$\frac{-15-3\sqrt{-3}}{2}$	ζ_3	$X^6+14X^3+63X^2-168X+161$
7	63	$\frac{1-3\sqrt{-3}}{2}$	$\{1,2,4,8,16,32\}$	$6-3\sqrt{-3}$	$\bar{\zeta}_3$	$X^6+14X^3+63X^2+210X+224$
21	9	1	$\{1,17,26,37,46,62\}$	$\frac{-3+3\sqrt{-3}}{2}$	1	$X^6-12X^4+5X^3+36X^2-30X+1$
21	63	$\frac{1+3\sqrt{-3}}{2}$	$\{1,5,25,38,58,62\}$	$\frac{-15-3\sqrt{-3}}{2}$	ζ_3	$X^6-21X^4+14X^3+63X^2-21X-35$
21	63	$\frac{1+3\sqrt{-3}}{2}$	$\{1,4,16,47,59,62\}$	$6-3\sqrt{-3}$	ζ_3	$X^6-21X^4+14X^3+63X^2-84X+28$

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