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ON A DUAL RELATION FOR ADDITION FORMULAS OF ADDITIVE GROUPS II

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Chapter 2. Sheffer Polynomials

Introduction

This paper is a continuation of our previous memoir [28], hereafter referred to as I, and constitutes the second chapter of this series. As stated in I, our aim in this series is to examine properties of a polynomial sequence with several variables satisfying an addition formula by means of the down-ladder, and to give a generalization of so called classical polynomials. In the present article, we study the two kinds of polynomial sequences:

(i) sequences $s_\alpha(x)$ of polynomials satisfying the identities

$$s_\alpha(x+y) = \sum_{\alpha=\beta+\gamma} s_\beta(x)p_\gamma(y)\,,$$

where $p_\alpha(x)$ is a given sequence of binomial type defined in I,

(ii) doubly indexed sequences $p_\alpha^{[\lambda]}(x)$ of polynomials satisfying

$$p_\alpha^{[\lambda+\mu]}(x+y) = \sum_{\alpha=\beta+\gamma} p_\beta^{[\lambda]}(x)p_\gamma^{[\mu]}(y)\,.$$

In the case of one variable (cf. [11], [22]), the addition formula (i) or (ii) holds for many well known polynomials, for example, Hermite, Laguerre, Euler, Bernoulli, Poisson-Charlier, Krawtchouk, and Stirling polynomials etc.. In Section 8, some of these polynomials are generalized to the case of several variables.

Let us give a brief description of contents of this paper.

Section 1 deals with fundamental properties of a polynomial sequence $s_\alpha(x)$ to satisfy the addition formula (i), that is called a Sheffer set. In this section, we have a relation between the Sheffer set $s_\alpha(x)$ and the polynomial sequence $p_\alpha(x)$ of binomial type. Also, an expansion formula

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in the Sheffer set $s_\alpha(x)$ is given.

Section 2 deals with a unipotent representation of a delta set defined in I on a Sheffer set. This representation uniquely determines a Sheffer set.

Section 3 deals with a recurrence formula of the Sheffer set $s_\alpha(x)$ with respect to the parameter α. Conversely, a class of difference systems resulting from the recurrence formula characterizes the Sheffer sets.

Section 4 deals with the umbral calculus. This gives a solution of so called "problem of connection constants" in the case of several variables.

Section 5 constructs a differential system of a Sheffer set as its eigenfunctions.

Section 6 deals with some properties of a polynomial sequence $p_\alpha^{[\lambda]}(x)$ satisfying the addition formula (ii). The polynomial sequence $p_\alpha^{[\lambda]}(x)$ turns out to be a special Sheffer set.

Section 7 deals with some special generating function associated with a Sheffer set. In the case of one variable, this concludes generating functions of some of classical polynomials examined by Carlitz [10].

Section 8 deals with a generalization of Hermite, Euler, Bernoulli and Laguerre polynomials.

Let us enumerate symbols and notations in this paper. The symbol Z_+^n is the subset in the n dimensional integral lattice Z^n, in which each point has all non-negative entries. Let the Greek letters $\alpha, \beta, \cdots$ be vectors in Z^n and, for example, the components of α be written in the form

$$\alpha = (\alpha_1, \cdots, \alpha_n)\,,$$

The polynomial sequence $p_\alpha(x)$ is a set of polynomials with the variable $x = (x_1, \cdots, x_n)$ depending on the parameter α in Z_+^n. For convenience of calculation, we regard $p_\alpha(x)$ as vanishing for $\alpha \notin Z_+^n$. The symbol $\boldsymbol{P}$ is the vector space of all polynomials with n variables. The origin is denoted by the notation $\mathbf{0}$. Let $\{e_1, \cdots, e_n\}$ be a unit coordinate system, that is,

$$e_1 = (1, 0, \cdots, 0), \cdots, e_n = (0, \cdots, 0, 1).$$

The inner product $\sum_{k=1}^n x_k y_k$ is written by $\langle x, y\rangle$. We use $e(\langle x, y\rangle)$ instead of $\exp\langle x, y\rangle$. For $\alpha \in Z_+^n$ and the variable $x = (x_1, \cdots, x_n)$, the length $\alpha_1 + \cdots + \alpha_n$ of α is denoted by $|\alpha|$, and the polynomial $x^\alpha/\alpha!$ is $x_1^{\alpha_1}/\alpha_1! \cdots x_n^{\alpha_n}/\alpha_n!$. Instead of the partial differential operators $\partial/\partial x_1, \cdots, \partial/\partial x_n$,

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we use the symbol $\partial_1, \cdots, \partial_n$. Also, the vector $(\partial/\partial x_1, \cdots, \partial/\partial x_n)$ is denoted by the symbol ∂. Let $\{P_1, \cdots, P_n\}$ be translation invariant operators. Then the multiple $P_1^{\alpha_1}\cdots P_n^{\alpha_n}$ is denoted by P^α. The notation $\delta_{\alpha\beta}$ is a generalization of the Kronecker's delta symbol δ_{ij} such that

$$\delta_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

§ 1. Definitions and fundamental properties

A polynomial sequence $s_\alpha(x)$ is called a *Sheffer set* or a *set of Sheffer polynomials for the delta set* $\{P_1, \cdots, P_n\}$ if

(i) $s_0(x) = c \neq 0$,

(ii) for each $i = 1, \cdots, n$,

$$P_i s_\alpha(x) = s_{\alpha - e_i}(x)\,.$$

If the delta set $\{P_1, \cdots, P_n\}$ is normal (as for the definition, see [28] § 1.), $s_\alpha(x)$ is called a *normal Sheffer set* or a *set of normal Sheffer polynomials.*

The following lemma is fundamental.

LEMMA 2.1.1. *A set of Sheffer polynomials $s_\alpha(x)$ is a base for the vector space* **P**.

Proof. By a change of the coordinate, we have only to prove the lemma in the case of a normal Sheffer set $s_\alpha(x)$. To prove the lemma, we use the induction on the parameter α of $s_\alpha(x)$. For each unit vector e_j, set

$$s_{e_j}(x) = \sum_{|\alpha| \leq l_j} a_j(\alpha) x^\alpha/\alpha!\,, \qquad j = 1, \cdots, n\,.$$

Since the normal delta set $\{P_1, \cdots, P_n\}$ has the following expressions:

$$P_i = \partial_i + \sum_{|\alpha|>1} c_i(\alpha)\partial^\alpha\,, \qquad i = 1, \cdots, n\,,$$

the homogeneous polynomial of the highest degree of $P_i s_{e_j}(x)$ is

$$\sum_{|\alpha| = l_j} a_j(\alpha) x^{\alpha - e_i}/(\alpha - e_i)!\,.$$

On the other hand, the definition of the Sheffer set gives

$$P_i s_{e_j}(x) = \delta_{ij} s_0(x) = \delta_{ij} c\,, \qquad i, j = 1, \cdots, n\,.$$

Therefore we obtain

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$$s_{e_j}(x) = cx_j + a_j, \qquad j = 1, \cdots, n,$$

for some constants a_j. By induction, we assume that for $|\alpha| < m$, the only monomial of the highest degree in $s_\alpha(x)$ is $c\,x^\alpha/\alpha!$. Set, for $|\alpha| = m$,

$$s_\alpha(x) = \sum_{|\beta| \leq M} a_\alpha(\beta)\, x^\beta/\beta!,$$

and operate P_i on $s_\alpha(x)$. Using the same argument in the case of $|\alpha| = 1$, we, also, see that for $|\alpha| = m$, the only monomial of the highest degree in $s_\alpha(x)$ is $c\,x^\alpha/\alpha!$. Q.E.D.

Remark. The only monomial of the highest degree in a normal Sheffer polynomial $s_\alpha(x)$ is $c\,x^\alpha/\alpha!$.

A Sheffer set relative to a delta set $\{P_1, \cdots, P_n\}$ is related to a set of basic polynomials for $\{P_1, \cdots, P_n\}$ by the following.

PROPOSITION 2.1.2. *Let $\{P_1, \cdots, P_n\}$ be a delta set with basic polynomials $p_\alpha(x)$. Then $s_\alpha(x)$ is a Sheffer set relative to $\{P_1, \cdots, P_n\}$ if and only if there exists an invertible translation invariant operator S such that*

(2.1.1) $$s_\alpha(x) = S^{-1}p_\alpha(x).$$

Proof. Suppose first (2.1.1). Then by Proposition 1.1.2 [28], $S^{-1}P_j = P_jS^{-1}$ and

$$P_js_\alpha(x) = P_jS^{-1}p_\alpha(x) = S^{-1}P_jp_\alpha(x) = S^{-1}p_{\alpha-e_j}(x) = s_{\alpha-e_j}(x), \quad j = 1, \cdots, n.$$

Further, since S is invertible, $S^{-1}1 = c \neq 0$, so that

$$s_0(x) = S^{-1}p_0(x) = S^{-1}1 = c\,.$$

Thus $s_\alpha(x)$ is a Sheffer set.

Conversely, to prove (2.1.1), we may assume that $s_\alpha(x)$ is a normal Sheffer set. Define S by setting

$$S;\ s_\alpha(x) \longrightarrow p_\alpha(x),$$

and extending S to all polynomials by the linearity and Lemma 2.1.1. Since, by Remark of Lemma 2.1.1, and Proposition 1.1.3 [28], the only monomial of the highest degree in $s_\alpha(x)$ and $p_\alpha(x)$ is $c\,x^\alpha/\alpha!$ and $x^\alpha/\alpha!$, respectively, S is invertible. It remains to show that S is translation invariant. To this end, note that S commutes with each P_j, $j = 1, \cdots, n$. Indeed,

$$SP_js_\alpha(x) = Ss_{\alpha-e_j}(x) = p_{\alpha-e_j}(x) = P_jp_\alpha(x) = P_jSs_\alpha(x), \quad j = 1, \cdots, n,$$

whence $SP^\alpha = P^\alpha S$. By the First Expansion Formula in [28], we conclude that S is translation invariant. Q.E.D.

By Proposition 1.1.4 [28], we see that every delta set has a unique sequence of basic polynomials. Hence every Sheffer polynomials $s_\alpha(x)$ is uniquely defined by a delta set $\{P_1, \cdots, P_n\}$ and an invertible translation invariant operator S. So we call $s_\alpha(x)$ a *Sheffer set* or a *set of Sheffer polynomials relative to* $\{P_1, \cdots, P_n; S\}$.

Now we get an expansion formula in the Sheffer polynomials.

THEOREM 2.1.3. (Second Expansion Formula). *Let $s_\alpha(x)$ be a set of Sheffer polynomials relative to $\{P_1, \cdots, P_n; S\}$. If T is any translation invariant operator, and $f(x)$ is any polynomial, the following identity holds*:

$$Tf(x+y) = \sum_\alpha s_\alpha(y) P^\alpha STf(x),$$

for every vectors x and y.

Proof. Let $p_\alpha(x)$ be basic polynomials for the delta set $\{P_1, \cdots, P_n\}$. By the First Expansion Formula in [28],

$$f(x+y) = \sum_\alpha p_\alpha(y) P^\alpha f(x).$$

Applying S^{-1}, regarding y as the variable and x as a parameter, this becomes

$$\begin{aligned} S^{-1}f(x+y) &= \sum_\alpha S^{-1}p_\alpha(y) P^\alpha f(x) \\ &= \sum_\alpha s_\alpha(y) P^\alpha f(x). \end{aligned}$$

Now, again, regarding y as the constant and x as the variable, and applying S followed by T, we obtain the second expansion formula. Q.E.D.

In the preceding theorem, setting $y = 0$ and $T = S^{-1}$, we obtain

COROLLARY 1. *If $s_\alpha(x)$ is a set of Sheffer polynomials relative to $\{P_1, \cdots, P_n; S\}$, then*

$$S^{-1} = \sum_\alpha s_\alpha(0) P^\alpha.$$

Let T be the identity operator and $f(x)$ be the Sheffer polynomial $s_\alpha(x)$ relative to $\{P_1, \cdots, P_n; S\}$. Then, COROLLARY 2. *The Sheffer polynomials $s_\alpha(x)$ satisfies*

$$s_\alpha(x+y) = \sum_{\alpha=\beta+\gamma} s_\beta(x)p_\gamma(y) .$$

Using the symbol of translation invariant operators defined in [27], we derive a generating function for the Sheffer polynomials.

COROLLARY 3. *Let $\{p_1^{-1}(\xi), \cdots, p_n^{-1}(\xi)\}$ be the formal inverse of the symbol $\{p_1(\xi), \cdots, p_n(\xi)\}$ for the delta set $\{P_1, \cdots, P_n\}$. Then the generating function for the Sheffer set $s_\alpha(x)$ is given by*

$$\sum_\alpha s_\alpha(x)\xi^\alpha = \frac{1}{S(p^{-1}(\xi))} e(\langle x, p^{-1}(\xi)\rangle) , \tag{2.1.2}$$

where $S(\xi)$ is the symbol of the operator S.

In the case of one variable, a polynomial sequence defined by the generating function (2.1.2) has been studied from of old (cf. [1], [6], [11] vol. 3, [23], [24], [26]).

The following converse of the Second Expansion Formula is proved.

PROPOSITION 2.1.4. *Let T be an invertible translation invariant operator, let $\{P_1, \cdots, P_n\}$ be a delta set, and let $s_\alpha(x)$ be a polynomial sequence. Suppose that*

$$f(x+a) = \sum_\alpha s_\alpha(a)P^\alpha Tf(x) \tag{2.1.3}$$

for all polynomials $f(x)$ and all vectors a. Then the set $s_\alpha(x)$ is the Sheffer set relative to $\{P_1, \cdots, P_n; T\}$.

Proof. Operating with T^{-1} and then with T after permuting variables, we have, from (2.1.3),

$$f(x+a) = \sum_\alpha Ts_\alpha(a)P^\alpha f(x) .$$

Setting $f(x) = p_\alpha(x)$, where $p_\alpha(x)$ is the basic set of the delta set $\{P_1, \cdots, P_n\}$, we obtain

$$p_\alpha(x+a) = \sum_{\alpha=\beta+\gamma} Ts_\beta(a)p_\gamma(x) .$$

Putting $x = \mathbf{0}$, this yields $p_\alpha(a) = Ts_\alpha(a)$ for every vectors a. Q.E.D.

In the above proposition, setting $f(x) = s_\alpha(x)$, we obtain the following result from Corollary 2 in Theorem 2.1.3.

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COROLLARY. *A polynomial sequence $s_\alpha(x)$ is a Sheffer set associated with a basic set $p_\alpha(x)$ if and only if*

$$s_\alpha(x+y) = \sum_{\alpha=\beta+\gamma} s_\beta(x) p_\gamma(y) .$$

As referred to the introduction, in the case of one variable, this addition formula holds for many classical polynomials.

§ 2. A unipotent representation of a delta set on a Sheffer set

In this section, we shall be concerned with a unipotent representation of a delta set on a set of Sheffer polynomials. Then, this representation uniquely determines a set of Sheffer polynomials as follows:

THEOREM 2.2.1. *Let $s_\alpha(x)$ be a polynomial sequence with $s_0(x) = 1$. Then:*

(i) *If $s_\alpha(x)$ is a Sheffer set, for every translation invariant operator T, there uniquely exists a sequence of constants c_α such that*

$$Ts_\alpha(x) = \sum_{\alpha=\beta+\gamma} s_\beta(x) c_\gamma . \tag{2.2.1}$$

(ii) *Let $\{P_1, \cdots, P_n\}$ be a delta set. For $\alpha \in Z^n_+$, let $\{c_1(\alpha), \cdots, c_n(\alpha)\}$ be a set of sequences such that the determinant $|c_i(e_j)|$ does not vanish. If it holds the identities*

$$P_j s_\alpha(x) = \sum_{\alpha=\beta+\gamma} s_\beta(x) c_j(\gamma) , \qquad j = 1, \cdots, n , \tag{2.2.2}$$

then $s_\alpha(x)$ is a Sheffer set.

Proof of (i). Let $p_\alpha(x)$ be a basic set associated with the Sheffer set $s_\alpha(x)$. Corollary of Proposition 2.1.4 gives

$$s_\alpha(x+y) = \sum_{\alpha=\beta+\gamma} s_\beta(x) p_\gamma(y) .$$

Applying T, regarding y as the variable and x as a parameter, this becomes

$$Ts_\alpha(x+y) = \sum_{\alpha=\beta+\gamma} s_\beta(x) Tp_\gamma(y) .$$

Setting $y = 0$ gives

$$Ts_\alpha(x) = \sum_{\alpha=\beta+\gamma} s_\beta(x) Tp_\gamma(0) .$$

Defining

$$Tp_\gamma(0) = c_\gamma ,$$

we obtain (2.2.1). Q.E.D.

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Proof of (ii). By a change of the coordinate, we have only to prove (ii) in the case of a normal delta set $\{P_1, \cdots, P_n\}$. To prove (ii), we use the induction on the parameter α of $s_\alpha(x)$. Putting $\alpha = e_i$ in (2.2.2), we have

(2.2.3) $$P_j s_{e_i}(x) = s_0(x) c_j(e_i) + s_{e_i}(x) c_j(0), \qquad i, j = 1, \cdots, n.$$

Comparing the homogeneous polynomials of the highest degree of the both sides of (2.2.3), and noting

(2.2.4) $$P_j = \partial_j + \sum_{|\alpha|>1} a_j(\alpha) \partial^\alpha ,$$

we have from $s_0(x) = 1$,

(2.2.5) $$s_{e_i}(x) = \sum_{j=1}^{n} c_j(e_i) x_j + b_i , \qquad i = 1, \cdots, n ,$$

for some constants b_i, and

(2.2.6) $$c_j(0) = 0 .$$

For $|\alpha| < m$, we assume that the homogeneous polynomial of the highest degree of $s_\alpha(x)$ is $(Cx)^\alpha/\alpha!$, where

$$(Cx)_i = \sum_{j=1}^{n} c_j(e_i) x_j , \qquad i = 1, \cdots, n .$$

For $|\alpha| = m$, set

$$s_\alpha(x) = \sum_{|\beta| \leqq M} a_\alpha(\beta) x^\beta / \beta! .$$

By (2.2.4) and (2.2.6), the homogeneous polynomials of the highest degree of the both sides in (2.2.2) are

(2.2.7) $$\sum_{|\beta| = M} a_\alpha(\beta) x^{\beta - e} (\beta - e_j)! ,$$

and

(2.2.8) $$\sum_{k=1}^{n} c_j(e_k) (Cx)^{\alpha - e_k} / (\alpha - e_k)! ,$$

respectively. Hence, comparing (2.2.7) and (2.2.8), we have $m = M$, and

$$\sum_{|\beta| = m} a_\alpha(\beta) x^\beta / \beta! = (Cx)^\alpha / \alpha! .$$

Thus, we see that the homogeneous polynomial of the highest degree of $s_\alpha(x)$ is $(Cx)^\alpha/\alpha!$. Therefore we conclude that the polynomial sequence $s_\alpha(x)$

generates all polynomials. Define a set of linear operators Q_j by

$$Q_j s_\alpha(x) = s_{\alpha - e_j}(x), \qquad j = 1, \cdots, n.$$

To prove that $\{Q_1, \cdots, Q_n\}$ is a delta set, first we show that each Q_i is translation invariant. Note that

$$\begin{aligned} Q_i P_j s_\alpha(x) &= Q_i \sum_{\alpha=\beta+\gamma} s_\beta(x) c_j(\gamma) = \sum_{\alpha=\beta+\gamma} s_{\beta - e_i}(x) c_j(\gamma) \\ &= \sum_{\alpha - e_i = \beta+\gamma} s_\beta(x) c_j(\gamma) = P_j s_{\alpha - e_i}(x) = P_j Q_i s_\alpha(x) . \end{aligned}$$

Since $s_\alpha(x)$ generates all polynomials, this implies

$$P_j Q_i = Q_i P_j \qquad i, j = 1, \cdots, n ,$$

whence

$$P^\alpha Q_i = Q_i P^\alpha \qquad i = 1, \cdots, n .$$

Using the First Expansion Formula in [28], we see that each Q_i is translation invariant. Hence, each Q_i has a differential expression by Proposition 1.1.2 in [28];

$$Q_i = \sum_\alpha a_i(\alpha) \partial^\alpha , \qquad i = 1, \cdots, n .$$

Since it holds

$$Q_i s_{e_j}(x) = \delta_{ij} , \qquad i, j = 1, \cdots, n .$$

we have, from (2.2.5), $a_i(\mathbf{0}) = 0$ and

$$\sum_{k=1}^{n} a_i(e_k) c_k(e_j) = \delta_{ij} , \qquad i, j = 1, \cdots, n .$$

Therefore, the set of operators $\{Q_1, \cdots, Q_n\}$ is a delta set. Now, we conclude that $s_\alpha(x)$ is a Sheffer set. Q.E.D.

§ 3. Recurrence formula

In this section we treat a partial difference system of Sheffer polynomials $s_\alpha(x)$ with respect to the parameter α, that is called a recurrence formula. In the theory of orthogonal polynomials with one variable [27], the recurrence formula holds for any three consecutive orthogonal polynomials. On the other hand, in the case of Sheffer polynomials, we can take the formula for any finite consecutive polynomials. Also we see that this recurrence formula determines a unique Sheffer set. For this purpose, we need an elementary theory of a completely integrable system of first order [20].

LEMMA 2.3.1. *Let $a_{ij}(\xi)$ be a formal power series with respect to $\xi = (\xi_1, \cdots, \xi_n)$ for each $i, j = 1, \cdots, n$. Let X_i be a vector field defined by*

$$X_i = \sum_{k=1}^{n} a_{ik}(\xi)\partial_k , \qquad i = 1, \cdots, n ,$$

where ∂_k is a partial derivation with respect to ξ_k. Then the partial differential system of first order

$$X_i q_j(\xi) = \delta_{ij} , \qquad i, j = 1, \cdots, n \tag{2.3.1}$$

is completely integrable if and only if

$$\sum_{k=1}^{n} a_{ik}(\xi)\partial_k a_{jl}(\xi) - a_{jk}(\xi)\partial_k a_{il}(\xi) = 0 . \tag{2.3.2}$$

Proof. Let $\tilde{X}_i$ be a prolongation of the vector field X_i to a space of variables $(\xi_1, \cdots \xi_n, q_1, \cdots, q_n)$ such that

$$\tilde{X}_i = \sum_{k=1}^{n} a_{ik}(\xi)\partial_k + \delta_{ik}\tilde{\partial}_k ,$$

where $\tilde{\partial}_k$ is the partial derivation with respect to the variable q_k;

$$\tilde{\partial}_k = \partial/\partial q_k , \qquad k = 1, \cdots, n .$$

As well known, if the complete solutions of (2.3.1) are given by the implicit functions

$$v_j(\xi_1, \cdots, \xi_n, q_1, \cdots, q_n) = c_j , \qquad j = 1, \cdots, n$$

for some constants c_j, to have the non-vanishing Jacobian

$$|\tilde{\partial}_i v_j| \neq 0 ,$$

the system (2.3.1) is equal to

$$\tilde{X}_i(v_j) = 0 , \qquad i, j = 1, \cdots, n . \tag{2.3.3}$$

Hence, (2.3.3) is a completely integrable system if and only if

$$[\tilde{X}_i, \tilde{X}_j] \equiv 0 \qquad \mathrm{mod}\,(\tilde{X}_1, \cdots, \tilde{X}_n) . \tag{2.3.4}$$

Since

$$\begin{aligned}[\tilde{X}_i, \tilde{X}_j] &= \Big(\sum_{k=1}^{n} a_{ik}(\xi)\partial_k + \delta_{ik}\tilde{\partial}_k\Big)\Big(\sum_{l=1}^{n} a_{jl}(\xi)\partial_l + \delta_{jl}\tilde{\partial}_l\Big) \\ &\quad - \Big(\sum_{k=1}^{n} a_{jk}(\xi)\partial_k + \delta_{jk}\tilde{\partial}_k\Big)\Big(\sum_{l=1}^{n} a_{il}(\xi)\partial_l + \delta_{il}\tilde{\partial}_l\Big) \\ &= \sum_{k,l=1}^{n} [a_{ik}(\xi)(\partial_k a_{jl}(\xi)) - a_{jk}(\xi)(\partial_k a_{il}(\xi))]\partial_l \,,\end{aligned}$$

the condition (2.3.4) is equal to (2.3.2). Q.E.D.

Let $\{q_1(\xi), \cdots, q_n(\xi)\}$ be a formal inverse of the symbol $\{p_1(\xi), \cdots, p_n(\xi)\}$ of the normal delta set $\{P_1, \cdots, P_n\}$, that is,

(2.3.5) $$q_j(p(\xi)) = \xi_j \,, \qquad j = 1, \cdots, n\,.$$

Operating with ∂_i on the both sides of (2.3.5), we obtain

(2.3.6) $$\sum_{k=1}^{n} (\partial_k q_j)(p(\xi))\partial_i p_k(\xi) = \delta_{ij} \,, \qquad i, j = 1, \cdots, n\,,$$

and, so

(2.3.7) $$\sum_{k=1}^{n} (\partial_i p_k)(q(\xi))\partial_k q_j(\xi) = \delta_{ij}, \qquad i, j = 1, \cdots, n.$$

Since $\{P_1, \cdots, P_n\}$ is a normal delta set, we have the following expansion

(2.3.8) $$(\partial_i p_j)(q(\xi)) = \delta_{ij} + \sum_{|\alpha|>1} c_{ij}(\alpha)\xi^\alpha, \qquad i, j = 1, \cdots, n$$

for some constants $c_{ij}(\alpha)$. Here, we treat the system (2.3.1) in the case of the following:

(2.3.9) $$a_{ij}(\xi) = \delta_{ij} + \sum_{|\alpha|>1} c_{ij}(\alpha)\xi^\alpha, \qquad i, j = 1, \cdots, n$$

for some constants $c_{ij}(\alpha)$. Then:

LEMMA 2.3.2. *The condition* (2.3.2) *is equal to*

(2.3.10) $$\begin{aligned}&(\alpha_i + 1)c_{jl}(\alpha + e_i) + \sum_{\alpha=\beta+\gamma} \sum_{k=1}^{n} (\gamma_k + 1)c_{ik}(\beta)c_{jl}(\gamma + e_k) \\ &\quad = (\alpha_j + 1)c_{il}(\alpha + e_j) + \sum_{\alpha=\beta+\gamma} \sum_{k=1}^{n} (\beta_k + 1)c_{jk}(\gamma)c_{il}(\beta + e_k), \\ &\qquad\qquad i, j, l = 1, \cdots, n.\end{aligned}$$

The proof is a direct verification. Secondly, we treat the other differential system.

LEMMA 2.3.3. *Let* $\{p_1(\xi), \cdots, p_n(\xi)\}$ *be a symbol of the normal delta*

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set $\{P_1, \cdots, P_n\}$ *given by* (2.3.8). *Then the partial differential system*

$$(\partial_i f)(\xi) = \sum_\alpha b_i(\alpha) p^\alpha(\xi), \qquad i = 1, \cdots, n \tag{2.3.11}$$

is completely integrable if and only if the coefficients $b_i(\alpha)$ *satisfy*

$$\begin{aligned} &\sum_{\alpha=\beta+\gamma} \sum_{k=1}^{n} c_{ik}(\gamma)(\beta_k + 1) b_j(\beta + e_k) \\ &\quad = \sum_{\alpha=\beta+\gamma} \sum_{k=1}^{n} c_{jk}(\gamma)(\beta_k + 1) b_i(\beta + e_k), \qquad i, j = 1, \cdots, n, \end{aligned} \tag{2.3.12}$$

where $c_{ij}(0) = \delta_{ij}$ *and* $c_{ij}(\alpha)$ *are defined by* (2.3.8).

Proof. As well known, the system (2.3.11) is completely integrable if and only if it holds

$$\partial_j \sum_\alpha b_i(\alpha) p^\alpha(\xi) = \partial_i \sum_\alpha b_j(\alpha) p^\alpha(\xi), \qquad i, j = 1, \cdots, n. \tag{2.3.13}$$

Inserting (2.3.8) into (2.3.13), we obtain (2.3.12). Q.E.D.

Now, we shall determine the form of a recurrence formula for a normal Sheffer set $s_\alpha(x)$. Since the homogeneous polynomials of the highest degree in $s_{\alpha+e_i}(x)$ and $s_\alpha(x)$ are the monomials $c\,x^{\alpha+e_i}/(\alpha+e_i)!$ and $c\,x^\alpha/\alpha!$, respectively, by Remark of Lemma 2.1.1, the recurrence formula must be the following shape:

$$(\alpha_i + 1) s_{\alpha+e_i}(x) = x_i s_\alpha(x) - \sum_{|\beta| \leqq |\alpha|} b_i(\alpha; \beta) s_\beta(x), \qquad i = 1, \cdots, n, \tag{2.3.14}$$

for some constants $b_i(\alpha; \beta)$ depending on the parameters α and β. To determine the constants $b_i(\alpha; \beta)$ in (2.3.14), we use the generating function of $s_\alpha(x)$ in Corollary of Theorem 2.1.3,

$$\sum_\alpha s_\alpha(x) p^\alpha(\xi) = S(\xi)^{-1} e(\langle x, \xi \rangle). \tag{2.3.15}$$

Operating with the partial derivation ∂_i with respect to ξ_i on the both sides of (2.3.15), we have

$$\begin{aligned} &\sum_\alpha \sum_{k=1}^{n} \partial_i p_k(\xi)(\alpha_k + 1) s_{\alpha+e_k}(x) p^\alpha(\xi) \\ &\quad = (x_i - (\partial_i(\log S))(\xi)) S(\xi)^{-1} e(\langle x, \xi \rangle). \end{aligned} \tag{2.3.16}$$

Setting $\xi_i = q_i(\eta)$ in (2.3.5) gives

$$\begin{aligned} &\sum_\alpha \sum_{k=1}^{n} (\partial_i p_k)(q(\eta))(\alpha_k + 1) s_{\alpha+e_k}(x) \eta^\alpha \\ &\quad = (x_i - (\partial_i(\log S))(q(\eta))) S(q(\eta))^{-1} e(\langle x, q(\eta) \rangle). \end{aligned} \tag{2.3.17}$$

Using (2.3.15), we get

$$\sum_\alpha \sum_{k=1}^n (\partial_i p_k)(q(\eta))(\alpha_k + 1)s_{\alpha+e_k}(x)\eta^\alpha = (x_i - (\partial_i(\log S))(q(\eta))) \sum_\alpha s_\alpha(x)\eta^\alpha . \tag{2.3.18}$$

Taking the expansions (2.3.8) and (2.3.11) with $f(\xi) = \log S(\xi)$, and the coefficients of the both sides of (2.3.18), we obtain

$$(\alpha_i + 1)s_{\alpha+e_i}(x) + \sum_{\alpha=\beta+\gamma} \sum_{k=1}^n c_{ik}(\beta)(\gamma_k + 1)s_{\gamma+e_k}(x) = x_i s_\alpha(x) - \sum_{\alpha=\beta+\gamma} b_i(\beta)s_\gamma(x) . \tag{2.3.19}$$

Thus we arrive at the final form of the recurrence formula of Sheffer sets:

$$(\alpha_i + 1)s_{\alpha+e_i}(x) = x_i s_\alpha(x) - \sum_\beta \left(b_i(\alpha - \beta) + \sum_{k=1}^n c_{ik}(\alpha - \beta + e_k)\beta_k\right)s_\beta(x), \quad i = 1, \cdots, n , \tag{2.3.20}$$

where $b_i(\alpha)$ and $c_{ij}(\alpha)$ are defined on Z_+^n, and

$$c_{ij}(\mathbf{0}) = 0, \qquad i, j = 1, \cdots, n.$$

Conversely, we see that the difference system (2.3.20) characterizes a Sheffer set as follows:

THEOREM 2.3.4. *Let $s_\alpha(x)$ be a polynomial sequence with $s_0(x) = c \neq 0$. Suppose $s_\alpha(x)$ is a base for the vector space* $\boldsymbol{P}$.

Then, for some given constants $b_i(\alpha)$ and $c_{ij}(\alpha)$ on Z_+^n to satisfy

$$c_{ij}(\mathbf{0}) = 0 , \qquad i, j = 1, \cdots, n ,$$

$s_\alpha(x)$ satisfies the difference system (2.3.20) if and only if $s_\alpha(x)$ is a Sheffer set. Then the differential systems (2.3.7) with (2.3.8) and (2.3.11) with $f(\xi) = \log S(\xi)$ are completely integrable. Take the formal inverse $\{p_1(\xi), \cdots, p_n(\xi)\}$ for the formal power series solution $\{q_1(\xi), \cdots, q_n(\xi)\}$ of (2.3.7) with (2.3.8) such that

$$q_j(\mathbf{0}) = 0 , \qquad j = 1, \cdots, n .$$

So, the sequence $s_\alpha(x)$ is a Sheffer set relative to $\{p_1(\partial), \cdots, p_n(\partial); S(\partial)\}$.

Proof. Our previous discussion has shown the necessary condition.

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So, we have only to prove the sufficient condition. Assume that a polynomial sequence $s_\alpha(x)$ satisfies (2.3.20) for some given constants $b_i(\alpha)$ and $c_{ij}(\alpha)$ on Z_+^n to hold

$$c_{ij}(0) = 0\,, \qquad i, j = 1, \cdots, n\,.$$

We calculate the compatibility condition for (2.3.20).

Firstly, we regard $s_{\alpha+e_i+e_j}(x)$, $i \neq j$, as a string of transformations:

$$s_\alpha(x) \longrightarrow s_{\alpha+e_i}(x) \longrightarrow s_{\alpha+e_i+e_j}(x)\,.$$

Then it holds

$$\begin{aligned}
s_{\alpha+e_i+e_j}(x) &= (\alpha_j+1)^{-1}\Big\{x_j s_{\alpha+e_i}(x) - \sum_{\alpha+e_i=\beta+\gamma} b_j(\beta)s_\gamma(x) \\
&\qquad\qquad - \sum_{k=1}^{n}\sum_{\alpha+e_i=\beta+\gamma} c_{jk}(\beta)(\gamma_k+1)s_{\gamma+e_k}(x)\Big\} \\
&= (\alpha_j+1)^{-1}(\alpha_i+1)^{-1}x_j\Big\{x_i s_\alpha(x) - \sum_{\alpha=\beta+\gamma} b_i(\beta)s_\gamma(x) \\
&\quad - \sum_{k=1}^{n}\sum_{\alpha=\beta+\gamma} c_{ik}(\beta)(\gamma_k+1)s_{\gamma+e_k}(x)\Big\} - (\alpha_j+1)^{-1}\Big\{\sum_{\alpha+e_i=\beta+\gamma} b_j(\beta)s_\gamma(x)\Big\} \\
&\quad - (\alpha_j+1)^{-1}\Big\{\sum_{k=1}^{n}\sum_{\alpha+e_i=\beta+\gamma} c_{jk}(\beta)(\gamma_k+1)s_{\gamma+e_k}(x)\Big\} \\
&= (\alpha_j+1)^{-1}(\alpha_i+1)^{-1}x_i x_j s_\alpha(x) - (\alpha_i+1)^{-1}(\alpha_j+1)^{-1} \\
&\quad \times \Big\{\sum_{\alpha=\beta+\gamma} b_i(\beta)\Big\{(\gamma_j+1)s_{\gamma+e_j}(x) + \sum_{\gamma=\mu+\nu} b_j(\mu)s_\nu(x) \\
&\quad + \sum_{k=1}^{n}\sum_{\gamma=\mu+\nu} c_{jk}(\mu)(\nu_k+1)s_{\nu+e_k}(x)\Big\}\Big\} - (\alpha_j+1)^{-1}(\alpha_i+1)^{-1} \\
&\quad \times \Big\{\sum_{k=1}^{n}\sum_{\alpha=\beta+\gamma} c_{ik}(\beta)(\gamma_k+1)\Big\{(\gamma_j+1+\delta_{jk})s_{\gamma+e_j+e_k}(x) \\
&\quad + \sum_{\gamma+e_k=\mu+\nu} b_j(\mu)s_\nu(x) + \sum_{l=1}^{n}\sum_{\gamma+e_k=\mu+\nu} c_{jl}(\mu)(\nu_l+1)s_{\nu+e_l}(x)\Big\}\Big\} \\
&\quad - (\alpha_j+1)^{-1}\Big\{\sum_{\alpha+e_i=\beta+\gamma} b_j(\beta)s_\gamma(x)\Big\} \\
&\quad - (\alpha_j+1)^{-1}\Big\{\sum_{k=1}^{n}\sum_{\alpha+e_i=\beta+\gamma} c_{jk}(\beta)(\gamma_k+1)s_{\gamma+e_k}(x)\Big\}.
\end{aligned}$$

Arranging the above identity, we have

$$\begin{aligned}
s_{\alpha+e_i+e_j}(x) &= (\alpha_j+1)^{-1}(\alpha_i+1)^{-1}x_j x_i s_\alpha(x) \\
&\quad - (\alpha_j+1)^{-1}(\alpha_i+1)^{-1}\Big\{\sum_{\alpha+e_j=\beta+\gamma} \gamma_j b_i(\beta)s_\gamma(x)\Big\}
\end{aligned}$$

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$$
\begin{aligned}
&- (\alpha_j + 1)^{-1}(\alpha_i + 1)^{-1}\Big\{\sum_{\alpha=\beta+\mu+\nu} b_i(\beta)b_j(\mu)s_\nu(x)\Big\} \\
&- (\alpha_j + 1)^{-1}(\alpha_i + 1)^{-1}\Big\{\sum_{k=1}^{n}\sum_{\alpha+e_k=\beta+\mu+\nu} \nu_k b_i(\beta)c_{jk}(\mu)s_\nu(x)\Big\} \\
&- (\alpha_j + 1)^{-1}(\alpha_i + 1)^{-1}\Big\{\sum_{k=1}^{n}\sum_{\alpha+e_j+e_k=\beta+\gamma} \gamma_j(\gamma_k - \delta_{jk})c_{ik}(\beta)s_\gamma(x)\Big\} \\
&- (\alpha_j + 1)^{-1}(\alpha_i + 1)^{-1}\Big\{\sum_{k=1}^{n}\sum_{\alpha+e_k=\beta+\mu+\nu} (\mu_k + \nu_k)b_j(\mu)c_{ik}(\beta)s_\nu(x)\Big\} \\
&- (\alpha_j + 1)^{-1}(\alpha_i + 1)^{-1}\Big\{\sum_{k,l=1}^{n}\sum_{\alpha+e_k+e_l=\beta+\mu+\nu} \nu_l(\mu_k + \nu_k - \delta_{kl}) \\
&\times c_{ik}(\beta)c_{jl}(\mu)s_\nu(x)\Big\} - (\alpha_j + 1)^{-1}\Big\{\sum_{\alpha+e_i=\beta+\gamma} b_j(\beta)s_\gamma(x)\Big\} \\
&- (\alpha_j + 1)^{-1}\Big\{\sum_{k=1}^{n}\sum_{\alpha+e_i+e_k=\beta+\gamma} \gamma_k c_{jk}(\beta)s_\gamma(x)\Big\}\,.
\end{aligned} \tag{2.3.21}
$$

On the other hand, $s_{\alpha+e_i+e_j}(x)$ may be regarded as the string of the transformations:

$$s_\alpha(x) \longrightarrow s_{\alpha+e_j}(x) \longrightarrow s_{\alpha+e_i+e_j}(x).$$

Then, by permuting i and j in (2.3.21), we obtain the identity:

$$
\begin{aligned}
s_{\alpha+e_i+e_j}(x) &= (\alpha_i + 1)^{-1}(\alpha_j + 1)^{-1}x_i x_j s_\alpha(x) \\
&- (\alpha_i + 1)^{-1}(\alpha_j + 1)^{-1}\Big\{\sum_{\alpha+e_i=\beta+\gamma} \gamma_i b_j(\beta)s_\gamma(x)\Big\} \\
&- (\alpha_i + 1)^{-1}(\alpha_j + 1)^{-1}\Big\{\sum_{\alpha=\beta+\mu+\nu} b_i(\beta)b_j(\mu)s_\nu(x)\Big\} \\
&- (\alpha_i + 1)^{-1}(\alpha_j + 1)^{-1}\Big\{\sum_{k=1}^{n}\sum_{\alpha+e_k=\beta+\mu+\nu} \nu_k b_j(\beta)c_{ik}(\mu)s_\nu(x)\Big\} \\
&- (\alpha_i + 1)^{-1}(\alpha_j + 1)^{-1}\Big\{\sum_{k=1}^{n}\sum_{\alpha+e_i+e_k=\beta+\gamma} \gamma_i(\gamma_k - \delta_{ik})c_{jk}(\beta)s_\gamma(x)\Big\} \\
&- (\alpha_i + 1)^{-1}(\alpha_j + 1)^{-1}\Big\{\sum_{k=1}^{n}\sum_{\alpha+e_k=\beta+\mu+\nu} (\mu_k + \nu_k)b_i(\mu)c_{jk}(\beta)s_\nu(x)\Big\} \\
&- (\alpha_i + 1)^{-1}(\alpha_j + 1)^{-1}\Big\{\sum_{k,l=1}^{n}\sum_{\alpha+e_k+e_l=\beta+\mu+\nu} \nu_l(\mu_k + \nu_k - \delta_{kl}) \\
&\times c_{il}(\mu)c_{jk}(\beta)s_\nu(x)\Big\} - (\alpha_i + 1)^{-1}\Big\{\sum_{\alpha+e_j=\beta+\gamma} b_i(\beta)s_\gamma(x)\Big\} \\
&- (\alpha_i + 1)^{-1}\Big\{\sum_{k=1}^{n}\sum_{\alpha+e_j+e_k=\beta+\gamma} \gamma_k c_{ik}(\beta)s_\gamma(x)\Big\}\,.
\end{aligned} \tag{2.3.22}
$$

Eliminating the symmetric terms in (2.3.21) and (2.3.22) with respect to the permutation (i, j), and multiplying $(\alpha_i + 1)(\alpha_j + 1)$ to (2.3.21) and (2.3.22), we obtain the identity

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$$\sum_{\alpha+e_j=\beta+\gamma}\gamma_j b_i(\beta)s_\gamma(x)+\sum_{k=1}^{n}\sum_{\alpha+e_k=\beta+\mu+\nu}\nu_k b_i(\beta)c_{jk}(\mu)s_\nu(x)$$
$$+\sum_{k=1}^{n}\sum_{\alpha+e_j+e_k=\beta+\gamma}\gamma_j(\gamma_k-\delta_{jk})c_{ik}(\beta)s_\gamma(x)$$
$$+\sum_{k=1}^{n}\sum_{\alpha+e_k=\beta+\mu+\nu}(\mu_k+\nu_k)b_j(\mu)c_{ik}(\beta)s_\nu(x)$$
$$+\sum_{k,l=1}^{n}\sum_{\alpha+e_k+e_l=\beta+\mu+\nu}\mu_k\nu_l c_{ik}(\mu)c_{jl}(\mu)s_\nu(x)$$
$$(2.3.23)\qquad +(\alpha_i+1)\Big\{\sum_{\alpha+e_i=\beta+\gamma}b_j(\beta)s_\gamma(x)\Big\}+(\alpha_i+1)\Big\{\sum_{k=1}^{n}\sum_{\alpha+e_i+e_k=\beta+\gamma}\gamma_k c_{jk}(\beta)s_\gamma(x)\Big\}$$
$$=\sum_{\alpha+e_i=\beta+\gamma}\gamma_i b_j(\beta)s_\gamma(x)+\sum_{k=1}^{n}\sum_{\alpha+e_k=\beta+\mu+\nu}\nu_k b_j(\beta)c_{ik}(\mu)s_\nu(x)$$
$$+\sum_{k=1}^{n}\sum_{\alpha+e_i+e_k=\beta+\gamma}\gamma_i(\gamma_k-\delta_{ik})c_{jk}(\beta)s_\gamma(x)$$
$$+\sum_{k=1}^{n}\sum_{\alpha+e_k=\beta+\mu+\nu}(\mu_k+\nu_k)b_i(\mu)c_{jk}(\beta)s_\nu(x)$$
$$+\sum_{k,l=1}^{n}\sum_{\alpha+e_k+e_l=\beta+\mu+\nu}\mu_k\nu_l c_{il}(\mu)c_{jk}(\beta)s_\nu(x)+(\alpha_j+1)\Big\{\sum_{\alpha+e_j=\beta+\gamma}b_i(\beta)s_\gamma(x)\Big\}$$
$$+(\alpha_j+1)\Big\{\sum_{k=1}^{n}\sum_{\alpha+e_j+e_k=\beta+\gamma}\gamma_k c_{ik}(\beta)s_\gamma(x)\Big\}.$$

Note that $s_\alpha(x)$ is a base for the vector space $\boldsymbol{P}$. Hence the coefficients of $s_{\alpha-\beta}(x)$ in (2.3.23) give

$$(\alpha_j-\beta_j)b_i(\beta+e_j)+\sum_{k=1}^{n}\sum_{\beta+e_k=\mu+\nu}(\alpha_k-\beta_k)b_i(\mu)c_{jk}(\nu)$$
$$+\sum_{k=1}^{n}(\alpha_j-\beta_j)(\alpha_k-\beta_k-\delta_{jk})c_{ik}(\beta+e_j+e_k)$$
$$+\sum_{k=1}^{n}\sum_{\beta+e_k=\mu+\nu}(\alpha_k+1-\nu_k)b_j(\mu)c_{ik}(\nu)$$
$$+\sum_{k,l=1}^{n}\sum_{\beta+e_k+e_l=\mu+\nu}\nu_k(\alpha_l-\beta_l)c_{ik}(\mu)c_{jl}(\nu)$$
$$(2.3.24)\qquad +(\alpha_i+1)b_j(\beta+e_i)+(\alpha_i+1)\Big\{\sum_{k=1}^{n}(\alpha_k-\beta_k)c_{jk}(\beta+e_i+e_k)\Big\}$$
$$=(\alpha_i-\beta_i)b_j(\beta+e_i)+\sum_{k=1}^{n}\sum_{\beta+e_k=\mu+\nu}(\alpha_k-\beta_k)b_j(\mu)c_{ik}(\nu)$$
$$+\sum_{k=1}^{n}(\alpha_i-\beta_i)(\alpha_k-\beta_k-\delta_{ik})c_{jk}(\beta+e_i+e_k)$$
$$+\sum_{k=1}^{n}\sum_{\beta+e_k=\mu+\nu}(\alpha_k+1-\nu_k)b_i(\mu)c_{jk}(\nu)$$
$$+\sum_{k,l=1}^{n}\sum_{\beta+e_k+e_l=\mu+\nu}\mu_k(\alpha_l-\beta_l)c_{il}(\mu)c_{jk}(\nu)$$

$$+ (\alpha_j + 1)b_i(\beta + e_j) + (\alpha_j + 1)\left\{\sum_{k=1}^{n} (\alpha_k - \beta_k)c_{ik}(\beta + e_j + e_k)\right\}.$$

Arranging (2.3.24), we obtain

$$\begin{aligned}(\beta_j + 1)b_i(\beta + e_j) &+ \sum_{k=1}^{n} \sum_{\beta+e_k=\mu+\nu} \mu_k b_i(\mu)c_{jk}(\nu) \\ &+ \sum_{k=1}^{n} (\beta_j + 1 + \delta_{jk})(\alpha_k - \beta_k)c_{ik}(\beta + e_j + e_k) \\ &- \sum_{k,l=1}^{n} \sum_{\beta+e_k+e_l=\mu+\nu} \nu_k(\alpha_l - \beta_l)c_{ik}(\mu)c_{jl}(\nu) \\ = (\beta_i + 1)b_j(\beta + e_i) &+ \sum_{k=1}^{n} \sum_{\beta+e_k=\mu+\nu} \mu_k b_j(\mu)c_{ik}(\nu) \\ &+ \sum_{k=1}^{n} (\alpha_k - \beta_k)(\beta_i + 1 + \delta_{ik})c_{jk}(\beta + e_i + e_k) \\ &- \sum_{k,l=1}^{n} \sum_{\beta+e_k+e_l=\mu+\nu} \nu_k(\alpha_l - \beta_l)c_{jk}(\mu)c_{il}(\nu) .\end{aligned} \tag{2.3.25}$$

Regarding α_l as variables and replacing $\beta + e_l$ with β, we get the following identity for the coefficient of α_l:

$$\begin{aligned}(\beta_j + 1)c_{il}(\beta + e_j) &- \sum_{k=1}^{n} \sum_{\beta+e_k=\mu+\nu} \nu_k c_{ik}(\mu)c_{jl}(\nu) \\ = (\beta_i + 1)c_{jl}(\beta + e_i) &- \sum_{k=1}^{n} \sum_{\beta+e_k=\mu+\nu} \nu_k c_{jk}(\mu)c_{il}(\nu) .\end{aligned} \tag{2.3.26}$$

This is exactly the completely integrable condition (2.3.10). Comparing the coefficient of the constant term, we have

$$\begin{aligned}(\beta_j + 1)b_i(\beta + e_j) &+ \sum_{k=1}^{n} \sum_{\beta+e_k=\mu+\nu} \mu_k b_i(\mu)c_{jk}(\nu) \\ &- \sum_{k=1}^{n} \beta_k(\beta_j + 1 + \delta_{jk})c_{ik}(\beta + e_j + e_k) \\ &+ \sum_{k,l=1}^{n} \sum_{\beta+e_k+e_l=\mu+\nu} \beta_l\nu_k c_{ik}(\mu)c_{jl}(\nu) \\ = (\beta_i + 1)b_j(\beta + e_i) &+ \sum_{k=1}^{n} \sum_{\beta+e_k=\mu+\nu} \mu_k b_j(\mu)c_{ik}(\nu) \\ &- \sum_{k=1}^{n} \beta_k(\beta_i + 1 + \delta_{ik})c_{jk}(\beta + e_i + e_k) \\ &+ \sum_{k,l=1}^{n} \sum_{\beta+e_k+e_l=\mu+\nu} \beta_l\nu_k c_{jk}(\mu)c_{il}(\nu) .\end{aligned} \tag{2.3.27}$$

Inserting (2.3.26) into (2.3.27), we obtain

$$(2.3.28)\qquad \begin{aligned}(\beta_j+1)b_i(\beta+e_j)&+\sum_{k=1}^{n}\sum_{\beta+e_k=\mu+\nu}\mu_k b_i(\mu)c_{jk}(\nu)\\ &=(\beta_i+1)b_j(\beta+e_i)+\sum_{k=1}^{n}\sum_{\beta+e_k=\mu+\nu}\mu_k b_j(\mu)c_{ik}(\nu)\,.\end{aligned}$$

This is, also, the completely integrable condition (2.3.12) with $f(\xi)=\log S(\xi)$. Therefore we get a unique set of formal solutions $\{q_1(\xi),\cdots,q_n(\xi)\}$ and $S(\xi)$ of (2.3.1) with initial conditions $q_j(\mathbf{0})=0$, $j=1,\cdots,n$, and (2.3.11) for $f(\xi)=\log S(\xi)$ with the initial condition $S(\mathbf{0})=c\neq 0$, respectively. We easily see that $\{q_1(\partial),\cdots,q_n(\partial)\}$ is a delta set. Let us give the formal inverse $\{p_1(\xi),\cdots,p_n(\xi)\}$ of $\{q_1(\xi),\cdots,q_n(\xi)\}$. Let $\tilde{s}_\alpha(x)$ be a Sheffer set relative to $\{p_1(\partial),\cdots,p_n(\partial);S(\partial)\}$. Since the partial difference system (2.3.20) with the initial conditions $s_0(x)=c$ has a unique polynomial solution, we obtain $\tilde{s}_\alpha(x)=s_\alpha(x)$. Q.E.D.

Remark. (i) The recurrence formula for a Sheffer set is obtained from the generating function (2.1.2).

(ii) If the coefficients $b_i(\alpha)$ and $c_{ij}(\alpha)$ in (2.3.8) and (2.3.11) with $f(\xi)=\log S(\xi)$, respectively, vanish without only finite terms, the consecutive Sheffer polynomials in the recurrence formula are finite.

§ 4. Umbral calculus

In its most primitive form, umbral notation or symbolic notation called in the past century is an algorithmic device for treating a sequence $a_1, a_2, a_3,\cdots$ as a sequence of powers $a, a^2, a^3,\cdots$. Computationally, this technique turned out to be very effective in the hands of Bilssard [5], Bell [4], and some invariant theorists [14] etc. In [21] and [22], G.C. Rota etc. built up a unified theory of this technique by considering the sequence a_n as defined by a linear functional on the space of polynomials: $a_n=L(x^n)$. In this section, we see that it is easy to generalize the unified theory in the case of several variables.

If a polynomial sequence $a_\alpha(x)$ is a base for the vector space $\boldsymbol{P}$, there exists a unique linear operator L on $\boldsymbol{P}$ such that $L(x^\alpha/\alpha!)=a_\alpha(x)$. We call L the *umbral representation* of $a_\alpha(x)$. We develop the umbral device in a form leading to a useful identities.

An *umbral operator* is an operator T which maps some basic sequence $p_\alpha(x)$ into another basic sequence $q_\alpha(x)$, that is, $Tp_\alpha(x)=q_\alpha(x)$. To motivate this definition, we require another definition, the *umbral composition* of

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two polynomial sequences:

$$a_\alpha(x) = \sum_\beta a(\alpha;\beta)x^\beta/\beta!\,,$$

and $b_\alpha(x)$. This is the sequence of polynomials $c_\alpha(x)$ defined by

$$c_\alpha(x) = \sum_\beta a(\alpha;\beta)b_\beta(x)\,.$$

We use for this umbral composition the notation

$$c_\alpha(x) = a_\alpha(\boldsymbol{b}(x))\,.$$

There is a simple connection between umbral operators and the umbral composition of basic sets. For, if T maps $x^\alpha/\alpha!$ to $b_\alpha(x)$, then

$$a_\alpha(\boldsymbol{b}(x)) = Ta_\alpha(x)\,.$$

We give some fundamental properties of the umbral operator.

PROPOSITION 2.4.1. *Let T be an umbral operator. Then, T^{-1} exists and*

(i) *the map $S\to TST^{-1}$ is an automorphism on the algebra Σ of translation invariant operators;*

(ii) *T maps every sequence of basic polynomials into a sequence of basic polynomials;*

(iii) *if $\{R_1, \cdots, R_n\}$ is a delta set, then $\{TR_1T^{-1}, \cdots, TR_nT^{-1}\}$ is a delta set;*

(iv) *T maps every Sheffer set into a Sheffer set;*

(v) *if $S = s(Q)$ where $Q = (Q_1, \cdots, Q_n)$ and $s(\xi)$ is a formal power series with n variables, then $TST^{-1} = s(TQT^{-1})$, where $TQT^{-1} = \{TQ_1T^{-1}, \cdots, TQ_nT^{-1}\}$ is as in* (iii).

Proof. Let $Tp_\alpha(x) = q_\alpha(x)$ for two basic polynomials $p_\alpha(x)$ and $q_\alpha(x)$. Since a basic set is a base for the vector space $\boldsymbol{P}$ (cf. Proposition 1.1.1 [28]), it is clear that T is invertible. To prove (i), let $\{P_1, \cdots, P_n\}$ and $\{Q_1, \cdots, Q_n\}$ be delta sets for $p_\alpha(x)$ and $q_\alpha(x)$, respectively. For each i, we have

$$TP_ip_\alpha(x) = Tp_{\alpha-e_i}(x) = q_{\alpha-e_i}(x) = Q_iTp_\alpha(x), \qquad i = 1, \cdots, n. \tag{2.4.1}$$

Hence $TP_i = Q_iT$, whence $TP^\alpha = Q^\alpha T$. Let S be any translation invariant operator and let the expansion of S in terms of $\{P_1, \cdots, P_n\}$ be

$$S = \sum_\alpha a_\alpha P^\alpha\,.$$

Then

$$TST^{-1} = \sum_{\alpha} a_\alpha TP^\alpha T^{-1} = \sum_{\alpha} a_\alpha Q^\alpha . \tag{2.4.2}$$

Thus TST^{-1} is a translation invariant operator. Also it is clear that $S \to TST^{-1}$ is onto. Therefore the map is an automorphism, as claimed.

Let $\{R_1, \cdots, R_n\}$ be a delta set. Set the expansion of R_i in terms of $\{P_1, \cdots, P_n\}$ such that

$$R_i = \sum_{\alpha} a_i(\alpha) P^\alpha, \qquad i = 1, \cdots, n .$$

Since $\{R_1, \cdots, R_n\}$ is a delta set,

$$a_i(0) = 0, \qquad i = 1, \cdots, n$$

and the determinant $|a_i(e_j)|$ does not vanish. From the identities similar to (2.4.2), we easily see that $\{TR_1T^{-1}, \cdots, TR_nT^{-1}\}$ is a delta set. We conclude (iii).

To prove (ii), let $r_\alpha(x)$ be a basic sequence with the delta set $\{R_1, \cdots, R_n\}$, Let $s_\alpha(x) = Tr_\alpha(x)$ and let $S_i = TR_iT^{-1}$ for each i. By (iii), $\{S_1, \cdots, S_n\}$ is a delta set. Now

$$S_i s_\alpha(x) = TR_iT^{-1}s_\alpha(x) = TR_i r_\alpha(x) = Tr_{\alpha - e_i}(x) = s_{\alpha - e_i}(x), \qquad i = 1, \cdots, n. \tag{2.4.3}$$

We need only to prove that for $|\alpha| \neq 0$, $s_\alpha(0) = 0$. Now we can write

$$r_\alpha(x) = \sum_{\beta} a(\alpha; \beta) p_\beta(x) .$$

Since $r_\alpha(0) = 0$, we have $a(\alpha; 0) = 0$. Hence

$$Tr_\alpha(x) = \sum_{\beta} a(\alpha; \beta) q_\beta(x) = s_\alpha(x ,$$

so that $s_\alpha(0) = 0$ for $|\alpha| \neq 0$.

To prove (iv) and (v), it is trivial. Q.E.D.

The next result determines the operator corresponding to an umbral composition.

THEOREM 2.4.2. *Let $s_\alpha(x)$ and $t_\alpha(x)$ be Sheffer sets relative to $\{P_1, \cdots, P_n; S\}$ and $\{Q_1, \cdots, Q_n; T\}$, respectively. Let $p_\alpha(x)$ and $q_\alpha(x)$ be basic sets for $\{P_1, \cdots, P_n\}$ and $\{Q_1, \cdots, Q_n\}$ and let the differential expressions of S, T, P_i and Q_j be*

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$$S = s(\partial),\ T = t(\partial),\ P_i = p_i(\partial) \quad \textit{and} \quad Q_j = q_j(\partial), \qquad i, j = 1, \cdots, n,$$

respectively. Define $r_\alpha(x)$ *to be the umbral composition of* $s_\alpha(x)$ *and* $t_\alpha(x)$, *in symbols*

$$r_\alpha(x) = s_\alpha(\boldsymbol{t}(x)).$$

Then $r_\alpha(x)$ *is a Sheffer set relative to*

$$\{p_1(q(\partial)), \cdots, p_n(q(\partial));\ t(\partial)s(q(\partial))\}.$$

Proof. First, we shall prove the result in the special case where S and T are the identity operators. We find a delta set for the sequence $u_\alpha(x) = p_\alpha(\boldsymbol{q}(x))$ to be a basic sequence. Let V: $x^\alpha/\alpha! \to q_\alpha(x)$ be an umbral operator. Then $u_\alpha(x) = Vp_\alpha(x)$, and by (iii) and (v) of Proposition 2.4.1, the delta set $\{VP_1V^{-1}, \cdots, VP_nV^{-1}\}$ of $u_\alpha(x)$ is the form $\{p_1(q(\partial)), \cdots, p_n(q(\partial))\}$ as desired. Next, suppose that T is the identity operator, but not S. We study the sequence $s_\alpha(\boldsymbol{q}(x))$. But

$$s_\alpha(\boldsymbol{q}(x)) = Vs_\alpha(x) = VS^{-1}p_\alpha(x),$$

and from $Vp_\alpha(x) = p_\alpha(\boldsymbol{q}(x))$, we obtain

$$s_\alpha(\boldsymbol{q}(x)) = VS^{-1}V^{-1}u_\alpha(x).$$

From (v) of Proposition 2.4.1, this proves that $s_\alpha(\boldsymbol{q}(x))$ is a Sheffer set relative to $\{p_1(q(\partial)), \cdots, p_n(q(\partial));\ s(q(\partial))\}$.

Now to the general case, we have

$$t_\alpha(x) = T^{-1}q_\alpha(x), \quad \text{and} \quad r_\alpha(x) = T^{-1}s_\alpha(\boldsymbol{q}(x));$$

thus, we are reduced to the previous case. Q.E.D.

Using the umbral notation, we give a representation of the general linear group $GL(n)$ on a Sheffer set.

COROLLARY 1. *Let* g *be in* $GL(n)$. *In the previous theorem, setting* $s_\alpha(x) = (gx)^\alpha/\alpha!$, *the sequence* $s_\alpha(\boldsymbol{t}(x))$ *is a Sheffer set relative to* $\{({}^tg^{-1}\hat{p}(\partial))_1, \cdots, ({}^tg^{-1}\hat{p}(\partial))_n;\ t(\partial)\}$, *where* ${}^tg^{-1}$ *is a transposed inverse of* g *and* $({}^tg^{-1}\hat{p}(\partial))_j$ *is the j-th component of the vector* ${}^tg^{-1}\hat{p}(\partial)$; *that is*

$$({}^tg^{-1}\hat{p}(\partial))_j = \sum_{k=1}^{n} ({}^tg^{-1})_{jk}p_k(\partial).$$

In this case, we denote $s_\alpha(\boldsymbol{t}(x))$ by $(g\boldsymbol{t})_\alpha(x)$.

The following result is useful.

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COROLLARY 2. *Let $s_\alpha(x)$ and $t_\alpha(x)$ be Sheffer sets as in Theorem* 2.4.2 *and let V be an umbral operator such that $Vs_\alpha(x) = t_\alpha(x)$. Then $r_\alpha(x) = V^{-1}s_\alpha(x)$ is uniquely defined by*

$$r_\alpha(x) = \sum_\beta s_\beta(x)[Q^\beta T s_\alpha(x)]_{x=0} . \tag{2.4.4}$$

Proof. By the Second Expansion Theorem, we have

$$s_\alpha(x) = \sum_\beta t_\beta(x)[Q^\beta T s_\alpha(x)]_{x=0} .$$

Applying the operator V^{-1} to the both sides, we obtain (2.4.4). Q.E.D.

Remark. In Corollary 2, setting $s_\alpha(x) = x^\alpha/\alpha!$, we obtain a unique Sheffer set $r_\alpha(x)$, called the inverse set, such that $r_\alpha(t(x)) = x^\alpha/\alpha!$.

COROLLARY 3. *Let $f(x)$ be any polynomial. Then in the notation of Corollary* 2, *we have*

$$f(r(x)) = \sum_\alpha x^\alpha/\alpha!\,[Q^\alpha T f(x)]_{x=0} .$$

The following result gives the solution of the so-called "problem of connection constants" in the case of several variables.

COROLLARY 4. *Given Sheffer sets $s_\alpha(x)$ and $t_\alpha(x)$ as in Theorem* 2.4.2, *the constants $a(\alpha;\beta)$ such that*

$$\sum_\beta a(\alpha;\beta)t_\beta(x) = s_\alpha(x)$$

are uniquely determined as follows. The polynomial sequence

$$u_\alpha(x) = \sum_\beta a(\alpha;\beta)x^\beta/\beta!$$

is the Sheffer set relative to $\{p_1(q^{-1}(\partial)), \cdots, p_n(q^{-1}(\partial)); s(q^{-1}(\partial))/t(q^{-1}(\partial))\}$, where $q^{-1}(\xi) = (q_1^{-1}(\xi), \cdots, q_n^{-n}(\xi))$ is a formal inverse of $(q_1(\xi), \cdots, q_n(\xi))$.

The following proposition gives a partial derivation with respect to each x_i of an umbral operator.

PROPOSITION 2.4.3. *Let W; $p_\alpha(x) \to x^\alpha/\alpha!$ be an umbral operator, and let $\{P_1, \cdots, P_n\}$ be the delta set of $p_\alpha(x)$. Then*

$$Wx_i f(x) = \sum_{k=1}^n x_k W(\partial_i p_k)(\partial) f(x) ,$$

for all polynomials $f(x)$. Here $p_i(\partial)$ is a differential expression of the operator P_i. Setting $Wx_i - x_i W = W^{(i)}$, we see

$$W^{(i)} = \sum_{k=1}^{n} x_k W((\partial_i p_k)(\partial) - \delta_{ik}).$$

Proof. Let the matrix $(b_{ij}(\xi))$ be the inverse matrix of $(\partial_j p_i(\xi))$. By Corollary 1.3.3 in [28], we have for each i

(2.4.5) $$p_{\alpha+e_i}(x) = (\alpha_i + 1)^{-1}\Big(\sum_{k=1}^{n} x_k b_{ki}(\partial)\Big)p_\alpha(x)\,.$$

Applying the umbral operator W on the both sides, it holds

$$(\alpha_i + 1)^{-1} W \sum_{k=1}^{n} x_k b_{ki}(\partial) p_\alpha(x) = W p_{\alpha+e_i}(x)\,.$$

Now, $Wp_{\alpha+e_i}(x) = x^{\alpha+e_i}/(\alpha+e_i)! = x_i(\alpha_i+1)^{-1}x^\alpha/\alpha! = x_i(\alpha_i+1)^{-1}Wp_\alpha(x)$, so that

(2.4.5) $$W \sum_{k=1}^{n} x_k b_{ki}(\partial) = x_i W$$

on the vector space $\boldsymbol{P}$. Applying the matrix operator $((\partial_j p_i)(\partial))$ to (2.4.5) from the right side, we obtain the result. Q.E.D.

§ 5. The Sheffer set as a system of eigenfunctions

In this section, we obtain Sheffer polynomials as eigenfunctions of some differential operator system. The key step consists in singling out a "natural" inner product associated with the Sheffer set over the real field. To this end, let $s_\alpha(x)$ be a Sheffer set relative to $\{P_1, \cdots, P_n; S\}$. Let W be an umbral operator mapping $s_\alpha(x)$ to $x^\alpha/\alpha!$. For any polynomials $f(x)$ and $g(x)$, we define a bilinear form associated with $s_\alpha(x)$;

(2.5.1) $$(f(x), g(x)) = [(Wf)(P)Sg(x)]_{x=0}\,, \qquad P = (P_1, \cdots, P_n)\,.$$

Then:

PROPOSITION 2.5.1. *The bilinear form* (2.5.1) *on the vector space* $\boldsymbol{P}$ *over the real field is a positive definite inner product.*

Proof. It suffices to show that $(s_\alpha(x), s_\beta(x)) = (s_\beta(x), s_\alpha(x)) = 0$ for $\alpha \neq \beta$, and $(s_\alpha(x), s_\alpha(x)) > 0$ for all α. Now

$$\begin{aligned}(s_\alpha(x), s_\beta(x)) &= (\alpha!)^{-1}[P^\alpha S s_\beta(x)]_{x=0} \\ &= (\alpha!)^{-1}[P^\alpha p_\beta(x)]_{x=0} = (\alpha!)^{-1}p_{\beta-\alpha}(\boldsymbol{0}) = (\alpha!)^{-1}\delta_{\alpha\beta}\,,\end{aligned}$$

where $p_\alpha(x)$ are basic polynomials for the delta set $\{P_1, \cdots, P_n\}$.

Q.E.D.

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We shall call (2.5.1) the *natural inner product associated with the Sheffer set* $s_\alpha(x)$. Now we construct an operator system to take $s_\alpha(x)$ as a system of eigenfunctions.

THEOREM 2.5.2. *For any Sheffer set* $s_\alpha(x)$ *relative to* $\{P_1, \cdots, P_n; S\}$, *there exists a unique set of operators* $\{T_1, \cdots, T_n\}$ *such that*

$$T_i = \sum_\alpha \left[\sum_{j=1}^n x_j u_{ij}(\alpha) + \sum_{j=1}^n v_{ij}(\alpha)\right] P^\alpha , \qquad i = 1, \cdots, n , \tag{2.5.2}$$

with the following properties:

(i) *for each* i, T_i *is essentially selfadjoint in the Hilbert space* $\boldsymbol{H}$ *obtained by completing the space* $\boldsymbol{P}$ *of polynomials in the associated inner product* (2.5.1);

(ii) *the spectrum of* $\{T_1, \cdots, T_n\}$ *consists of simple eigenvalues at each vectors in* Z_+^n *and the eigenfunction associated with the vector* α *is the polynomial* $s_\alpha(x)$, *that is*

$$T_i s_\alpha(x) = \alpha_i s_\alpha(x) , \qquad \alpha = (\alpha_1, \cdots, \alpha_n) ;$$

(iii) *setting the inverse matrix* $(b_{ij}(\xi))$ *of the matrix* $(\partial_j p_i$
stants $u_{ij}(\alpha)$ *and* $v_{ij}(\alpha)$ *in* (2.5.2) *are given by*

$$u_{ij}(\alpha) = [b_{ji}(\partial) p_{\alpha - e_i}(x)]_{x=0}$$

and

$$v_{ij}(\alpha) = [-(\partial_j(\log S))(\partial) b_{ji}(\partial) p_{\alpha - e_i}(x)]_{x=0} ,$$

where $p_\alpha(x)$ *are the basic polynomials for the delta set* $\{P_1, \cdots, P_n\}$.

Proof. Taking the partial derivations $\partial/\partial\xi_i = \partial_i$ on the both sides of the generating function

$$S(\xi)^{-1} e(\langle a, \xi\rangle) = \sum_\alpha s_\alpha(a) p^\alpha(\xi) ,$$

we have

$$\partial_j [S(\xi)^{-1} e(\langle a, \xi\rangle)] = \sum_\alpha s_\alpha(a) \sum_{i=1}^n \alpha_i p^{\alpha - e_i}(\xi)(\partial_j p_i)(\xi) . \tag{2.5.3}$$

The left side of (2.5.3) is

$$\begin{aligned} \partial_j [S(\xi)^{-1} e(\langle a, \xi\rangle)] &= [- S(\xi)^{-1} \partial_j S(\xi) + a_j] S(\xi)^{-1} e(\langle a, \xi\rangle) \\ &= [-\partial_j(\log S)(\xi) + a_j] S(\xi)^{-1} e(\langle a, \xi\rangle) . \end{aligned}$$

Operating the matrix $(b_{ij}(\xi))$ from the right hand, and multiplying by $p_i(\xi)$ on (2.5.3), we obtain

$$(2.5.4)\qquad \sum_{j=1}^{n} p_i(\xi)b_{ji}(\xi)(a_j - \partial_j(\log S)(\xi))S(\xi)^{-1}e(\langle a, \xi\rangle) = \sum_{\alpha} \alpha_i s_\alpha(a)p^\alpha(\xi)\,.$$

For each i, define the operator T_i to be

$$T_i = \sum_{j=1}^{n} p_i(\partial)b_{ji}(\partial)(a_j - (\partial_j(\log S))(\partial))\,.$$

Next, expand T_i in multiple powers of $\{P_1, \cdots, P_n\}$, that is,

$$(2.5.5)\qquad T_i = \sum_{\alpha} c_i(\alpha)P^\alpha\,, \qquad c_i(\alpha) = [T_i p_\alpha(x)]_{x=0}$$

Compute $c_i(\alpha)$ for each i as follows:

$$\begin{aligned} c_i(\alpha) &= [T_i p_\alpha(x)]_{x=0} \\ &= \left[\sum_{j=1}^{n} (a_j - (\partial_j(\log S))(\partial))b_{ji}(\partial)p_i(\partial)p_\alpha(x)\right]_{x=0} \\ &= \sum_{j=1}^{n} a_j[b_{ji}(\partial)p_{\alpha-e_i}(x)]_{x=0} - \sum_{j=1}^{n} [(\partial_j(\log S))(\partial)b_{ji}(\partial)p_{\alpha-e_i}(x)]_{x=0} \\ &= \sum_{j=1}^{n} a_j u_{ij}(\alpha) + \sum_{j=1}^{n} v_{ij}(\alpha)\,, \end{aligned}$$

where

$$u_{ij}(\alpha) = [b_{ji}(\partial)p_{\alpha-e_i}(x)]_{x=0}\,,$$

and

$$v_{ij}(\alpha) = -[(\partial_j(\log S))(\partial)b_{ji}(\partial)p_{\alpha-e_i}(x)]_{x=0}\,.$$

From (2.5.5) we have for any polynomial $f(x)$

$$(2.5.6)\qquad T_i S^{-1}f(x+a) = \sum_{\alpha} c_i(\alpha)P^\alpha S^{-1}f(x+a)\,, \qquad i = 1, \cdots, n\,.$$

Also by the Second Expansion Theorem

$$S^{-1}f(x+a) = \sum_{\alpha} s_\alpha(x)P^\alpha f(a)\,,$$

so that placing the right side in (2.5.6), we obtain

$$(2.5.7)\qquad \begin{aligned} T_i S^{-1}f(x+a) &= \sum_{\alpha} c_i(\alpha)P^\alpha \sum_{\beta} s_\beta(x)P^\beta f(a) \\ &= \sum_{\alpha} [\sum_{\beta} c_i(\beta)P^\beta s_\alpha(x)]P^\alpha f(a)\,. \end{aligned}$$

Comparing (2.5.7) with the right side of (2.5.4), we obtain

$$\sum_{\beta} c_i(\beta)P^{\beta}s_\alpha(x) = \alpha_i s_\alpha(x)\,.$$

Therefore every Sheffer polynomial $s_\alpha(x)$ is an eigenfunction of the operators $\{T_1, \cdots, T_n\}$. Since the Sheffer set generates all polynomials, it is obvious to prove (i). Q.E.D.

§ 6. Cross sequences

In this section we shall be concerned with a polynomial sequence $p_\alpha^{[\lambda]}(x)$ depending on a parameter λ to satisfy the identity

$$p_\alpha^{[\lambda+\mu]}(x+y) = \sum_{\alpha=\beta+\gamma} p_\beta^{[\lambda]}(x)p_\gamma^{[\mu]}(y)\,. \tag{2.6.1}$$

In the case of one variable, Steffensen [26] and Rota etc. [22] treated the polynomial sequence to satisfy (2.6.1). We can easily give a generalization of their properties as stated in [22].

Let $(\boldsymbol{Q}^c)^m$ be the m tensor products of complex rational numbers $\boldsymbol{Q}^c$. Then, a *cross sequence of polynomials*, written by $p_\alpha^{[\lambda]}(x)$ where λ ranges over $(\boldsymbol{Q}^c)^m$ and α over Z_+^n, is defined by the following properties:

(i) $p_\alpha^{[0]}(x)$ is a polynomial sequence of binomial type defined in Section 1 [28];

(ii) for any λ and μ in $(\boldsymbol{Q}^c)^m$, it holds the identity (2.6.1).

Now we begin stating the well known lemma, so we omit the proof.

LEMMA 2.6.1. *Let $p(\lambda;\xi)$ be a non zero formal power series depending on $\lambda = (\lambda_1, \cdots, \lambda_m)$ in $(\boldsymbol{Q}^c)^m$ to satisfy*

$$p(\lambda+\mu;\xi) = p(\lambda;\xi)p(\mu;\xi)\,.$$

Then there exists a set of the invertible formal power series $\{q_1(\xi), \cdots, q_m(\xi)\}$ such that

$$p(\lambda;\xi) = q_1^{\lambda_1}(\xi)\cdots q_m^{\lambda_m}(\xi)\,.$$

Using the above lemma, we have a fundamental property of the cross sequence.

THEOREM 2.6.2. *A polynomial sequence $p_\alpha^{[\lambda]}(x)$ is a cross sequence if and only if there exists a m parameter group $Q_1^{\lambda_1}\cdots Q_m^{\lambda_m}$ of translation invariant operators $\{Q_1, \cdots, Q_m\}$ and a sequence $p_\alpha(x)$ of binomial type such that*

$$p_\alpha^{[\lambda]}(x) = Q_1^{-\lambda_1}\cdots Q_m^{-\lambda_m}p_\alpha(x)\,.$$

Proof. The necessary condition is clear. So we prove the sufficient

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condition. First, note that the sequence $p_\alpha(x) = p_\alpha^{[0]}(x)$ is of binomial type. Setting $\mu = \mathbf{0}$ in (2.6.1) and applying Corollary of Proposition 2.1.4, we infer that $p_\alpha^{[\lambda]}(x)$ is a Sheffer set relative to a invertible operator $Q(\lambda)$. From (2.6.1), we have

$$Q(\lambda)^{-1}p_\alpha(x+y) = \sum_{\alpha=\beta+\gamma} p_\beta^{[\lambda]}(x)p_\gamma(y)\,,$$

and applying $Q(\mu)^{-1}$ to the both sides, we get

$$Q(\mu)^{-1}Q(\lambda)^{-1}p_\alpha(x+y) = \sum_{\alpha=\beta+\gamma} p_\beta^{[\lambda]}(x)p_\gamma^{[\mu]}(y)\,.$$

Since $p_\alpha(x)$ is a base for the vector space $\boldsymbol{P}$, we obtain

$$Q(\lambda+\mu) = Q(\lambda)Q(\mu)\,.$$

From Lemma 2.6.1, we conclude the result. Q.E.D.

From the above theorem, the polynomial sequence $p_\alpha^{[\lambda]}(x)$ is a cross sequence relative to $\{P_1, \cdots, P_n; Q_1, \cdots, Q_m\}$, where $\{P_1, \cdots, P_n\}$ is the delta set for the basic set $p_\alpha^{[0]}(x)$. If the dimension m of the parameter λ is equal to the dimension n of the variable x, we obtain an interesting proposition.

PROPOSITION 2.6.3. *Suppose that for each i,* $I - Q_i = P_i$, *where* $\{P_1, \cdots, P_n\}$ *is a delta set. Then for fixed a and a cross sequence* $p_\alpha^{[\lambda]}(x)$ *relative to* $\{P_1, \cdots, P_n; Q_1, \cdots, Q_n\}$, *it holds that*

$$p_\alpha^{[x-\alpha]}(a)$$

is a Sheffer set relative to a set of difference operators $\Delta_i = \exp\partial_i - I$, $i = 1, \cdots, n$.

Proof. We have

$$\begin{aligned} p_\alpha^{[\lambda+e_i-\alpha]}(x) - p_\alpha^{[\lambda-\alpha]}(x) &= Q^{-\lambda-e_i+\alpha}(I-Q_i)p_\alpha(x) \\ &= Q^{-\lambda-e_i+\alpha}p_{\alpha-e_i}(x) \\ &= p_{\alpha-e_i}^{[\lambda+e_i-\alpha]}(x)\,. \end{aligned}$$

Q.E.D.

Every invertible translation invariant operator T can be written in the form $T = \exp F$ for some translation invariant operator F. Indeed, setting $T = I + S$, where $S\,1 = 0$, $F = \log(I+S)$ is well defined and $T = \exp F$. Hence, putting

$$Q_i = \exp F_i\,, \qquad i = 1, \cdots, m\,,$$

we call $\{F_1, \cdots, F_m\}$ the generator of the cross sequence $p_\alpha^{[\lambda]}(x)$.

PROPOSITION 2.6.4. (i) *If* $\{F_1, \cdots, F_m\}$ *and* $\{G_1, \cdots, G_m\}$ *are the generators of cross sequences* $p_\alpha^{[\lambda]}(x)$ *and* $q_\alpha^{[\lambda]}(x)$ *having the same basic sequence, then* $\{F_1 + G_1, \cdots, F_m + G_m\}$ *is the generator of the cross sequence*

$$\exp\Big(-\sum_{i=1}^{m} \lambda_i G_i\Big) p_\alpha^{[\lambda]}(x) = \exp\Big(-\sum_{i=1}^{m} \lambda_i F_i\Big) q_\alpha^{[\lambda]}(x)\,.$$

(ii) *If each* R_i, $i = 1, \cdots, m$ *is any invertible translation invariant operator, then* $R^{-\lambda} p_\alpha^{[\lambda]}(x)$ *is a cross sequence when* $p_\alpha^{[\lambda]}(x)$ *is one.*

The proof is trivial, so we omit the proof.

§ 7. A class of generating functions

Recently, some new generating functions for classical polynomials have been studied in several papers [7], [9], [10], [25]. We give a few typical examples:

(i) for Hermite polynomials $H_n(x)$,

$$\sum_{n=0}^{\infty} H_n(x + ny)\xi^n/n! = e^{(2x\eta - \eta^2)}/(1 - 2y\eta)$$

where $\xi = \eta e^{-y\eta}$;

(ii) for the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$,

$$\sum_{n=0}^{\infty} L_n^{(\alpha+\beta n)}(x)\xi^n = (1 + \eta)^{\alpha+1} e^{-x\eta}/(1 - \beta\eta)$$

where α, β are arbitrary complex numbers and η is defined by

$$\xi = \eta(1 + \eta)^{-\beta-1}\,, \qquad \eta(0) = 0\,.$$

In this section, we shall give the similar generating functions to the case of our Sheffer polynomials. In the case of one variable, it concludes above results of the generating functions.

THEOREM 2.7.1. *Let* $s_\alpha(x)$ *be a Sheffer set relative to the operators* $\{P_1, \cdots, P_n; S\}$. *Let* $\{p_1^{-1}(\xi), \cdots, p_n^{-1}(\xi)\}$ *be the formal inverse of the symbol* $\{p_1(\xi), \cdots, p_n(\xi)\}$ *of the delta set* $\{P_1, \cdots, P_n\}$. *Let* $S(\xi)$ *be the symbol of* S. *Then it holds the following identity*

$$\text{(2.7.1)} \qquad \sum_\alpha s_\alpha(A\alpha + b) \prod_{i=1}^{n} (\xi_i e(-\langle a_i, p^{-1}(\xi)\rangle))^{\alpha_i}$$
$$= S(p^{-1}(\xi))^{-1} e(\langle b, p^{-1}(\xi)\rangle) |\delta_{ij} - \xi_i \langle a_i, (\partial_j p^{-1})(\xi)\rangle|^{-1}\,,$$

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where A is a $n\times n$ matrix (a_{ij}), b is an n dimensional vector $(b_1, \cdots, b_n)$, $|\delta_{ij}-\xi_i\langle a_i, (\partial_j p^{-1})(\xi)\rangle|$ is the determinant, and the inner product

$$\langle a_i, p^{-1}(\xi)\rangle = \sum_{k=1}^{n} a_{ki}p_k^{-1}(\xi)$$
$$\langle b, p^{-1}(\xi)\rangle = \sum_{k=1}^{n} b_k p_k^{-1}(\xi)\,.$$

Proof. Let $F_i(\partial)$ be defined by

$$F_i(\partial) = \partial_i e(-\langle a_i, p^{-1}(\partial)\rangle) \qquad i = 1, \cdots, n.$$

Let $f_\alpha(x)$ be a set of basic polynomials for the delta set $\{F_1(\partial), \cdots, F_n(\partial)\}$. Since

$$\begin{aligned}\partial_j F_i(\xi) &= \delta_{ij} e(-\langle a_i, p^{-1}(\xi)\rangle) - \xi_i\langle a_i, \partial_j p^{-1}(\xi)\rangle e(-\langle a_i, p^{-1}(\xi)\rangle)\\ &= \{\delta_{ij} - \xi_i\langle a_i, \partial_j p^{-1}(\xi)\rangle\} e(-\langle a_i, p^{-1}(\xi)\rangle)\,,\end{aligned}$$

by the Transfer Formula (i) in [28], $f_\alpha(x)$ is given as follows:

$$f_\alpha(x) = |\delta_{ij} - \partial_i\langle a_i, (\partial_j p^{-1})(\partial)\rangle|\, e(\langle A\alpha, p^{-1}(\partial)\rangle)x^\alpha/\alpha!.$$

Setting the sequence c_α by

$$S(p^{-1}(\xi))^{-1} e(\langle b, p^{-1}(\xi)\rangle)|\delta_{ij} - \xi_i\langle a_i, (\partial_j p^{-1})(\xi)\rangle|^{-1} = \sum_\alpha c_\alpha F^\alpha(\xi),$$

and using the First Expansion Formula in [28], we have

$$c_\alpha = [S(p^{-1}(\partial))^{-1} e(\langle b, p^{-1}(\partial)\rangle)|\delta_{ij} - \partial_i\langle a_i, (\partial_j p^{-1})(\partial)\rangle|^{-1} f_\alpha(x)]_{x=0}.$$

By the Second Expansion Formula, this gives

$$\begin{aligned}c_\alpha &= [S(p^{-1}(\partial))^{-1} e(\langle A\alpha + b, p^{-1}(\partial)\rangle)x^\alpha/\alpha!]_{x=0}\\ &= s_\alpha(A\alpha + b).\end{aligned}$$

Q.E.D.

§ 8. Examples

In this section, we shall be concerned with a generalization of some classical polynomials; Hermite, Laguerre, Euler and Bernoulli polynomials. And we discuss some properties of each polynomial.

(a) A generalization of Hermite polynomials.

Since the publication of the book [2] by Appell and Kampé de Fériet, a generalization of Hermite polynomials with several variables has been studied by many investigators (cf. [11] vol. 2, p. 285). Now, we examine properties of the generalized Hermite polynomials from our operational point of view.

Let g be a real symmetric and nondegenerate $n\times n$ matrix (g_{ij}). Let's give Sheffer polynomials $H_\alpha(g; x)$ relative to $\{\partial_1, \cdots, \partial_n; e(-\langle g\partial, \partial\rangle/2)\}$:

$$H_\alpha(g; x) = e(\langle g\partial, \partial\rangle/2)x^\alpha/\alpha!\,. \tag{2.8.1}$$

From Corollary 1 of Theorem 2.4.2, we see that $(g^{-1}H)_\alpha(g; x)$ is the Sheffer set relative to $\{(g\partial)_1, \cdots, (g\partial)_n; e(-\langle g\partial, \partial\rangle/2)\}$;

$$(g^{-1}H)_\alpha(g; x) = e(\langle g\partial, \partial\rangle/2)(g^{-1}x)^\alpha/\alpha!\,. \tag{2.8.2}$$

where $(g\partial)_i = \sum_{k=1}^n g_{ik}\partial_k$. The generating functions of $H_\alpha(g; x)$ and $(g^{-1}H)_\alpha(g; x)$ are given by Corollary 3 of Theorem 2.1.3, respectively;

$$\sum_\alpha H_\alpha(g; x)\xi^\alpha = e(\langle x, \xi\rangle + 2^{-1}\langle g\xi, \xi\rangle)\,, \tag{2.8.3}$$

$$\sum_\alpha (g^{-1}H)_\alpha(g; x)\xi^\alpha = e(\langle x, g^{-1}\xi\rangle + 2^{-1}\langle g^{-1}\xi, \xi\rangle)\,. \tag{2.8.4}$$

Hence, in the notation of Appell and Kampé de Fériet,

$$H_\alpha(-A^{-1}; x) = G_\alpha(x)/\alpha!, \qquad (-AH)_\alpha(-A^{-1}; x) = (-1)^{|\alpha|}H_\alpha(x)/\alpha!\,,$$

where A is a real symmetric matrix (a_{ij}).

We enumerate some properties of the generalized Hermite polynomials with a brief proof.

(i) Rodrigues' formula (cf. [11] vol. 2, p. 285). Since

$$e(\langle g\partial, \partial\rangle/2)x_i - x_i e(\langle g\partial, \partial\rangle/2) = (g\partial)_i e(\langle g\partial, \partial\rangle/2)\,,$$

and

$$(g\partial)_i e(-\langle g^{-1}x, x\rangle/2) = -x_i e(-\langle g^{-1}x, x\rangle/2) + e(-\langle g^{-1}x, x\rangle/2)(g\partial)_i\,,$$

by the iteration, it holds the identities

$$\begin{aligned} &e(\langle g\partial, \partial\rangle/2)[x^\alpha/\alpha!]\,e(-\langle g\partial, \partial\rangle/2)\\ &\qquad = (g\partial + x)^\alpha/\alpha!\\ &\qquad = e(-\langle g^{-1}x, x\rangle/2)[(g\partial)^\alpha/\alpha!]\,e(\langle g^{-1}x, x\rangle/2)\,. \end{aligned} \tag{2.8.5}$$

Hence, we obtain the Rodrigues' formula

$$H_\alpha(g; x) = e(-\langle g^{-1}x, x\rangle/2)[(g\partial)^\alpha/\alpha!]\,e(\langle g^{-1}x, x\rangle/2)\,. \tag{2.8.6}$$

In like manner, we get

$$(2.8.7)\qquad \begin{aligned}(g^{-1}H)_\alpha(g;x) &= (\partial + g^{-1}x)^\alpha/\alpha! \\ &= e(-\langle g^{-1}x, x\rangle/2)[\partial^\alpha/\alpha!]\,e(\langle g^{-1}x, x\rangle/2)\,.\end{aligned}$$

(ii) Recurrence formula.

From (2.3.20), we obtain the recurrence formulas for $H_\alpha(g;x)$ and $(g^{-1}H)_\alpha(g;x)$, respectively;

$$(2.8.8)\qquad (\alpha_i + 1)H_{\alpha+e_i}(g;x) = x_i H_\alpha(g;x) + \sum_{k=1}^{n} g_{ik}H_{\alpha-e_k}(g;x)\,,$$

$$(2.8.9)\qquad \begin{aligned}&(\alpha_j + 1)(g^{-1}H)_{\alpha+e_j}(g;x) \\ &\quad = (g^{-1}x)_j(g^{-1}H)_\alpha(g;x) + \sum_{k=1}^{n}(g^{-1})_{jk}(g^{-1}H)_{\alpha-e_k}(g;x)\,, \\ &\qquad\qquad i, j = 1, \cdots, n\,.\end{aligned}$$

(iii) Partial differential systems to determine the generalized Hermite polynomials.

By Theorem 2.5.2, we obtain

$$(2.8.10)\qquad \{(g\partial)_i\partial_i + x_i\partial_i\}H_\alpha(g;x) = \alpha_i H_\alpha(g;x)$$

and

$$(2.8.11)\qquad \{\partial_j(g\partial)_j + (g^{-1}x)_j(g\partial)_j\}(g^{-1}H)_\alpha(g;x) = \alpha_j(g^{-1}H)_\alpha(g;x)\,, \quad i, j = 1, \cdots, n\,.$$

(vi) A generalization of the formula of Burchnall-Feldheim-Watson (cf. [8], [12], [13], [29], [30]).

From (2.8.5), it holds the identities

$$\begin{aligned}&(g\partial + x)^\alpha/(\alpha!)f(x) \\ &\quad = e(-\langle g^{-1}x, x\rangle/2)(g\partial)^\alpha/(\alpha!)[e(\langle g^{-1}x, x\rangle/2)f(x)] \\ &\quad = e(-\langle g^{-1}x, x\rangle/2)\sum_{\alpha=\mu+\nu}[(g\partial)^\mu/(\mu!)e(\langle g^{-1}x, x\rangle/2)][(g\partial)^\nu/(\nu!)f(x)] \\ &\quad = \sum_{\alpha=\mu+\nu} H_\mu(g;x)(g\partial)^\nu/(\nu!)f(x)\,.\end{aligned}$$

Setting $f(x) = (g^{-1}H)_\beta(g;x)$, we have

$$(2.8.12)\qquad (g\partial + x)^\alpha/(\alpha!)(g^{-1}H)_\beta(g;x) = \sum_{\alpha=\mu+\nu} H_\mu(g;x)(g^{-1}H)_{\beta-\nu}(g;x)/\nu!\,.$$

We introduce the following umbral notation: Setting

$$(gx)^\alpha/\alpha! = \sum_\mu c_\alpha(\mu)x^\mu/\mu!\,,$$

we define

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$$(g(\boldsymbol{g}^{-1}\boldsymbol{H})_\beta)_\alpha(g; x) = \sum_\mu c_\alpha(\mu)(g^{-1}\boldsymbol{H})_{\beta+\mu}(g; x).$$

Then (2.8.12) is written as follows:

(2.8.13) $$(g(\boldsymbol{g}^{-1}\boldsymbol{H})_\beta)_\alpha(g; x) = \sum_{\alpha=\mu+\nu} H_\mu(g; x)(g^{-1}\boldsymbol{H})_{\beta-\nu}(g; x)/\nu!.$$

In the same way, we obtain

(2.8.14) $$(g^{-1}\boldsymbol{H}_\beta)_\alpha(g; x) = \sum_{\alpha=\mu+\nu} (g^{-1}\boldsymbol{H})_\mu(g; x)H_{\beta-\nu}(g; x)/\nu!.$$

Secondly, we represent $H_\alpha(g; x)(g^{-1}\boldsymbol{H})_\beta(g; x)$ as a linear combination of $H_\alpha(g; x)$ or $(g^{-1}\boldsymbol{H})_\alpha(g; x)$. By the Second Expansion Formula, we have

(2.8.15) $$\begin{aligned} &H_\alpha(g; x)(g^{-1}\boldsymbol{H})_\beta(g; x) \\ &\quad = \sum_\mu H_\mu(g; x)[\partial^\mu \boldsymbol{e}(-\langle g\partial, \partial\rangle/2)H_\alpha(g; x)(g^{-1}\boldsymbol{H})_\beta(g; x)]_{x=0}. \end{aligned}$$

Noting

$$\begin{aligned} &[\boldsymbol{e}(-\langle g\partial, \partial\rangle/2)(\sum_{\alpha,\beta} H_\alpha(g; x)(g^{-1}\boldsymbol{H})_\beta(g; x)\xi^\alpha\eta^\beta)]_{x=0} \\ &\quad = [\boldsymbol{e}(-\langle g\partial, \partial\rangle/2)\boldsymbol{e}(2^{-1}\langle g\xi, \xi\rangle + 2^{-1}\langle g^{-1}\eta, \eta\rangle + \langle x, \xi + g^{-1}\eta\rangle)]_{x=0} \\ &\quad = \boldsymbol{e}(-\langle \xi, \eta\rangle), \end{aligned}$$

we get

$$[\boldsymbol{e}(-\langle g\partial, \partial\rangle/2)H_\alpha(g; x)(g^{-1}\boldsymbol{H})_\beta(g; x)]_{x=0} = (-1)^{|\alpha|}(\alpha!)^{-1}\delta_{\alpha\beta}.$$

Hence, in the right side of (2.8.15), the term in the bracket is calculated as follows:

$$\begin{aligned} &[\partial^\mu \boldsymbol{e}(-\langle g\partial, \partial\rangle/2)H_\alpha(g; x)(g^{-1}\boldsymbol{H})_\beta(g; x)]_{x=0} \\ &\quad = \sum_{\mu=\nu+\omega} \left[\boldsymbol{e}(-\langle g\partial, \partial\rangle/2)\frac{\mu!}{\nu!}(\partial^\nu H_\alpha(g; x))\frac{1}{\omega!}(\partial^\omega(g^{-1}\boldsymbol{H})_\beta(g; x))\right]_{x=0} \\ &\quad = \sum_{\mu=\nu+\omega} \left[\frac{\mu!}{\nu!}\boldsymbol{e}(-\langle g\partial, \partial\rangle/2)H_{\alpha-\nu}(g; x)\frac{1}{\omega!}((g^{-1}\cdot g\partial)^\omega(g^{-1}\boldsymbol{H})_\beta(g; x))\right]_{x=0} \end{aligned}$$

(2.8.16) $$\begin{aligned} &\quad = \sum_{\mu=\nu+\omega} \left[\frac{\mu!}{\nu!}\boldsymbol{e}(-\langle g\partial, \partial\rangle/2)H_{\alpha-\nu}(g; x)\right. \\ &\qquad\qquad \left. \times(\sum_{\omega_i=|\lambda_i|} \prod_{i,j} (g^{-1})_{ij}^{\lambda_{ij}}(\lambda_{ij}!)^{-1}(g^{-1}\boldsymbol{H})_{\beta-\Lambda}(g; x))\right]_{x=0} \\ &\quad = \sum_{\mu=\nu+\omega} \sum_{\omega_i=|\lambda_i|,} \sum_{\alpha-\nu=\beta-\Lambda} \prod_{ij} (g^{-1})_{ij}^{\lambda_{ij}}(\lambda_{ij}!)^{-1}(-1)^{|\alpha-\nu|}\mu!(\nu!(\alpha-\nu)!)^{-1} \\ &\quad \equiv c_\mu, \end{aligned}$$

where $\lambda_i = (\lambda_{i1}, \cdots, \lambda_{in})$, $\Lambda = (\Lambda_1, \cdots, \Lambda_n)$ and $\Lambda_j = \sum_{k=1}^n \lambda_{kj}$. Then we obtain

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(2.8.17) $$H_\alpha(g;x)(g^{-1}H)_\beta(g;x)=\sum_\mu c_\mu H_\mu(g;x),$$

where c_μ is the constant defined by (2.8.16). In the same way, we can represent $H_\alpha(g;x)(g^{-1}H)_\beta(g;x)$ as a linear sum of $(g^{-1}H)_\alpha(g;x)$, but we omit it.

(b) A generalization of Laguerre polynomials.

As well known (cf. [11] vol. 2, p. 189), the down-ladder of Laguerre polynomials with one variable is $(d/dx)/(d/dx-1)$. So we take a linear fractional transformation of the derivation ∂;

$$\{\partial_1/1+\langle a,\partial\rangle,\cdots,\partial_n/1+\langle a,\partial\rangle\},\qquad a=(a_1,\cdots,a_n)$$

for a delta set of a generalized Laguerre polynomials $L_\alpha(x)$.

Setting

$$p_i(\xi)=\xi_i/1+\langle a,\xi\rangle,\qquad i=1,\cdots,n,$$

we have the matrix

$$((\partial_i p_j)(\xi))=(\delta_{ij}(1+\langle a,\xi\rangle)^{-1}-a_i\xi_j(1+\langle a,\xi\rangle)^{-2}).$$

Since the determinant $|((\partial_i p_j)(\xi))|$ is equal to $(1+\langle a,\xi\rangle)^{-n-1}$, by the Transfer Formula [28], we obtain

(2.8.18) $$L_\alpha(x)=(1+\langle a,\partial\rangle)^{|\alpha|-1}x^\alpha/\alpha!.$$

Now, using the Heisenberg-Weyl relation, we change the representation (2.8.18) or $L_\alpha(x)$ as follows:

$$\begin{aligned}L_\alpha(x)&=(1+\langle a,\partial\rangle)^{|\alpha|}x^\alpha/\alpha!-\sum_{k=1}^n a_k(1+\langle a,\partial\rangle)^{|\alpha|-1}x^{\alpha-e_k}/(\alpha-e_k)!\\&=x_j(1+\langle a,\partial\rangle)^{|\alpha|}x^{\alpha-e_j}/\alpha!+a_j|\alpha|(1+\langle a,\partial\rangle)^{|\alpha|-1}x^{\alpha-e_j}/\alpha!\\&\qquad-\sum_{k=1}^n a_kx_j(1+\langle a,\partial\rangle)^{|\alpha|-1}x^{\alpha-e_j-e_k}/(\alpha-e_k)!\\&\qquad-(|\alpha|-1)a_j\sum_{k=1}^n a_k(1+\langle a,\partial\rangle)^{|\alpha|-2}x^{\alpha-e_j-e_k}/(\alpha-e_k)!\\&=x_j(1+\langle a,\partial\rangle)^{|\alpha|}x^{\alpha-e_j}/\alpha!\\&\qquad+\sum_{k=1}^n a_j(1+\langle a,\partial\rangle)^{|\alpha|-1}x_kx^{\alpha-e_j-e_k}/(\alpha-e_k)!\\&\qquad-\sum_{k=1}^n a_kx_j(1+\langle a,\partial\rangle)^{|\alpha|-1}x^{\alpha-e_j-e_k}/(\alpha-e_k)!\\&\qquad-\sum_{k=1}^n(|\alpha|-1)a_ja_k(1+\langle a,\partial\rangle)^{|\alpha|-2}x^{\alpha-e_j-e_k}/(\alpha-e_k)!\end{aligned}$$

$$= x_j(1 + \langle a, \partial\rangle)^{|\alpha|}x^{\alpha-e_j}/\alpha!$$
$$+ \sum_{k=1}^{n} \begin{vmatrix} a_j, & a_k \\ x_j, & x_k \end{vmatrix} (1 + \langle a, \partial\rangle)^{|\alpha|-1}x^{\alpha-e_j-e_k}/(\alpha - e_k)!$$

for each $j = 1, \cdots, n$. Hence, we obtain the other representation of $L_\alpha(x)$:

$$(2.8.19)\qquad L_\alpha(x) = \frac{1}{n}\Big\{\sum_{j=1}^{n} x_j(1 + \langle a, \partial\rangle)^{|\alpha|}x^{\alpha-e_j}/\alpha! + \sum_{k,j=1}^{n} \begin{vmatrix} a_j, & a_k \\ x_j, & x_k \end{vmatrix} (1 + \langle a, \partial\rangle)^{|\alpha|-1}x^{\alpha-e_j-e_k}/(\alpha - e_k)!\Big\}.$$

This gives a formula similar to the classical Rodrigues' formula. Fixing the vector b to satisfy

$$1 + \langle a, b\rangle = 0,$$

we obtain the formula

$$(2.8.20)\qquad \begin{aligned} L_\alpha(x) = \frac{1}{n}\Big\{&\sum_{j=1}^{n} x_j e(\langle b, x\rangle)\langle a, \partial\rangle^{|\alpha|} e(-\langle b, x\rangle)x^{\alpha-e_j}/\alpha! \\ &+ \sum_{k,j=1}^{n} \begin{vmatrix} a_j, & a_k \\ x_j, & x_k \end{vmatrix} e(\langle b, x\rangle)\langle a, \partial\rangle^{|\alpha|-1} \\ &\times e(-\langle b, x\rangle)x^{\alpha-e_j-e_k}/(\alpha - e_k)!\Big\}. \end{aligned}$$

Now we deal with the Sheffer set $L_\alpha^\lambda(x)$ respect to $\{\partial_1/1 + \langle a, \partial\rangle, \cdots, \partial_n/1 + \langle a, \partial\rangle; (1 + \langle a, \partial\rangle)^{-\lambda-1}\}$:

$$L_\alpha^\lambda(x) = (1 + \langle a, \partial\rangle)^{\lambda+|\alpha|}x^\alpha/\alpha!.$$

(i) Partial differential system to determine the Sheffer set $L_\alpha^\lambda(x)$. Since the inverse matrix of $((\partial_i p_j)(\xi))$ is

$$(\delta_{ij}(1 + \langle a, \xi\rangle) + a_i\xi_j(1 + \langle a, \xi\rangle)),$$

we can easily calculate the numbers $u_{ij}(\alpha)$ and $v_{ij}(\alpha)$ in Theorem 2.5.2:

$$u_{ij}(\alpha) = \delta_{ij}\frac{(|\alpha| - 1)!}{(\alpha - e_i)!}a^{\alpha-e_i} + \frac{(|\alpha| - 1)!}{(\alpha - e_i - e_j)!}a^{\alpha-e_j},$$

$$v_{ij}(\alpha) = (\lambda + 1)\delta_{ij}\delta_{\alpha e_i}a_j + (\lambda + 1)\frac{(|\alpha| - 2)!}{(\alpha - e_i - e_j)!}a^\alpha.$$

We have

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$$\sum_{j=1}^{n}\sum_{\alpha} x_j u_{ij}(\alpha)P^\alpha$$
$$= \sum_{j=1}^{n}\sum_{\alpha} x_j\delta_{ij}\frac{(|\alpha|-1)!}{(\alpha-e_i)!}a^{\alpha-e_i}P^\alpha + \sum_{j=1}^{n}\sum_{\alpha} x_j\frac{(|\alpha|-1)!}{(\alpha-e_i-e_j)!}a^{\alpha-e_j}P^\alpha$$
$$= x_iP_i(1-\langle a, P\rangle)^{-1} + \sum_{j=1}^{n} x_ja_iP_iP_j(1-\langle a, P\rangle)^{-2},$$

where $\langle a, P\rangle = \sum_{k=1}^{n} a_kP_k$. Setting

$$P_i = \partial_i/1 + \langle a, \partial\rangle, \qquad i = 1, \cdots, n,$$

we get the identity

$$\text{(2.8.21)}\qquad \sum_{j=1}^{n}\sum_{\alpha} x_ju_{ij}(\alpha)P^\alpha = x_i\partial_i + \sum_{j=1}^{n} x_ja_i\partial_i\partial_j, \qquad i = 1, \cdots, n.$$

Also, it holds

$$\text{(2.8.22)}\qquad \sum_{j=1}^{n}\sum_{\alpha} v_{ij}(\alpha)P^\alpha = (\lambda+1)a_i\partial_i, \qquad i = 1, \cdots, n.$$

Hence, we obtain the partial differential system to determine $L_\alpha^\lambda(x)$:

$$\text{(2.8.23)}\qquad \left[\sum_{j=1}^{n} x_ja_i\partial_i\partial_j + (x_i + (\lambda+1)a_i)\partial_i\right]L_\alpha^\lambda(x) = \alpha_iL_\alpha^\lambda(x), \qquad i = 1, \cdots, n.$$

(ii) Recurrence formula.

Since the formal inverse of $\{\xi_1/1 + \langle a, \xi\rangle, \cdots, \xi_n/1 + \langle a, \xi\rangle\}$ is $\{\xi_1/1 - \langle a, \xi\rangle, \cdots, \xi_n/1 - \langle a, \xi\rangle\}$, in the notations of Section 3, we get the identities

$$(\partial_ip_j)(q(\xi)) = \delta_{ij} - \delta_{ij}\langle a, \xi\rangle - a_i\xi_j + a_i\xi_j\langle a, \xi\rangle,$$
$$(\partial_i \log S)(q(\xi)) = -(\lambda+1)a_i + (\lambda+1)a_i\langle a, \xi\rangle.$$

Using Theorem 2.3.4, we obtain the recurrence formula:

$$\begin{aligned}(\alpha_i+1)L_{\alpha+e_i}^\lambda(x) &= [x_i + (|\alpha|+\lambda)a_i]L_\alpha^\lambda(x)\\ \text{(2.8.24)}\qquad &+ (\alpha_i+1)\sum_{k=1}^{n} a_kL_{\alpha+e_i-e_k}^\lambda(x)\\ &- (|\alpha|+\lambda)\sum_{k=1}^{n} a_ia_kL_{\alpha-e_k}^\lambda(x), \qquad i = 1, \cdots, n.\end{aligned}$$

(iii) Composition formulas.

Let $M_\alpha^\lambda(x)$ be a Sheffer set relative to $\{\partial_1/1 - \langle a, \partial\rangle, \cdots, \partial_n/1 - \langle a, \partial\rangle;$ $(1 - \langle a, \partial\rangle)^{-\lambda-1}\}$:

$$M_\alpha^\lambda(x) = (1 - \langle a, \partial\rangle)^{|\alpha|+\lambda}x^\alpha/\alpha!.$$

Give the other Sheffer set $N_\alpha^\lambda(x)$ relative to $\{\partial_1, \cdots, \partial_n; (1-\langle a, \partial\rangle)^\lambda\}$;

$$N_\alpha^\lambda(x) = (1 - \langle a, \partial\rangle)^{-\lambda} x^\alpha/\alpha!\,.$$

Then from Theorem 2.4.2, we have the composition formulas:

$$L_\alpha^{\lambda_1}(M^{\lambda_2}(L^{\lambda_3}(\cdots(L^{\lambda_{2n+1}})\cdots)))(x) = L_\alpha^{\lambda_1-\lambda_2+\lambda_3-\cdots+\lambda_{2n+1}}(x)\,,$$
$$L_\alpha^{\lambda_1}(M^{\lambda_2}(L^{\lambda_3}(\cdots(M^{\lambda_{2n}})\cdots)))(x) = N_\alpha^{\lambda_1-\lambda_2+\cdots-\lambda_{2n}}(x)\,.$$

In the case of one variable, this formula is well known as the second composition law (cf. [22] p. 729]).

(c) A generalization of Euler and Bernoulli polynomials.

As well known (cf. [19], [11] vol. 1, [15] Chapter V, VI), Euler $E_n(x)$ and Bernoulli polynomials $B_n(x)$ are defined by the operations of the mean on the one dimensional convex cell [0, 1] and its vertex {0, 1}:

$$\frac{1}{2}(E_n(x+1) + E_n(x)) = x^n\,, \qquad \int_x^{x+1} B_n(t)dt = x^n\,.$$

Now, we construct Euler and Bernoulli polynomials with several variables by using operations of the mean on an n dimensional convex cell and its faces. Let e_i be unit vectors

$$e_1 = (1, 0, \cdots, 0)\,, \qquad e_2 = (0, 1, 0, \cdots, 0), \cdots, e_n = (0, \cdots, 0, 1)\,.$$

Let V be a convex cell spanned by the origin $\mathbf{0}$ and the set of vectors

$$\left\{\sum_{k=1}^{l} e_{i_k};\ i_k,\ l = 1, \cdots, n,\ i_{k_1} \neq i_{k_2} \text{ if } k_1 \neq k_2\right\}.$$

Set V_l for a set of all l dimensional faces of V. We define the operation J_l of the mean associated with V_l as follows:

$$\begin{aligned}
J_l p(x) = k_l^n \sum_{i_1<\cdots<i_l} \int_0^1 \cdots \int_0^1 [&p(x_1, \cdots, x_{i_1}+\xi_{i_1}, \cdots, x_{i_l}+\xi_{i_l}, \cdots, x_n) \\
&+ p(x_1+1, x_2, \cdots, x_{i_1}+\xi_{i_1}, \cdots, x_{i_l}+\xi_{i_l}, \cdots, x_n) \\
&+ p(x_1, x_2+1, x_3, \cdots, x_{i_1}+\xi_{i_1}, \cdots, x_{i_l}+\xi_{i_l}, \cdots, x_n) \\
&\qquad \vdots \\
&+ p(x_1, \cdots, x_{i_1}+\xi_{i_1}, \cdots, x_{i_l}+\xi_{i_l}, \cdots, x_n+1) \\
&+ p(x_1+1, x_2+1, x_3, \cdots, x_{i_1}+\xi_{i_1}, \cdots, x_{i_l}+\xi_{i_l}, \cdots, x_n) \\
&\qquad \vdots \\
&+ p(x_1+1, x_2+1, \cdots, x_{i_1-1}+1, x_{i_1}+\xi_{i_1}, \cdots, x_{i_l}+\xi_{i_l}, \\
&\qquad x_{i_l+1}+1, \cdots, x_n+1)]d\xi_{i_1}\cdots d\xi_{i_l}\,,
\end{aligned} \tag{2.8.25}$$

where k_l^n is the inverse of the total volume $2^{n-l}\binom{n}{l}$ of V_l. Note that in the case of V_0, that is the set of all vertices of V, the operator J_0 is defined by

$$\begin{aligned} J_0 p(x) = 2^{-n}\Big\{ p(x) + \sum_{i=1}^{n} p(x+e_i) \\ + \sum_{i_1<i_2} p(x+e_{i_1}+e_{i_2}) + \cdots + p(x+e_1+\cdots+e_n)\Big\}. \end{aligned} \tag{2.8.26}$$

Then J_l is written as follows:

$$\begin{aligned} J_l p(x) = \sum_{i_1<\cdots<i_l} \frac{(e(\partial_1)+1)\cdots(e(\partial_n)+1)}{(e(\partial_{i_1})+1)\cdots(e(\partial_{i_l})+1)} \\ \times \int_0^1 \cdots \int_0^1 p(x_1, \cdots, x_{i_1}+\xi_{i_1}, \cdots, x_{i_l}+\xi_{i_l}, \cdots, x_n) d\xi_{i_1}\cdots d\xi_{i_l}. \end{aligned} \tag{2.8.27}$$

Operating the both side of (2.8.27) by $\partial_1 \cdots \partial_n$, we have

$$\begin{aligned} \partial_1 \cdots \partial_n J_l p(x) = \sum_{i_1<\cdots<i_l} \frac{(e(\partial_1)+1)\cdots(e(\partial_n)+1)}{(e(\partial_{i_1})+1)\cdots(e(\partial_{i_l})+1)} \\ \times \frac{\partial_1\cdots\partial_n}{\partial_{i_1}\cdots\partial_{i_l}}(e(\partial_{i_1})-1)\cdots(e(\partial_{i_l})-1)p(x). \end{aligned} \tag{2.8.28}$$

Hence, we obtain the differential representation of J_l

$$\begin{aligned} J_l p(x) = \sum_{i_l<\cdots<i_l} \frac{(e(\partial_1)+1)\cdots(e(\partial_n)+1)}{(e(\partial_{i_1})+1)\cdots(e(\partial_{i_l})+1)} \\ \times \frac{(e(e_{i_1})-1)}{\partial_{i_1}}\cdots\frac{(e(\partial_{i_l})-1)}{\partial_{i_l}}p(x). \end{aligned} \tag{2.8.29}$$

We consider the general n dimensional convex cell $(gV)_n$ spanned by the origin $\mathbf{0}$ and

$$\Big\{\sum_{k=1}^{l} g_{i_k};\ l, i_k = 1, \cdots, n,\ i_{k_1} \neq i_{k_2} \text{ if } k_1 \neq k_2\Big\},$$

where a set of the vectors $\{g_1, \cdots, g_n\}$ is linear independent. Let $(gV)_l$ be a set of all l dimensional faces in $(gV)_n$. As the right side of (2.8.25) is denoted by

$$k_l^n \int \underset{V_l}{\cdots} \int p(x+\xi)d\xi^{(l)},$$

we define the operation of the mean J_l^g associated with $(gV)_l$:

$$J_l^g p(x) = k_l^n \int \underset{V_l}{\cdots} \int p(x+g\xi)d\xi^{(l)}. \tag{2.8.30}$$

Then in the similar way, we get the differential representation of J_l^g:

$$J_l^g p(x) = k_l^n \sum_{i_1<\cdots<i_l} \frac{(e(\langle g_1, \partial\rangle)+1)\cdots(e(\langle g_n, \partial\rangle)+1)}{(e(\langle g_{i_1}, \partial\rangle)+1)\cdots(e(\langle g_{i_l}, \partial\rangle)+1)} \times \frac{(e(\langle g_{i_1}, \partial\rangle)-1)}{\langle g_{i_1}, \partial\rangle}\cdots\frac{(e(\langle g_{i_l}, \partial\rangle)-1)}{\langle g_{i_l}, \partial\rangle} p(x)\,. \tag{2.8.31}$$

We note that J_l^g is a symmetric function with respect to $\{g_1, \cdots, g_n\}$. Noting the operator J_l^g has the inverse, we define a generalization of Euler and Bernoulli polynomials $B_\alpha(x; (gV)_l)$ associated with $(gV)_l$ as follows:

$$J_l^g B_\alpha(x; (gV)_l) = x^\alpha/\alpha!\,. \tag{2.8.32}$$

Hence, the polynomials $B_\alpha(x; (gV)_l)$ is a Sheffer set related to $\{\partial_1, \cdots, \partial_n; J_l^g\}$. In the special case, we have the following relations with the classical Euler and Bernoulli polynomials:

$$B_\alpha(x;\, V_0) = E_{\alpha_1}(x_1)\cdots E_{\alpha_n}(x_n)/\alpha!\,,$$
$$B_\alpha(x;\, V_n) = B_{\alpha_1}(x_1)\cdots B_{\alpha_n}(x_n)/\alpha!\,.$$

Let g be the matrix $\begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}$. By using the umbral notation in Corollary 1 of Theorem 2.5.2, we obtain

$$B_\alpha({}^t g x; (gV)_l) = ({}^t g B)_\alpha(x;\, V_l)\,. \tag{2.8.33}$$

From Corollary 3 of Theorem 2.1.3, we have the generating function of $B_\alpha(x; (gV)_l)$:

$$\sum_\alpha B_\alpha(x; (gV)_l)\xi^\alpha = e(\langle x, \xi\rangle)\Big(\sum_{i_1<\cdots<i_l} \frac{(e(\langle g_1, \xi\rangle)+1)\cdots(e(\langle g_n, \xi\rangle)+1)}{(e(\langle g_{i_1}, \xi\rangle)+1)\cdots(e(\langle g_{i_l}, \xi\rangle)+1)} \times \frac{(e(\langle g_{i_1}, \xi\rangle)-1)}{\langle g_{i_1}, \xi\rangle}\cdots\frac{(e(\langle g_{i_l}, \xi\rangle)-1)}{\langle g_{i_l}, \xi\rangle}\Big)^{-1}. \tag{2.8.34}$$

Also, the Second Expansion Formula gives a generalization of the Euler-MacLaurin sum formula.

Now, we examine the other properties of $B_\alpha(x; (gV)_l)$ similar to the classical Euler and Bernoulli polynomials (cf. [19]).

(i) Symmetry

Let R_l be the following reflexion:

$$R_i g_j = (-1)^{\delta^{ij}} g_j, \qquad i, j = 1, \cdots, n.$$

Setting

$$c = \tfrac{1}{2}(g_1 + \cdots + g_n)$$

for the center of $(gV)_n$, we have

$$\sum_\alpha B_\alpha(c + R_i x; (gV)_l)\xi^\alpha = \sum_\alpha B_\alpha(c + x; (gV)_l)({}^tR_i\xi)^\alpha, \qquad i = 1, \cdots, n,$$

where ${}^tR_i$ is the transposed matrix of R_i. Hence, using the umbral notation of Corollary 1 of Theorem 2.5.2, we obtain

$$B_\alpha(c + R_i x; (gV)_l) = (R_i B)_\alpha(c + x; (gV)_l), \qquad i = 1, \cdots, n.$$

In the classical case, this is equal to

$$B_n(1 - x) = (-1)^n B_n(x)$$

and

$$E_n(1 - x) = (-1)^n E_n(x).$$

(ii) Multiplication formulas.

Set for the given positive integers l_i, $i = 1, \cdots, n$.

$$G(x; (gV)_n) = \sum_{k_1=0}^{l_1-1} \cdots \sum_{k_n=0}^{l_n-1} B_\alpha\Big(x + \frac{k_1}{l_1} g_1 + \cdots + \frac{k_n}{l_n} g_n; (gV)_n\Big).$$

Then it holds

$$\begin{aligned}\Big(e\Big(\frac{1}{l_1}\langle g_1, \partial\rangle\Big) - 1\Big) \cdots \Big(e\Big(\frac{1}{l_n}\langle g_n, \partial\rangle\Big) - 1\Big) G(x; (gV)_n) \\ = (e(g_1, \partial\rangle) - 1) \cdots (e(\langle g_n, \partial\rangle) - 1) B_\alpha(x; (gV)_n) \\ = \langle g_1, \partial\rangle \cdots \langle g_n, \partial\rangle x^\alpha/\alpha!.\end{aligned}$$

Setting the matrix g^l

$$g^l = \begin{pmatrix} (g^l)_1 \\ \vdots \\ (g^l)_n \end{pmatrix} = \begin{pmatrix} l_1 & & 0 \\ & \ddots & \\ 0 & & l_n \end{pmatrix}^{-1} \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} \begin{pmatrix} l_1 & & 0 \\ & \ddots & \\ 0 & & l_n \end{pmatrix} \quad \text{and} \quad l^{-1}\partial = (l_1^{-1}\partial_1, \cdots, l_n^{-1}\partial_n),$$

we have

$$\begin{aligned}(e(\langle (g^l)_1, l^{-1}\partial\rangle) - 1) \cdots (e(\langle (g^l)_n, l^{-1}\partial\rangle) - 1) G(x; (gV)_n) \\ = l_1 \cdots l_n \langle (g^l)_1, l^{-1}\partial\rangle \cdots \langle (g^l)_n, l^{-1}\partial\rangle x^\alpha/\alpha!.\end{aligned}$$

Hence, we obtain the multiplication formula for $B_\alpha(x; (gV)_n)$:

$$B_\alpha(lx; (g^l V)_n) = l_1^{-\alpha_1+1} \cdots l_n^{-\alpha_n+1} \sum_{k_1=0}^{l_1} \cdots \sum_{k_n=0}^{l_n} \times B_\alpha\left(x + \frac{k_1}{l_1} g_1 + \cdots + \frac{k_n}{l_n} g_n; (gV)_n\right).$$

Similarly, we have the formulas

$$B_\alpha(lx; (g^l V)_0) = l_1^{-\alpha_1+1} \cdots l_n^{-\alpha_n+1} \sum_{k_1=0}^{l_1} \cdots \sum_{k_n=0}^{l_n} \times (-1)^{|k|} B_\alpha\left(x + \frac{k_1}{l_1} g_1 + \cdots + \frac{k_n}{l_n} g_n; (gV)_0\right),$$

for every odd number l_i,

and

$$B_\alpha(lx; (g^l V)_n) = l_1^{-\alpha_1+1} \cdots l_n^{-\alpha_n+1} \sum_{k_1=0}^{l_1} \cdots \sum_{k_n=0}^{l_n} \times (-1)^{|k|} B_\alpha\left(x + \frac{k_1}{l_1} g_1 + \cdots + \frac{k_n}{l_n} g_n; (gV)_0\right),$$

for every even number l_i.

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