There is a workmanlike chapter on counting techniques before continuous random variables, including the Normal distribution, are introduced. This leads to the central limit theorem and its various uses. The next chapter is on confidence intervals, and now we encounter the thorny issue of estimating variance. In order to confront the fact that the sample variance is a biased estimator for population variance, one needs a theoretical framework which, unless a teacher is prepared to go well beyond the syllabus, is beyond the scope of a textbook. The current author simply says what happens.

The general theoretical framework of hypothesis testing is not taken beyond the level of Year 1 core until the penultimate chapter. My own practice was to introduce the concepts of Type 1 and Type 2 errors in the context of binomial tests, so that the calculations are relatively straightforward and yet the concepts can be understood thoroughly. However, the author's treatment of this, and non-parametric testing, seems to be very sound.

This is not the book I would have written if anyone had the misfortune to commission one from me. Perhaps I am expecting a somewhat more theoretical approach to what is, after all, a difficult subject to teach. However, it contains plenty of excellent questions and everything you need to know is there, even it is presented as a *fait accompli* rather than justified from first principles. Use it by all means, but be prepared for some searching questions from that awkward pupil who insists on knowing why things work rather than just how to do the problems.

*Reference*

1. Owen Toller, *The mathematics of A-level statistics*, The Mathematical Association (2009).

10.1017/mag.2023.37 

GERRY LEVERSHA
*15 Maunder Road, Hanwell,*
*London W7 3PN*
e-mail: *g.leversha@btinternet.com*

**OCR A level Further Mathematics Discrete (A)** by Nick Geere, pp. 162, £21, ISBN 978-1-5104-3337-3, Hodder Education (2018)

This slim volume completes the set of five books which cover the syllabus for the OCR Further Mathematics examination. I have to admit that discrete mathematics is not an area I have taught and, apart from combinatorics, not one I am expert in. Moreover, my interest is motivated by problem-solving rather than by familiarity with particular aspects of discrete mathematics such as network algorithms, critical path analysis, linear programming or game theory. So I shall take it as read—I think with reason—that the author of the book knows the syllabus intimately, has covered all the necessary topics and has offered much useful advice on tackling the questions which will appear on the examination paper. The focus of this review, therefore, will mainly be on the first two chapters and, in particular, on concept development and problem-solving.

The first chapter provides a synopsis of what this subject is about, beginning with a useful classification of problems into the four (possibly overlapping) categories of existence, construction, enumeration and optimisation. The specific items covered include the pigeonhole principle, set theory with Venn diagrams (and the inclusion-exclusion principle) and permutations and combinations. Derangements are mentioned but not pursued beyond four objects. I have no criticisms of this treatment.

The second chapter is on graph theory, and what emerges immediately is the need for precise technical language. It is essential in the theory of networks, for example, to distinguish between a *walk*, a *trail* and a *path*, all of which are examples of *routes*, and some of which are *cycles*. In the first section alone, there are at least 23 such definitions. The remainder of the chapter covers Eulerian graphs, adjacency matrices, complete bipartite graphs, *k*-colourability, Hamiltonian cycles, planarity and Euler's formula, and it ends with Kuratowski's theorem on the characterisation of non-planar graphs. My reason for supplying this intimidating list is that it highlights a characteristic of discrete mathematics—the number of unfamiliar concepts and relationships which must be assimilated before a pupil feels confident even in *understanding* the problems which they are asked to solve. Naturally there are going to be aspects of this in any area of mathematics—for example in mechanics and statistics which are the parallel disciplines in the OCR set-up—but I think I am justified in claiming that the burden is particularly irksome in this field.

The author, to do him credit, is well aware of this, and he uses the exercises to clarify issues which might easily be overlooked. The first question in Exercise 2.1 reminds you that a cycle is a closed *path*, so it cannot visit the same vertex (apart from the endpoints) twice. A second exercise emphasises that a loop contributes two to the degree of its vertex. Much of the novel vocabulary is highlighted in this way, although I did wonder if it was obvious that a graph which is ‘simply connected’ is *both* simple (with no multiple edges or loops) *and* connected (with no isolated vertices). After all, the adverb simply might be referring to a type of connection.

The author also makes effective use of Activities, whose purpose is to help readers get into the thought process of new work they encounter, and Discussion Points, which invite pupils to discuss particular issues with their fellow students and teacher. An example of the first asks novices to show that, in any graph, the number of odd vertices is always even, and one of the second is to find—in the course of dialogue with their peers—the smallest number of edges for a simply-connected graph with  vertices. The tangible benefit of tackling such problems is that students are encouraged to talk in the language of graph theory so that it eventually becomes as familiar as the algebra or calculus they mastered earlier in their school career.

There is a peculiar flavour to the types of argument used in discrete mathematics—a good deal of argument by contradiction and a need to identify, in the midst of quite a complex problem, some key feature which permits progress towards a solution. I strongly suspect that many teachers are likely to be just as unfamiliar with many aspects of this course as are their students, and it will be a steep learning curve for them as well. A real danger is that discrete mathematics might degenerate into a collection of clever tricks without an overarching mathematical structure, and the author, to his credit, works hard to avoid that.

Next I move on to the exercises in the book, as it is here that I have some reservations (which may well be more pertinent to the subject than to this particular text). As an example, take the concept of isomorphic graphs, defined as follows:-

> Two graphs said to be *isomorphic* if one can be distorted in some way to produce the other. Isomorphic graphs must have the same number of vertices, each of the same degree, and their vertices must be connected in the same way.

So to prove that two graphs *are* isomorphic, one displays a correspondence which respects the criterion above. There might well be more than one, but that does not matter. Much harder, it seems to me, is to show that they are *not* isomorphic. There might well be obvious reasons for this—for example, graph A has a vertex of order 4 and graph B does not. However, what if there is no such egregious distinction? The author suggests examining the adjacency matrices, which might look different, but then you might be able to relabel the vertices and they turn out to be the same. There were several places in this chapter where I looked at the solutions and wondered about their rigour. One example asks the student to determine whether a graph is planar, fully justifying their answer, and the solution states that it is not, using Kuratowski's theorem, since it does not contain a subgraph which is a *subdivision* (formed when you insert extra vertices into one for more edges) of $K_5$. How do you know that? Have you looked at all possible subdivisions? This rather worries me, wearing my Olympiad hat, as a sort of 'Trust Me' argument.

However, this sort of issue strikes me as generic for many of the subject areas covered by the mantle of Discrete Mathematics, which probably has as many undetermined problems as number theory, and one can hardly censure the author for it. So, despite this reservation, I am sure that his book will turn out to be very useful to those who are teaching the discipline for the first time, as well as whose who aren't, and, of course, their students.

10.1017/mag.2023.38 

GERRY LEVERSHA
*15 Maunder Road, Hanwell,*
*London W7 3PN*
e-mail: *g.leversha@btinternet.com*

**Bounded gaps between primes** by Kevin Broughan, pp. 590, £39.99 (paper), ISBN 978-1-108-79920-1, Cambridge University Press (2021).

Let $p_n$ denote the $n$th prime, and $g_n$ the $n$th gap between successive primes. The twin-primes conjecture states that $g_n = 2$ for infinitely many $n$. Being essentially an additive problem concerned with primes which are defined, ostensibly, in terms of multiplication only, such a basic problem is bound to be difficult. Indeed there was no method to tackle the problem until G. H. Hardy and J. E. Littlewood created their powerful circle method, which is particularly suitable for the investigation of such additive problems. The method shows that knowledge of the distribution of primes in arithmetic progressions, and exponential sums, will be central to the investigation of the conjecture. If the argument can be completed successfully then not only is the conjecture true, but the counting function for twin-primes has the asymptotic value $C_2 x/(\log x)^2$, where the twin-primes constant $C_2 \approx 1.32$ is an explicitly defined number. Computer counts of prime-twins up to various $x$ of modest size show that there is good agreement with the formula.

The prime number theorem shows that, for large $n$, the gap $g_n$ has the average value $\log p_n$, so that $c = \liminf \left(g_n / \log p_n\right) \leqslant 1$. From their circle method, together with the assumption of the Generalised Riemann Hypothesis (GRH), which implies that the distribution of primes in arithmetic progressions is uniform with respect to the moduli of the progressions, Hardy and Littlewood established in 1923 that $c < 1$. The dependence on GRH was removed by P. Erdős in 1940 and, for the remaining part of the twentieth century, the upper estimate for $c$ was duly brought down, but 'only' to slightly better than $c < \frac{1}{4}$. We say 'only' because of the spectacular achievements in the present century, and we put it in quotes because such achievements are based on the many important ideas and results from distinguished mathematicians in their approaches to the reduction of the bound. Thus, besides the development in sieve methods, there is the important Bombieri-Vinogradov theorem which shows that, in the distribution of primes up to $x$ in arithmetic progressions, there is uniformity with respect to the modulus varying up to nearly $\sqrt{x}$, at least in