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A quantitative version of the Kupka–Smale theorem

Y. YOMDIN
Ben Gurion University of the Negev, Beer-Sheva 84120, *Israel*

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Abstract. Let M^m be a compact, m-dimensional smooth manifold. The n-periodic point x of a diffeomorphism $f: M \to M$ is called γ-hyperbolic, for $\gamma \geq 0$, if the eigenvalues λ_j of $df^n(x)$ satisfy $||\lambda_j| - 1| \geq \gamma$, $j = 1, \ldots, m$. We prove that any C^k-diffeomorphism $f: M \to M$, $k \geq 3$, for any $\varepsilon > 0$ can be ε-approximated in C^k-norm by $f_\varepsilon : M \to M$ such that for any n each n-periodic point of f_ε is $(a(\varepsilon))^{n^\alpha}$-hyperbolic. Here $\alpha = \log_2 (m^2 + mk + k - 1) + 1$, $a(\varepsilon) = a_0 \cdot \varepsilon^{\frac{24}{11}(m+1)(m^2+mk+k-1)}$ and $a_0 > 0$ depends on f.

0. *Introduction*

A theorem of Kupka and Smale ([**4**], [**7**]) or, more precisely, one part of this theorem, asserts that all the periodic points of a generic diffeomorphism (or closed orbits of a generic flow) are hyperbolic. In many cases it is important to have more precise information of this type. First of all, sometimes there are no periodic points at all (or their existence is not known), while there are many recurrent trajectories. Thus the natural question is whether one has generically some hyperbolicity of these almost closed trajectories?

Another question, related to the first one, is the following: how does the 'hyperbolicity' (measured in one way or another) of periodic orbits of a typical flow depend on the length of the period?

From the Kupka–Smale theorem it follows that given a flow v we can obtain, by an arbitrary small perturbation, a new flow v' with all the closed orbits hyperbolic. This property also leads to a natural quantitative question: how big a 'hyperbolicity' of orbits of v' can we achieve, if the perturbations allowed should be bounded (in some C^k-metric) by given $\varepsilon > 0$?

The theorem of Kupka and Smale does not answer questions of this type, first of all because the main tool in its proof – the transversality theorem (see [**1**], [**8**]) – is essentially qualitative. In any application of transversality we obtain existence (and genericity in one or another sense) of 'non-degenerate' mappings, but no quantitative information about the 'measure of non-degeneracy'. We find that the source of this situation is the 'qualitativeness' of the Morse–Sard theorem (see [**6**]): it claims that the set of critical values of a differentiable mapping is small, but gives no information about the 'measure of regularity' of non-critical values.

In [**10**] the quantitative version of the Morse–Sard theorem was obtained. It gives the sharp geometric restrictions on the set of 'near-critical', rather than exactly critical, values of a differentiable mapping. Thus it allows one to describe the distribution of the values of this mapping with respect to the degree of their regularity. In [11] the corresponding general 'quantitative transversality theorem' is obtained.

In the present paper we use this quantitative transversality theorem to obtain a quantitative version of (the first part of) the Kupka–Smale theorem, which, in particular, answers the above-stated questions. As a consequence we obtain some additional geometric information about closed (and almost-closed) orbits of a typical flow. In particular, we give a lower bound for the distance between any two closed trajectories of periods, not exceeding a given T, and the upper bound for the number of such trajectories.

In fact, in this paper we need only the simplest case of a quantitative transversality theorem (and we give its simple proof in this special case in the addendum). The main difficulties in the proof of our version of Kupka–Smale theorem are of a 'dynamical' nature.

We do not touch in this paper the second part of the theorem of Kupka and Smale, namely, the question of transversality of stable and unstable manifolds of closed orbits. Here the quantitative results can be also obtained and they will appear separately.

The approach to the study of closed orbits, based on quantitative transversality, was proposed by M. Gromov in [3]. I would like to thank M. Gromov for suggesting this question to me and for numerous useful discussions. I would also like to thank the Max-Planck-Institut für Mathematik, where this paper was written, for its kind hospitality.

1. *Statement of main results and the sketch of the proof*

In this section we formulate our results only in the case of dynamical systems with discrete time (and the detailed proofs in §§ 2–5 below are also given only in this case). However, in § 6 we state the main theorems in the case of flows and describe the necessary (rather minor) modifications of the proofs in this case.

Let X be a compact differentiable (C^∞) manifold of dimension m. We fix some finite atlas (U_s, Ψ_s), $s=1,\ldots,p$, on X, $\Psi_s : B_1^m \tilde{\to} U_s \subset X$, where B_1^m is the unit ball in $\mathbb{R}^m$, such that all the derivatives of any fixed order of $\Psi_s^{-1}\circ\Psi_{s'}$ are bounded. We assume also that the images $\Psi_s(B_{\frac{1}{2}}^m)$, $s=1,\ldots,p$ of the open ball in $\mathbb{R}^m$ of radius $\frac{1}{2}$, cover X.

In addition, let some Riemannian metric on X be fixed. We denote by $\hat{\delta}$ the distance on X, defined by this metric. Denote by δ_0 the Lebesgue number of the covering $\Psi_s(B_{\frac{1}{2}}^m)$, $s=1,\ldots,p$, of X in the metric $\hat{\delta}$; thus any two points $x_1, x_2 \in X$ with $\hat{\delta}(x_1,x_2)\le\delta_0$ belong to $\Psi_s(B_{\frac{1}{2}}^m)$ for some $s=1,\ldots,p$, and, in particular, $x_1, x_2 \in U_s$.

For $k=1,2,\ldots$, let $D^k(X)$ be the space of k-times continuously differentiable diffeomorphisms $f: X\to X$, with the metric d_k defined by the atlas (U_s, Ψ_s).

For $f\in D^k(X)$ we define the constants $M_1(f),\ldots,M_k(f)$ as

$$M_i(f)=\max_{s,s'}\sup_{x\in U_s, f(x)\in U_{s'}}\|d^i(\Psi_{s'}^{-1}\circ f\circ\Psi_s(\Psi_s^{-1}(x))\|,$$

$$M_1'(f)=M_1(f^{-1}).$$

In our quantitative version of the theorem of Kupka and Smale we consider not only periodic, but also 'almost-periodic' points of a given diffeomorphism. In fact, even if the final results are stated for periodic points only, in the proof we must estimate deviations of orbits considered from a periodic behaviour. Thus we give the following definition:

Definition (1.1). Let $f \in D^k(X)$, $\delta \geq 0$ and $n \in \mathbb{N}$ be given. The point $x \in X$ is called *(n, δ)-periodic* for f, if $\hat{\delta}(x, f^n(x)) \leq \delta$.

In particular, for $\delta = 0$, an $(n, 0)$-periodic point is a periodic point of f of period n in the usual sense.

We also need some measure of hyperbolicity of almost periodic points. We obtain this using the charts of the atlas (U_s, Ψ_s). Of course, for the usual periodic points the definition below is the standard one.

Definition (1.2). For a linear mapping $L:\mathbb{R}^m \to \mathbb{R}^m$ let $\gamma(L) = \min_{1 \leq j \leq m} \|\lambda_j| - 1|$, where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of L.

Thus the linear mapping L is hyperbolic in the usual sense if and only if $\gamma(L) > 0$.

Definition (1.3). Let $f \in D^k(X)$ and let $x \in X$ be an (n, δ)-periodic point of f, $\delta \leq \delta_0$. For $\gamma > 0$, the point x is called an *(n, γ)-hyperbolic* (or simply *γ-hyperbolic*) point of f, if for any chart U_s, containing both x and $f^n(x)$,

$$\gamma(d(\Psi_s^{-1} \circ f^n \circ \Psi_s)(\Psi_s^{-1}(x))) \geq \gamma.$$

Now we can formulate our main results. For $m, k = 1, 2, \ldots$ denote by $\alpha(m, k)$ the constant

$$\alpha(m, k) = \log_2(m^2 + mk' + k' - 1) + 1,$$

where $k' = \max(k, 3)$.

THEOREM (1.4). *Let X be a compact smooth manifold of dimension m. In each space $D^k(X)$, $k = 1, 2, \ldots$, there is a dense subset W_k, such that diffeomorphisms $f \in W_k$ have the following property*:

For some constant $a > 0$ (depending on f) and each natural n, any (n, a^{n^α})-periodic point of f is (n, a^{n^α})-hyperbolic, where $\alpha = \alpha(m, k)$.

COROLLARY (1.5). *For any $f \in W_k$ there are constants $b > 0$ and C, depending on f, such that*:

(1) *For any two periodic points $x_1 \neq x_2$ of f with periods $\leq n$, the distance $\hat{\delta}(x_1, x_2)$ is at least b^{n^α}.*

(2) *The number of periodic points of f of period $\leq n$ does not exceed C^{n^α}.*

These results are implied by the following more precise statement:

THEOREM (1.6). *Let $k \geq 3$ and let $f \in D^k(X)$ be given. Then there exist constants $a_0 > 0$ and $\varepsilon_0 > 0$, depending only on M, on the atlas (U_s, Ψ_s) and on $M_1(f), \ldots, M_k(f)$, $M_1'(f)$, such that for any $\varepsilon > 0$, $\varepsilon \leq \varepsilon_0$, one can find $f' \in D^k(X)$, $d_k(f', f) \leq \varepsilon$, with the following property: for each natural n, any $(n, a(\varepsilon)^{n^\alpha})$-periodic point of f is $(n, a(\varepsilon)^{n^\alpha})$-hyperbolic. Here $a(\varepsilon) = a_0 \cdot \varepsilon^{\frac{24}{11}(m+1)(m^2+mk+k-1)}$, $\alpha = \alpha(m, k)$.*

Thus theorem 1.4, corollary 1.5 and theorem 1.6 answer the above stated questions, concerning the measure of hyperbolicity of periodic and almost periodic points of a typical, in some sense, diffeomorphism.

The main open question concerning the results above is related to the following fact: the order of decrease of the hyperbolicity with the growth of the period, we obtain, is overexponential. In particular, our bound C^{n^α} for the number of periodic points of periods $\leq n$ increases overexponentially with n. (Our $\alpha = \alpha(m, k)$ is greater than 1 for any $m, k = 1, 2, \ldots$. The first values of $\alpha(m, k)$ are the following:

$$\alpha(1,1) = \alpha(1,2) = \alpha(1,3) \simeq 3.585, \qquad \alpha(1,4) = 4, \ldots$$
$$\alpha(2,1) = \alpha(2,2) = \alpha(2,3) \simeq 4.585, \qquad \alpha(2,4) \simeq 4.907, \ldots).$$

On the other hand, the theorem of Artin and Masur [**2**] guarantees the exponential growth of the number of periodic points with the period for a dense set of diffeomorphisms.

(Notice, however, that in the case of flows no bound seems to be known for the number of periodic orbits of period $\leq T$; thus the bound of the form C^{T^α}, which we obtain in § 6 for a dense set of vector fields, seems to be new.)

In some points of the proof, given in this paper, we use, for the sake of simplicity, rather rough estimates. This concerns, first of all, the variant of the quantitative transversality theorem we use: it takes into account only three times differentiability of the diffeomorphism f.

Thus the value of the 'overexponentiality index' $\alpha(m, k)$ can essentially be improved, at least for big k. However, our method does not allow one to get $\alpha = 1$, i.e. the exponential rate, even if we use the best *a priori* possible estimates on each step. The technical reason is that we use some variant of the so-called Peixoto induction on the length of the period, and computations at this point lead to overexponentiality.

In more geometric terms we can say, that overexponentiality in our estimates appears as a result of the same difficulty as in many other questions in dynamical systems: it is difficult to control the influence of perturbations on recurrent trajectories.

In the case $X = S^1$ and for the space $D_0^k(S^1)$ of orientation-preserving diffeomorphisms this difficulty can be settled, and we obtain the following result (which is given here, as well as theorem 6.5, only to illustrate the above discussion; of course, in the one-dimensional case, much stronger results can be obtained).

THEOREM (1.7). *In each $D_0^k(S^1)$, $k = 1, 2, \ldots$, there is a dense subset W_k, such that diffeomorphisms $f \in W_k$ have the following property: for some $a > 0$, depending on f, any (n, a^n)-periodic point of f is (n, a^n)-hyperbolic.*

Also in the general situation there is a possibility of controlling the influence of perturbations on some special kind of recurrent trajectories. This allows us to improve significantly our bounds and, presumably, to get exponential rate of the decreasing of hyperbolicity, in some additional situations. We hope to publish these results separately.

Another important remark concerns the notion of genericity, appropriate for the quantitative results above. If we consider the periodic points with periods not exceeding some given number, then the set of diffeomorphisms, satisfying inequalities of theorem 1.4 with some fixed $a > 0$ (and with signs $<, >$ instead of $\leq, \geq$) is open, but not dense. Hence we cannot expect the set of 'good' diffeomorphisms to be the countable intersection of everywhere dense open sets. In this paper we prove only that the set of 'good' diffeomorphisms is dense. However, a much more precise description of the geometry of this set is possible. This description requires the infinite-dimensional version of the quantitative transversality theorem, as well as some new notions concerning the geometry of infinite-dimensional spaces, and it will appear separately.

In §§ 2–5 below we prove theorem 1.6, and then at the end of § 5 we obtain, as easy consequences, theorem 1.4 and corollary 1.5. We do not prove in this paper theorem 1.7 and the corresponding result for flows – theorem 6.5.

Since the proof of theorem 1.6 is rather long, we give here a short sketch of the main steps.

First of all, we consider the family of perturbations f_t^ρ of a given diffeomorphism $f: X \to X$. Here $\rho > 0$ is a real parameter and t is a collection of affine transformations of $\mathbb{R}^m$. Roughly, to obtain f_t^ρ, we cover X by some family of balls of radius ρ, perform on each ball the diffeomorphism, which is identical near the boundary and coincides with the corresponding component of t on some smaller ball. Then we take a composition of f with these diffeomorphisms.

The main property of these perturbations is the following: assume that $x \in X$ belongs to one of the balls of the family above, while $f(x), f^2(x), \ldots, f^{n-1}(x)$ lie outside of it.

Let t_j be the component of t, corresponding to our ball. Then t_j acts non-degenerately on $f^n(x)$, $df^n(x)$, and the measure of this non-degeneracy decreases exponentially with n (see lemma 2.3 below).

Now the proof of theorem 1.6 goes through the induction on the length of the period, similar to the Peixoto induction (see [1], [5]). Assume that for a given diffeomorphism $f \in D^k(X)$, we can find $f_1 \in D^k(X)$, with $d_k(f_1, f) \leq \varepsilon/2$, such that the property of theorem 1.6 is satisfied for all the almost periodic points of f_1 with periods $\leq n$.

Now we want to perturb f_1 slightly into $f_2 \in D^k(X)$, such that $d_k(f_1, f_2) \leq \varepsilon/4$, the 'good' behaviour of points with periods $\leq n$ is preserved, and all the almost periodic points of f_2 with periods between n and $2n$ satisfy the required conditions.

To do this we subdivide all the almost periodic points of f_1 with periods between n and $2n$ into two parts: those which are 'simple', i.e. their 'intermediate' iterations do not return too close to the initial point, and those whose orbits are 'almost iterations' of shorter almost-closed orbits. Now the perturbations act non-degenerately on the points of the first type, and by transversality arguments we can find a perturbation, making them hyperbolic. The points of the second type are hyperbolic *a priori*, as iterations of points with shorter period, which are hyperbolic by the induction assumption.

This is the main step of the proof, where all the estimates come together and where the rate of a hyperbolicity decrease is determined, so we describe it more accurately.

For $\eta > 0$ we call the point $x \in X$ (q, η)-*simple*, if $\hat{\delta}(x, f^j(x)) \geq \eta$, $j = 1, \ldots, q-1$. The main 'dynamical' ingredient in our proof is the following statement (see lemma 3.1 below): if the almost periodic point of period q is not (q, η)-simple (for sufficiently small η), it is an 'almost iteration' of an almost periodic point with period $l < q$, dividing q, and the 'accuracy' of this almost iteration is of order $C^q \eta$.

Now denote the hyperbolicity of almost periodic points with periods $\leq n$ of f_1 by γ_1. If we want almost iterations of these points to be hyperbolic, the accuracy $C^{2n}\eta$ should be sufficiently small with respect to γ_1. This condition determines the value of the parameter η as a function of γ_1. Fixing this η, we obtain the hyperbolicity of all the points with periods between n and $2n$, which are not η-simple.

If we want our perturbations f_t^ρ to act non-degenerately on the η-simple points, we should have ρ sufficiently small with respect to η, and this condition determines the value of ρ as a function of γ_1.

Now the maximal value $\tilde{\nu}$ of the parameter t in our perturbations f_t^ρ is determined by the condition $d_k(f_2, f_1) \leq \varepsilon/4$, which transforms into $\tilde{\nu} \leq C_1^{2n} \rho^k \cdot \varepsilon$ (the smaller value is the radius ρ of the balls, on which the perturbation is concentrated, the smaller value should be t to keep the C^k norm $\varepsilon/4$ of the perturbation). Thus in turn we obtain $\tilde{\nu}$ as a function of γ_1 and ε.

Here we apply the quantitative transversality theorem (theorem 4.2 and its conclusion in our situation: lemma 4.4 below). We obtain the existence of the value t_0 of the parameter t, such that $\|t_0\| \leq \tilde{\nu}$, and all the (q, γ_2)-periodic and (q, η)-simple points of $f_2 = f_{1,t_0}^\rho$ are γ_2-hyperbolic, for $n \leq q \leq 2n$ where γ_2 is given as an expression in terms of the maximal size $\tilde{\nu}$ of the allowed perturbations. Thus we obtain at last γ_2 as a function of γ_1 and ε.

Now proceeding by induction we build the sequence of diffeomorphisms $f_1, f_2, \ldots \in D^k(X)$, converging in the C^k-topology to some $f' \in D^k(X)$ such that $d_k(f', f) \leq \varepsilon$ and for all $i = 1, 2, \ldots$ and any q, $2^{i-1} < q \leq 2^i$, each (q, γ_i)-periodic point of f' is γ_i-hyperbolic. In the sequence γ_i each term γ_i is given by the above-described expression through γ_{i-1} and ε. Solving this recurrent relation we obtain the bounds for hyperbolicity, given in theorem 1.6.

The paper is organized in the following way: in § 2 we describe the perturbations f_t^ρ and their action on the diffeomorphism f and its iterations. In § 3 we prove that the trajectory which is not 'simple' is an iteration of a shorter trajectory. In § 4 we formulate the quantitative transversality theorem and apply it in our situation. In § 5 we complete the proof of the main results for the case of discrete time. In § 6 we formulate our results for the case of flows and indicate the necessary alterations in the proofs. In the addendum we prove the special version of the quantitative transversality theorem used in this paper.

2. *Construction of perturbations and some preliminary results*

First we construct some family of diffeomorphisms of the Euclidean space $\mathbb{R}^m$. Let L_m be the space of linear mappings of $\mathbb{R}^m$ with the standard norm, and let

$$L'_m = \{L \in L_m : \|L\| \leq \tfrac{1}{2}\}.$$

Denote by T the direct product $T = B^m_{\frac{1}{2}} \times L'_m$, where $B^m_{\frac{1}{2}}$ is, as above, the ball of radius $\frac{1}{2}$ centred at the origin of $\mathbb{R}^m$.

Let us fix some C^∞-smooth function $\omega:[0,\infty)\to[0,\infty)$ such that $\omega(x)=1$ for $0\le x\le 1$ and $\omega(x)=0$ for $x\ge 7$.

Now for any $t=(v, L)\in T$ let $h_t:\mathbb{R}^m\to\mathbb{R}^m$ be defined by

$$h_t(x) = x + \omega(\|x\|)(v + L(x)).$$

One can choose ω in such a way that for any $t\in T$, $h_t:\mathbb{R}^m\to\mathbb{R}^m$ is a diffeomorphism.

Now we translate diffeomorphisms h_t to the manifold X. Let some $\rho>0$, $\rho\le\frac{1}{20}$, be given. Consider in $B^m_{\frac{1}{2}}\subset\mathbb{R}^m$ a regular $\frac{1}{10}\rho$-net ξ_i, $i=1,2,\ldots$. Now for a given $\nu>0$, $\nu\le 1$ and for $i=1,2,\ldots, s=1,\ldots,p$, define the diffeomorphism $h^{\rho,\nu}_{i,s,t}: X\to X$, $t\in T$, as follows:

$$h^{\rho,\nu}_{i,s,t}(x) = \Psi_s\left((\xi_i + \rho h_{\nu t}\left(\frac{1}{\rho}(\Psi_s^{-1}(x)-\xi_i)\right)\right)$$

for $x\in U_s$, and $h^{\rho,\nu}_{i,s,t}(x)=x$ for $x\notin U_s$. (Here (U_s,Ψ_s), $s=1,\ldots,p$, is the above fixed atlas on the manifold X.)

Thus the $h^{\rho,\nu}_{i,s,t}$ are correctly defined diffeomorphisms, concentrated in the images (under all the coordinate mappings Ψ_s) of the balls of radius 7ρ, centred at the points ξ_i.

The additional parameter ν allows us to scale the perturbations without changing the space of parameters.

Now let us fix some ordering $h^{\rho,\nu}_{q,t}$, $q=1,\ldots,N(\rho)$, of all the diffeomorphisms $h^{\rho,\nu}_{i,s,t}$. Let $T_\rho = T^{N(\rho)}$. For any $t=(t_1,\ldots,t_{N(\rho)})\in T_\rho$ define the diffeomorphism $h^{\rho,\nu}_t: X\to X$ as the composition

$$h^{\rho,\nu}_t = h^{\rho,\nu}_{N(\rho),t_{N(\rho)}}\circ\cdots\circ h^{\rho,\nu}_{1,t_1}.$$

We perturb diffeomorphisms $f: X\to X$, composing them with $h^{\rho,\nu}_t$. Let ρ and ν be fixed, $0<\rho\le\frac{1}{20}$, $0<\nu\le 1$, and let $f\in D^k(X)$. For any $t\in T_\rho$ we denote by $f^{\rho,\nu}_t$ (or, shortly, by f_t) the diffeomorphism $f\circ h^{\rho,\nu}_t\in D^k(X)$.

The following properties of perturbations f_t can be proved by straightforward computations:

LEMMA (2.1). *Let ρ and ν as above be fixed, and let $f\in D^k(X)$, $k=1,2,\ldots$. Then there is a constant K_1, depending only on $M_1(f),\ldots,M_k(f)$, such that for any $t\in T_\rho$ and for each natural n,*

$$d_k((f^{\rho,\nu}_t)^n, f^n)\le K_1^n\nu(1/\rho)^{k-1}.$$

In particular, for any $x\in X$,

$$\hat{\delta}(f^n_t(x), f^n(x))\le K_1^n\cdot\nu\cdot\rho,$$

$$\|df^n_t(x)-df^n(x)\|\le K_1^n\nu,$$

where the norm is computed in any chart U_s containing both $f^n_t(x)$ and $f^n(x)$.

(Here and below our notation is chosen in the following way: given a diffeomorphism $f\in D^k(X)$, we denote by K_j 'big', and by a_j 'small' constants, depending only on $M_1(f)$, $M'_1(f),\ldots, M_k(f)$, or on a part of these data, which will be used in the course of the paper; C_j and c_j denote, respectively, 'big' and 'small' constants, depending on the same data, which are used only inside the proof of some specific estimate.)

Now we show that our family of perturbations is big enough to act non-degenerately on any trajectory of f, which is sufficiently 'non-recurrent'. It is convenient to define some auxiliary mapping to the first order jet-space, associated with $f: X \to X$.

Let $f \in D^k(X)$ be given and let $\rho > 0$, $\rho \le \frac{1}{20}$, and $\nu \le 1$ be fixed. Assume that for a given subset $Q \subset X$ and for a natural n, Q and $f^n(Q)$, as well as $f_t^n(Q)$ for any $t \in T_\rho$, are contained in the same coordinate neighbourhood U_s. We fix this s and define the mapping Φ_s with respect to the local coordinates in U_s: for $x \in Q$, $\Phi_s(n, x, t) = (f_t^n(x), df_t^n(x))$.

We consider Φ_s as the mapping

$$\Phi_s(n, x, \cdot): T_\rho \to \mathbb{R}^m \times L_m,$$

computing $f_t^n(x)$ in local coordinates in U_s. To simplify the notation we omit indices ρ, ν in the notation for Φ_s.

The restriction of Φ_s on any factor T in $T_\rho = T^{n(\rho)}$ is the mapping of the spaces of the same dimension. We show that in our situation, for at least one factor T in T_ρ this restriction is non-degenerate. As usual in our quantitative approach, we need some measure of this non-degeneracy:

Definition (2.2). For a linear mapping $L: \mathbb{R}^p \to \mathbb{R}^q$ define $\kappa(L)$ as the minimal semiaxis of the ellipsoid $L(B_1^p) \subset \mathbb{R}^q$, where B_1^p is the unit ball centred at the origin of $\mathbb{R}^p$.

Let $S > 0$ be a constant such that for any $x_1, x_2 \in X$, contained in some U_s,

$$\frac{1}{S}\hat{\delta}(x_1, x_2) \le \|\Psi_s^{-1}(x_1) - \Psi_s^{-1}(x_2)\| \le S\hat{\delta}(x_1, x_2).$$

LEMMA (2.3). *Let $f \in D^k(X)$, $k \ge 3$. There are constants $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and K_2, depending only on $M_1(f)$, $M_1'(f)$, $M_2(f)$, $M_3(f)$, with the following property:*

Let $\rho > 0$, $\rho \le \frac{1}{20}$, be fixed, and let $Q \subset X$ be a subset of diameter $\le (1/10S)\rho$ in metric $\hat{\delta}$.

Assume that for some n and for any $x \in Q$, $\hat{\delta}(f^i(x), x) \ge 20\ S\rho$, $i = 1, 2, \ldots, n-1$. Then there exists j, $1 \le j \le N(\rho)$, such that for any $\nu \le a_1^n$, $x \in Q$ and $t \in T$,

(1) $\kappa(d_{t_j}\Phi_s(n, x, t)) \ge a_2^n \nu\rho$, where U_s is any chart containing Q and $f_t^n(Q)$ for $t \in T_\rho$. Moreover, if $\nu \le a_3^n \rho$, we have in addition:

(2) Denote by $\bar{\Phi}: T \to \mathbb{R}^m \times L_m$ the restriction of Φ_s to the j'th factor T in T_ρ. Then $\bar{\Phi}$ is one to one and for any $\tau_1, \tau_2 \in T$,

$$\|\tau_2 - \tau_1\| \le K_2^n(1/\nu\rho)\|\bar{\Phi}(\tau_2) - \bar{\Phi}(\tau_1)\|.$$

Proof. First of all, we note that if we put $a_1 = 1/K_1$, where K_1 is the constant defined in lemma 2.1, and if $\nu \le a_1^n$, then for any $x \in Q$ and $t \in T$,

$$\delta(f_t^i(x), x) \ge 19\ S\rho, \qquad i = 1, \ldots, n-1.$$

Since the diameter of Q is at most $(1/10S)\rho$, the set $\Psi_s^{-1}(Q)$ is contained in the ball B of radius ρ, centred at some point ξ_i of the net, introduced in the definition of the perturbations f_t^ρ, while all the points $\Psi_s^{-1}(f_t^i(x))$, $x \in Q$, $t \in T_\rho$, $i = 1, \ldots, n-1$, lie outside the ball of radius 10ρ, centred at the same ξ_i.

Now let $h_{j,t_j}^{\rho,\nu} = h_{i,s,t_j}^{\rho,\nu}$ be the diffeomorphism of X corresponding to the point ξ_i and to the chart U_s. By the definition of h we obtain that $h_{j,t_j}^{\rho,\nu}$ acts as the affine transformation $t_j \in T$ on the initial point of any trajectory $x, f_t(x), \ldots, f_t^n(x)$, $x \in Q$, and acts trivially on all the iterations. Straightforward computations of the differentials now prove the inequality (1) of lemma 2.3.

To prove the property (2), we note that the norm of the second derivative of $\bar{\Phi}$ with respect to $\tau \in T$, does not exceed $C^n\nu^2$ for any $x \in Q$, $t \in T_\rho$, where C depends only on $M_1(f)$, $M_2(f)$, $M_3(f)$. This follows by direct computations of the derivatives of $f_t^{\rho,\nu}$.

Now if the inequality $C^n\nu^2 \le \frac{1}{10}a_2^n \nu\rho$ is satisfied, which is implied by the stronger inequality $\nu \le a_3^n\rho$, where $a_3 = a_2/10C$, then the second derivative of Φ does not exceed $\frac{1}{10}$ of the 'non-degeneracy' of the first differential of $\bar{\Phi}$. The standard application of the inverse function theorem now proves the second part of lemma 2.3, with $K_2 = 2/a_2$. □

We also need some estimates concerning the behaviour of the 'hyperbolicity' of a given mapping under perturbations. Recall that for a linear mapping $L:\mathbb{R}^m \to \mathbb{R}^m$ the 'hyperbolicity' $\gamma(L)$ is defined as

$$\gamma(L) = \min_{1\le j\le m} \left| |\lambda_j| - 1 \right|,$$

where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of L. The following two inequalities can be proved by elementary linear algebra considerations:

LEMMA (2.4). *Let $L \in L_m$, $\gamma(L) > 0$. Denote by $M(L)$ the maximum of $\|L\|$, $\|L^{-1}\|$. Then for any $\Delta \in L_m$,*

$$\gamma(L+\Delta) \ge \gamma(L) - (4M(L)/\gamma(L))\|\Delta\|.$$

Now let $Z_m \subset L_m$ be the set of non-hyperbolic mappings, $Z_m = \{L \in L_m : \gamma(L) = 0\}$. Clearly, Z_m is a semi-algebraic subset in L_m of codimension 1.

LEMMA (2.5). *For any $L \in L_m$,*

$$\gamma(L) \ge \operatorname{dist}(L, Z_m),$$

where dist (L, Z_m) *is the distance from L to the set Z_m in the usual norm in L_m.*

We return to diffeomorphisms $f: X \to X$. If a closed orbit of f is hyperbolic, then all the iterations of this orbit are also hyperbolic. The following lemma gives conditions under which 'almost iterations' of a hyperbolic almost closed orbit remain hyperbolic.

LEMMA (2.6). *Let $f \in D^k(X)$, $k \ge 2$, and let $x \in X$ be an (l, δ)-periodic point of f, which is (l, γ)-hyperbolic, $\delta > 0$, $\gamma > 0$.*

For some $n = pl$ and for any i, $0 \le i \le n$, $i = ql + r$, $r < l$, let the following inequality be satisfied: $\hat{\delta}(f^i(x), f^r(x)) \le \delta$. (I.e. the trajectory $x, f(x), \ldots, f^n(x)$ is the pth 'almost iteration' of the trajectory $x, f(x), \ldots, f^l(x)$.)

Then the point x, which is, by assumption, (n, δ)-periodic for f, is (n, γ')-hyperbolic, with $\gamma' = \gamma - K_3^n\delta/\gamma$, where K_3 depends only on $M_1(f)$, $M_2(f)$. In particular, for $\delta \le \xi \cdot \gamma^2/K_3^n$, $\gamma' \ge (1-\xi)\gamma$.

Proof. It is sufficient to make the computations in a fixed coordinate neighbourhood U_s, containing all the points

$$x_q = f^{ql}(x), \qquad q = 0, 1, \ldots, p, \quad x_0 = x.$$

We have: $df^n(x) = df^l(x_{p-1}) \circ df^l(x_{p-2}) \circ \cdots \circ df^l(x)$. Since $\|d^2 f^l\| \leq C^l$, where the constant C depends only on $M_1(f)$, $M_2(f)$, and since, by assumption, $\hat{\delta}(x_q, x) \leq \delta$, $q = 1, \ldots, p$, we obtain:

$$\|df^l(x_q) - df^l(x)\| \leq C^l \delta,$$

and we can write $df^l(x_q) = df^l(x) + \Delta_q$, where $\|\Delta_q\| \leq C^l \delta$. Hence

$$\begin{aligned} df^n(x) &= (df^l(x) + \Delta_{p-1}) \circ \cdots \circ (df^l(x) + \Delta_1) \circ df^l(x) \\ &= [df^l(x)]^p + \Delta', \end{aligned}$$

where

$$\|\Delta'\| \leq 2^p \|df^l(x)\|^p \cdot C^l \delta \leq C_1^n \delta,$$

with $C_1 = 2C \cdot M_1(f)$. But

$$\gamma([df^l(x)]^p) \geq \gamma(df^l(x)) \geq \gamma,$$

and $\|[df^l(x)]^p\| \leq M_1^n(f)$, therefore by lemma 2.4,

$$\gamma(df^n(x) \geq \gamma - (4M_1^n(f)/\gamma) C_1^n \delta \geq \gamma - K_3^n \delta / \gamma = \gamma',$$

where $K_3 = 4M_1(f) C_1$. Substituting $\delta = \xi \cdot \gamma^2 / K_3^n$, we obtain $\gamma' = (1 - \xi)\gamma$. □

3. *Lemma on iterated almost periodic trajectories*

This result, although elementary, is the main 'dynamical' ingredient of our proof.

The following statement is evident for usual periodic trajectories: if $f^n(x) = x$ and if $f^i(x) = f^j(x)$ for some $0 \leq i < j \leq n$, $(i, j) \neq (0, n)$, then for some $l < n$, dividing n, $f^l(x) = x$, and for any i, $0 \leq i \leq n$, $i = ql + r$, $r < l$, $f^i(x) = f^r(x)$; in other words, the orbit $x, f(x), \ldots, f^n(x)$ is the n/l'th iteration of the orbit $x, f(x), \ldots, f^l(x)$.

But in the case of almost periodic trajectory and 'almost closing' on some intermediate step, we cannot expect *a priori* the behaviour similar to that described above. Clearly there can be recurrent trajectories, which are not 'almost iterations' of some shorter trajectory.

The following lemma shows that if the 'closing' of our trajectory at the end and in the 'middle' is exponentially small with respect to the length of the trajectory, then it behaves, essentially, as in the case of exactly closed trajectories, described above.

LEMMA (3.1). *Let $f \in D^k(X)$, $k \geq 1$. There exists a constant K_4, depending only on $M_1(f)$, $M_1'(f)$, such that the following alternative is satisfied:*

For $\delta > 0$ and $n \in \mathbb{N}$, let $x \in X$ be an (n, δ)-periodic point of f. Then for any $\eta \geq \delta$, either

(a) $\hat{\delta}(f^i(x), f^j(x)) \geq \eta$ *for any* $0 \leq i < j \leq n$, $(i, j) \neq (0, n)$; *or*

(b) *there is $l < n$, dividing n, such that x is an $(l, K_4^n \eta)$-periodic point of f, and for any i, $0 \leq i \leq n$, $i = ql + r$, $r < l$,*

$$\hat{\delta}(f^i(x), f^r(x)) \leq K_4^n \eta.$$

Proof. Let the assumption (a) be false. Then there are $i<j$, $(i,j)\neq(0,n)$, such that $\hat{\delta}(x_i,x_j)\leq\eta$. (We write x_i for $f^i(x)$.) We find $l<n$, dividing n, such that $\hat{\delta}(x_0,x_l)\leq K_4^n\eta$, using the Euclidean division algorithm. Denote $j-i$ by b and let $n=qb+r$, $r<b$.

LEMMA (3.2). *In the situation considered,*

$$\hat{\delta}(x_0,x_b)\leq C^n\eta,$$
$$\hat{\delta}(x_0,x_r)\leq C^n\eta,$$

where C depends only on $M_1(f)$, $M_1'(f)$.

Proof. First of all, we note that if $\hat{\delta}(x_i,x_j)\leq\alpha$, then for any s, positive or negative, $\hat{\delta}(x_{i+s},x_{j+s})\leq M^{|s|}\alpha$, where $M=\max(M_1(f),M_1'(f))$. Substituting $s=-i$, we get

$$\hat{\delta}(x_0,x_b)\leq M^i\eta\leq M^n\eta.$$

Now, for any $p\leq n/b$,

$$\hat{\delta}(x_0,x_{pb})\leq(2M)^n\eta.$$

Indeed, we have

$$\begin{aligned}\hat{\delta}(x_0,x_b)&\leq M^n\eta,\\ \hat{\delta}(x_b,x_{2b})&\leq M^n\eta,\\ &\vdots\\ \hat{\delta}(x_{(p-1)b},x_{pb})&\leq M^n\eta.\end{aligned}$$

Adding these inequalities we obtain

$$\hat{\delta}(x_0,x_{pb})\leq nM^n\eta\leq(2M)^n\eta.$$

Now,

$$\hat{\delta}(x_{qb},x_n)\leq\hat{\delta}(x_{qb},x_0)+\hat{\delta}(x_0,x_n)\leq(2M)^n\eta+\delta\leq(3M)^n\eta,$$

since, by assumption, $\delta\leq\eta$.

Finally, applying f^{-qb}, we obtain $\hat{\delta}(x_0,x_r)\leq M^n(3M)^n\eta$, and the inequalities of lemma 3.2 follow, if we put $C=3M^2$. □

Now we apply the Euclidean algorithm to find the greatest common divisor of numbers n and b:

$$\begin{aligned}n&=qb+r,\\ b&=q_1r+r_1,\\ r&=q_2r_1+r_2,\\ &\vdots\\ r_{s-2}&=q_sr_{s-1}+r_s,\\ r_{s-1}&=q_{s+1}r_s.\end{aligned}$$

Here r_s is the gcd of n and b. Put $l=r_s$. By lemma 3.2,

$$\hat{\delta}(x_0,x_b)\leq C^n\eta,$$
$$\hat{\delta}(x_0,x_r)\leq C^n\eta.$$

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Applying lemma 3.2 once more to the orbit $x_0, \ldots, x_b$, with $(i, j) = (0, r)$, we obtain

$$\hat{\delta}(x_0, x_r) \le C^n \eta,$$
$$\hat{\delta}(x_0, x_{r_1}) \le C^b \cdot C^n \eta,$$

and then, successively,

$$\begin{aligned} \hat{\delta}(x_0, x_{r_1}) &\le C^{n+b}\eta, \\ \hat{\delta}(x_0, x_{r_2}) &\le C^{n+b+r}\eta, \\ &\vdots \\ \hat{\delta}(x_0, x_{r_s}) &\le C^{n+b+r+\cdots+r_{s-2}}\eta. \end{aligned}$$

Since the sum of the remainders $r + r_1 + \cdots + r_{s-2}$ in the Euclidean algorithm does not exceed n, we obtain

$$\hat{\delta}(x_0, x_l) \le C^{3n}\eta.$$

Hence for any j, $\hat{\delta}(x_j, x_{j+l}) \le M^n C^{3n}\eta$, and for the same reason as above, $\hat{\delta}(x_j, x_{j+pl}) \le (2M)^n C^{3n}\eta$, for any p, j such that $0 \le j, j+pl \le n$.

If we put $K_4 = 2MC^3$, we have $\hat{\delta}(x_0, x_l) \le K_4^n \eta$, $\hat{\delta}(x_i, x_r) \le K_4^n \eta$ for any i, $0 \le i \le n$, $i = ql + r$, $r < l$.

Lemma 3.1 is proved. □

4. *Hyperbolization of simple trajectories*

In this section we show how to perturb a given diffeomorphism $f: X \to X$ in order to obtain a new one f' with all the 'simple' almost periodic points, up to some fixed period, hyperbolic.

To get the required perturbation we apply the quantitative transversality theorem in its simplest form, concerning the case of 'empty intersections'. So first of all we state this theorem here.

Although in our applications of quantitative transversality we work with the usual Lebesgue measure, it is convenient to formulate (and to prove) the theorem, using another geometric tool: the metric entropy (or capacity).

Definition (4.1). Let $A \subset \mathbb{R}^s$ be a bounded subset. For any $\xi > 0$ define $M(\xi, A)$ as the minimal number of balls of radius ξ covering A. ($H_\xi(A) = \log_2 M(\xi, A)$ is called the ξ-entropy of A.)

Let $Q \subset \mathbb{R}^m$ be a closed domain with the following property:

for any $x_1, x_2 \in Q$ there is a curve in Q, connecting x_1 and x_2, of length $\le S_1 \|x_2 - x_1\|$.

Let $F: Q \times B^q \to \mathbb{R}^q$ be a continuously differentiable mapping (where B^q is the unit ball in $\mathbb{R}^q$), satisfying the following conditions:

(1) For any $(x, t) \in Q \times B^q$,

$$\|d_x F(x, t)\| \le R_1.$$

(2) For any $x \in Q$ the mapping $F(x, \cdot): B^q \to \mathbb{R}^q$ is one to one and for any $t_1, t_2 \in B^q$,

$$\|t_2 - t_1\| \le R_2 \|F(x, t_2) - F(x, t_1)\|.$$

Let the bounded subsets $A \subset Q$ and $A' \subset \mathbb{R}^q$ be given. Define $\Delta_F(A; A') \subset B^q$ as the set of all $t \in B^q$ such that for some $x \in A$, $F(x, t) \in A'$.

THEOREM (4.2). *For any $\xi > 0$ and for $\xi' = 2(S_1R_2(1+R_1)+1)\xi$,*

$$M(\xi', \Delta_F(A, A')) \leq M(\xi, A)M(\xi, A').$$

The proof of this theorem is given in the addendum. Roughly, theorem 4.2 is related to the usual transversality theorem in the following way: in the above situation the usual transversality results assert that if $\dim A + \dim A' < q$, then we can find $t \in B^q$ such that $F(\cdot, t)(A) \cap A' = \emptyset$, while theorem 4.2 allows us to find $\xi > 0$, such that for some $t \in B^q$ the image under $F(\cdot, t)$ of the ξ-neighbourhood of A does not intersect the ξ-neighbourhood of A'.

Definition (4.3). For $\eta > 0$ the point $x \in X$ is called an (n, η)*-simple point* of a diffeomorphism $f: X \to X$, if $\hat{\delta}(f^j(x), x) \geq \eta$ for $j = 1, 2, \ldots, n-1$.

Below we fix some $f \in D^k(X)$, $k \geq 3$. Let $\eta > 0$, $\eta \leq S$ be given. We fix $\rho = \eta/100S$. (Here S is the transfer constant from metric $\hat{\delta}$ to metrics in coordinate neighbourhoods U_s, defined in § 2.)

For any ν, $0 < \nu \leq 1$, and for any $t \in T_\rho$, let $f_t^\nu = f_t^{\rho,\nu}$ be the perturbation of f, defined in § 2.

LEMMA (4.4). *Let η, $0 < \eta \leq S$, be given, and let $\rho = \eta/100S$. There is a constant $a_4 > 0$, depending only on $M_1(f)$, $M_1'(f)$, $M_2(f)$, $M_3(f)$, such that for any natural N and for any ν, $0 < \nu \leq a_3^N\rho$ (where a_3 is the constant defined in lemma 2.3), there exists $t_0 \in T_\rho$ for which the diffeomorphism $f' = f_{t_0}^\nu$ has the following property:*

for $\gamma = a_4^N \nu^{m+1} \rho^{m^2+m}$, and for any $n \leq N$, each (n, η)-simple and (n, γ)-periodic point of f' is (n, γ)-hyperbolic.

Proof. First, let us fix some $n \leq N$. Let $\Omega_n' \subset X$ be the set of $(n, \frac{1}{2}\eta)$-simple points of f.

Consider the covering of X by the sets Q_i of the following form: we subdivide $\mathbb{R}^m$ into regular cubes with the edge $\eta\nu/1000\sqrt{m}\, S^3 M_1^n(f)$, take the images of those cubes which are contained in B_1^m under all the coordinate mappings Ψ_s, and fix some ordering Q_i of these images. (We assume that $\nu > 0$, $\nu \leq a_3^n\rho$, is fixed.)

Let Ω_n be the union of those Q_i which intersect Ω_n'. Thus any $(n, \frac{1}{2}\eta)$-simple point of f belongs to Ω_n. On the other hand, since $\|df^j\| \leq M_1(f)^j \leq M_1(f)^n$, by the choice of the diameter of sets Q_i we obtain that any point of Ω_n is $(n, \frac{1}{3}\eta)$-simple for f.

Let us consider the measure $\mathcal{m}$ in $\mathbb{R}^m \times L_m$, proportional to the usual Lebesgue measure and such that $\mathcal{m}(T) = 1$, where T, as above, is the direct product of the balls of radius $\frac{1}{2}$ in $\mathbb{R}^m$ and L_m, respectively. By the same symbol $\mathcal{m}$ we denote the corresponding product measure in $T_\rho = T^{N(\rho)}$. Thus $\mathcal{m}(T_\rho) = 1$. We also denote by μ the Lebesgue measure on X, associated with the above fixed Riemannian metric.

Let us fix some $Q_i \subset \Omega_n$.

LEMMA (4.5). *For $\lambda > 0$, $\lambda \leq a_5^n\rho\nu$, let $\Delta_i(\lambda) \subset T_\rho$ denote the set of $t \in T_\rho$, for which there is some $x \in Q_i$, such that x is an (n, λ)-periodic but not an (n, λ)-hyperbolic point of f_t^ν. Then*

$$\mathcal{m}(\Delta_i(\lambda)) \leq K_5^n(1/\nu)^{m+1}(1/\rho)^{m^2+m}\mu(Q_i) \cdot \lambda,$$

where the constants a_5 and K_5 depend on the same parameters of f as above.

Proof. First we note that the conditions of lemma 2.3 are satisfied for f and any set Q_i as above. Indeed, by construction, the diameter of each Q_i in the metric $\hat{\delta}$ does not exceed $\eta/1000S^2 = \rho/10S$. On the other hand, each point of Q_i belongs to Ω_n and hence is $(n, \frac{1}{3}\eta)$-simple, and $\frac{1}{3}\eta = (100/3)S\rho > 20\ S\rho$.

Lemma 2.3 now guarantees the existence of the index j, such that the j'th component $t_j \in T$ of the parameter $t \in T_\rho$ acts non-degenerately on the n'th iteration of f at points $x \in Q_i$.

Let us fix this j and represent each $t \in T_\rho$ as $t = (t', t_j)$. Clearly, it is sufficient to prove that for any t' the measure $m(\Delta_{i,t'}(\lambda))$ in T does not exceed the required value, where $\Delta_{i,t'}(\lambda) \subset T$ is the set of $\tau = t_j \in T$, for which $(t', \tau) \in \Delta_i(\lambda)$.

Let us fix some t' and for a given $\tau = t_j \in T$ denote by $f_\tau : X \to X$ the diffeomorphism $f_\tau = f_t^{\rho,\nu}$, $t = (t', \tau)$.

The following computations are made in some fixed coordinate neighbourhood U_s, containing Q_i and $f_\tau^n(Q_i)$, $\tau \in T$.

Define the mapping $\Phi : Q_i \times T \to \mathbb{R}^m \times L_m$ by

$$\Phi(x, \tau) = (f_\tau^n(x) - x, df_\tau^n(x)).$$

We want to apply theorem 4.2 to the mapping Φ. We have:

(1) $\|d_x\Phi\| \le C^n$, where the constant C depends only on $M_1(f)$, $M_2(f)$.

For any $x \in Q_i$ the mapping $\Phi(x, \cdot) : T \to \mathbb{R}^m \times L_m$ coincides, up to a parallel translation, with the mapping $\bar{\Phi}$, defined in lemma 2.3(2). Since the condition $\nu \le a_3^n \rho$ of lemma 2.3 is also satisfied by our assumptions, this lemma gives us the following:

(2) $\Phi(x, \cdot) : T \to \mathbb{R}^m \times L_m$ is one to one and for any $\tau_1, \tau_2 \in T$,

$$\|\tau_2 - \tau_1\| \le K_2^n(1/\nu\rho)\|\Phi(x, \tau_2) - \Phi(x, \tau_1)\|.$$

Thus the assumptions of theorem 4.2 are satisfied for Φ with constants $R_1 = C^n$ and $R_2 = K_2^n(1/\nu\rho)$. The constant S_1, characterizing the geometry of Q_i, in our case, clearly, does not exceed S^4.

As the set A we take all the Q_i. Clearly, $M(\xi, Q_i) \le C_1\mu(Q_i)\cdot(1/\xi)^m$, assuming that $\xi \le \operatorname{diam} Q_i$, which is implied by the stronger inequality $\xi \le a_5^n\rho\nu$, $a_5 = 1/10S^2M_1(f)$. Here C_1 depends only on m and S.

As the set $A' \subset \mathbb{R}^m \times L_m$ we take some part of the $2\lambda'$-neighbourhood of $0 \times Z_m$, where Z_m is the set of non-hyperbolic linear mappings, defined in lemma 2.5, and $\lambda' = S\lambda$.

The image $\Phi(Q_i \times T)$ is contained in some ball B in $\mathbb{R}^m \times L_m$ of radius $C^n \cdot \operatorname{diam} Q_i + K_1^n\nu \le C_2^n\nu$. Indeed, $\|d_x\Phi\| \le C^n$, and, on the other hand, for each $\tau \in T$,

$$\|df_\tau^n(x) - df^n(x)\| \le K_1^n\nu,$$

by lemma 2.1. Finally, the diameter of any Q_i, by construction, does not exceed ν.

So we take as A' the $2\lambda'$-neighbourhood of $(0 \times Z_m) \cap B$ in $\mathbb{R}^m \times L_m$.

$0 \times Z_m \subset \mathbb{R}^m \times L_m$ is a semialgebraic set of dimension $m^2 - 1$, defined by a fixed number of polynomial equations and inequalities of fixed degrees, depending only on m. Hence for the metric entropy of $0\times Z_m$ we have the following inequality (see e.g. [9], [10]):

LEMMA (4.6). *For any ball B of radius r in $\mathbb{R}^m\times L_m$, and for any $\xi>0$, $\xi\le r$,*

$$M(\xi,(0\times Z_m)\cap B_r)\le C_3(r/\xi)^{m^2-1},$$

where the constant C_3 depends only on m.

COROLLARY (4.7). $M(3\lambda',A')\le C_4^n(\nu/\lambda')^{m^2-1}$.

Proof. Take a covering of $(0\times Z_m)\cap B$ by balls of radius λ'. Since the radius of the ball B is equal to $C_2^n\nu$, and since, by assumption, $\lambda'\le Sa_5^n\rho\nu<C_2^n\nu$, we can find such a covering with the number of balls not exceeding

$$C_3(C_2^n\nu/\lambda')^{m^2-1}\le C_4^n(\nu/\lambda')^{m^2-1},$$

where $C_4=C_3C_2^{m^2-1}$.

But then the balls of radius $3\lambda'$, centred at the same points, cover the $2\lambda'$-neighbourhood A' of $(0\times Z_m)\cap B$.

Now we are ready to apply theorem 4.2. Put ξ in this theorem equal to $3\lambda'$. We obtain:

$$\begin{aligned}M(\xi',\Delta_\Phi(A,A'))&\le M(\xi,A)\cdot M(\xi,A')\le C_1\mu(Q_i)(1/3\lambda')^m\cdot C_4^n(\nu/\lambda')^{m^2-1}\\&\le C_5^n\nu^{m^2-1}(1/\lambda')^{m^2+m-1}\mu(Q_i),\end{aligned}$$

where

$$\begin{aligned}\xi'&=2(S_1R_2(1+R_1)+1)3\lambda'\\&\le 2(S^4K_2^n(1/\nu\rho)(1+C^n)+1)3\lambda'\le C_6^n\lambda'/\nu\rho.\end{aligned}$$

Now let C_7 be the measure of the unit ball in $\mathbb{R}^m\times L_m$. The measure of the ball of radius ξ' is hence equal to

$$C_7\xi'^{m^2+m}\le C_7(C_6^n\lambda'/\nu\rho)^{m^2+m}\le C_8^n(\lambda')^{m^2+m}(1/\nu\rho)^{m^2+m}.$$

Therefore we obtain:

$$\begin{aligned}m(\Delta_\Phi(A,A'))&\le C_5^n\nu^{m^2-1}(1/\lambda')^{m^2+m-1}\mu(Q_i)\cdot C_8^n(\lambda')^{m^2+m}(1/\nu\rho)^{m^2+m}\\&\le K_5^n(1/\nu)^{m+1}(1/\rho)^{m^2+m}\mu(Q_i)\cdot\lambda.\end{aligned}$$

To prove lemma 4.5 it remains to note that the set $\Delta_{i,t'}(\lambda)$, introduced above, is contained in $\Delta_\Phi(A,A')$. Indeed, $\tau\in T$ belongs to $\Delta_{i,t'}(\lambda)$ if and only if there exists $x\in Q_i$ which is (n,λ)-periodic but not (n,λ)-hyperbolic for f_τ. This means that $\hat\delta(f_\tau^n(x),x)\le\lambda$ or $\|f_\tau^n(x)-x\|\le S\lambda=\lambda'$ in our fixed coordinate neighbourhood U_s and that the hyperbolicity $\gamma(df_\tau^n(x))\le\lambda<\lambda'$, and by lemma 2.5 the distance of $df_\tau^n(x)$ to Z_m in L_m does not exceed λ'.

Hence $\Phi(x,\tau)=(f_\tau^n(x)-x,df_\tau^n(x))$ belongs to the $2\lambda'$-neighbourhood A' of $0\times Z_m$ in $\mathbb{R}^m\times L_m$, and by definition of $\Delta_\Phi(A,A')$, τ belongs to this set.

Lemma 4.5 is proved. □

COROLLARY (4.8). *Let $\Delta^n(\lambda)$ be the set of $t\in T_\rho$ for which there exists a point $x\in\Omega_n$, which is (n,λ)-periodic but not (n,λ)-hyperbolic for f_t. Then, for $\lambda\le a_5^n\rho\nu$,*

$$m(\Delta^n(\lambda))\le K_6^n(1/\nu)^{m+1}(1/\rho)^{m^2+m}\lambda.$$

Proof. $\Delta^n(\lambda)=\bigcup_{i\in I}\Delta_i(\lambda)$, where I is the set of those i for which $Q_i\subset\Omega_n$. Hence

$$m(\Delta^n(\lambda))\le\sum_{i\in I}m(\Delta_i(\lambda))\le K_5^n(1/\nu)^{m+1}(1/\rho)^{m^2+m}\lambda\sum_{i\in I}\mu(Q_i)$$
$$\le K_6^n(1/\nu)^{m+1}(1/\rho)^{m^2+m}\lambda,$$

since by definition of the covering Q_i, $\sum_{i\in I}\mu(Q_i)$ does not exceed some constant, depending only on the compact manifold X and the atlas (U_s,Ψ_s). □

COROLLARY (4.9). *The measure of the set $\bar{\Delta}^N(\lambda)$, consisting of those $t\in T_\rho$ for which there is at least one $n\le N$ and an (n,λ)-periodic point $x\in\Omega_n$ of f_t, which is not (n,λ)-hyperbolic, does not exceed $K_7^N(1/\nu)^{m+1}(1/\rho)^{m^2+m}\lambda$.*

Proof. $\bar{\Delta}^N(\lambda)$ is the union of $\Delta^n(\lambda)$, $n=1,2,\ldots,N$. The additional factor N, which appears in the bound for the measure of this union, enters in K_7^N. □

Now we can complete the proof of lemma 4.4. By definition of our measure m on T, $m(T_\rho)=1$. Hence if we take γ so small that the measure of the 'bad' set $\bar{\Delta}^N(\gamma)$ is strictly less than 1, we find the required t_0.

Thus we put $\gamma=a_4^N\nu^{m+1}\rho^{m^2+m}$, where $a_4=\frac{1}{2}K_7$, and take some $t_0\in T_\rho\setminus\bar{\Delta}^N(\gamma)$.

Then, by definition of $\bar{\Delta}^N(\gamma)$, any (n,γ)-periodic point of $f'=f_{t_0}$, belonging to Ω_n, is (n,γ)-hyperbolic for f'. It remains to observe that if $x\in X$ is (n,η)-simple for f', then (since, by assumption, $\nu\le a_3^N\rho$) lemma 2.1 implies that x is $(n,\frac{1}{2}\eta)$-simple for f, and hence $x\in\Omega_n$. Lemma 4.4 is proved. □

We can summarize our application of quantitative transversality as follows: the set W of periodic and non-hyperbolic points in the first jet space has codimension $m+1$. Since $\dim X=m$, the usual transversality theorem asserts that the measure of those $t\in T_\rho$, for which $\tilde{f}_t(X)$ intersects W, is zero. (Here $\tilde{f}$ is the first jet extension of f.)

The quantitative transversality theorem gives an upper bound for the measure of those $t\in T_\rho$ for which the distance between $\tilde{f}_t(X)$ and W is at most γ. The main point is that in this bound the factor γ appears in the first power (which corresponds to codim $W-\dim X=1$), and in particular, for $\gamma=0$ we once more obtain measure zero. But we can find exactly the biggest $\gamma=\bar{\gamma}$, for which the measure of the 'bad' set of t is still strictly less than $m(T_\rho)$. Then taking some 'good' t_0, we obtain f_{t_0} with distance between $f_{t_0}(X)$ and W at least $\bar{\gamma}$.

5. *Proof of main results*

In this section we prove first theorem 1.6 and then, as easy consequences, theorem 1.4 and corollary 1.5.

Let $f\in D^k(X)$, $k\ge3$, be given. Define $\varepsilon_0>0$ as $\varepsilon_0=a_3$, where the constant a_3, depending on $M_1(f)$, $M_1'(f)$, $M_2(f)$, $M_3(f)$ was defined above.

Now let $\varepsilon>0$, $\varepsilon\le\varepsilon_0$ be given. We define inductively the sequence $\gamma_r(\varepsilon)$, $r=0,1,\ldots$, as follows:

$$\gamma_0(\varepsilon)=a_6\varepsilon^{m+1};\qquad \gamma_{r+1}(\varepsilon)=a_6^{2^{r+1}}\varepsilon^{m+1}\gamma_r^\beta(\varepsilon),$$

where $\beta=2(m^2+mk+k-1)$ and

$$a_6=\tfrac{1}{2}a_4(1/200SK_3K_4M_1(f))^{\beta/2}(a_3/2K_1)^{m+1}>0,$$

with the constants a_3, a_4, K_1, K_2, K_3, K_4 and S, depending on X, $M_1(f)$, $M_1'(f)$, $M_2(f), \ldots, M_k(f)$, as defined above. (Below we shall shortly write γ_r instead of $\gamma_r(\varepsilon)$.)

We subdivide all the periods of almost periodic points considered into the parts between 2^{r-1} and 2^r, $r=0,1,\ldots$, and prove theorem 1.6 by induction on r. The following lemma forms the initial step of our induction:

LEMMA (5.1). *There exists $f_0 \in D^k(X)$, such that:*

(1) $d_k(f_0, f) \leq \varepsilon/2$;

(2) *any $(1, 2\gamma_0)$-periodic (or, in other words, almost fixed) point of f_0 is $(1, 2\gamma_0)$-hyperbolic.*

The proof will be given below. The next lemma forms the main step of the induction: passing from r to $r+1$ (or from periods $\leq 2^r$ to periods $\leq 2^{r+1}$):

LEMMA (5.2). *Let $f_r \in D^k(X)$ be given, satisfying the following conditions:*

(1) $d_k(f_r, f) \leq \varepsilon$;

(2) *for any i, $0 \leq i \leq r$, and for any n, $2^{i-1} < n \leq 2^i$, each (n, ξ_i)-periodic point of f_r is (n, ξ_i)-hyperbolic, for some $\xi_i \geq \gamma_i$, $i=0,1,\ldots,r$.*

Then there exists a diffeomorphism $f_{r+1} \in D^k(X)$ with the following properties:

(a) $d_k(f_{r+1}, f_r) \leq \varepsilon/2^{r+2}$;

(b) *for any i, $0 \leq i \leq r$, and for any n, $2^{i-1} < n \leq 2^i$, each $(n, (i-2^{-r-2})\xi_i)$-periodic point of f_{r+1} is $(n, (1-2^{-r-2})\xi_i)$-hyperbolic;*

(c) *for any n, $2^r < n \leq 2^{r+1}$, each $(n, 2\gamma_{r+1})$-periodic point of f_{r+1} is $(n, 2\gamma_{r+1})$-hyperbolic.*

The proof of lemma 5.2 is also given below. Now we complete the proof of theorem 1.6.

First let us take f_0, whose existence is provided by lemma 5.1. Then we build, starting from f_0, and repeatedly applying lemma 5.2, the sequence of diffeomorphisms $f_r \in D^k(X)$, $r=1,2,\ldots$

This is possible, since at each step the conditions of lemma 5.2 are satisfied. Indeed, assume that $f_0, \ldots, f_r$ can be built. By the property (a)

$$d_k(f_r, f) \leq d_k(f_r, f_{r-1}) + \cdots + d_k(f_0, f) \leq (2^{-r-1} + \cdots + 2^{-2})\varepsilon < \varepsilon.$$

So the condition (1) of lemma 5.2 is satisfied for f_r.

Now fix some i, $0 \leq i \leq r$. By the property (c) of lemma 5.2, applied on the i'th step, for any n, $2^{i-1} < n \leq 2^i$, each $(n, 2\gamma_i)$-periodic point of f_i is $(n, 2\gamma_i)$-hyperbolic. In turn, by property (b), any $(n, \bar{\xi}_i)$-periodic point of f_r is $(n, \bar{\xi}_i)$-hyperbolic, where

$$\bar{\xi}_i = 2\gamma_i(1-2^{-i-2})(1-2^{-i-3}) \cdots (1-2^{-r-1}) > \gamma_i.$$

Hence the condition (2) of lemma 5.2 is also satisfied for f_r, and, applying this lemma, we can find f_{r+1} with the required properties.

Now, by property (a) of lemma 5.2, the sequence $f_0, f_1, \ldots, f_r, \ldots$ converges in $D^k(X)$ to some diffeomorphism $f' \in D^k(X)$ with $d_k(f', f) \leq \varepsilon$.

By the estimates above, for any $i=0,1,2,\dots$ and for any n, $2^{i-1}<n\le 2^i$, each $(n,\hat{\xi}_i)$-periodic point of f' is $(n,\hat{\xi}_i)$-hyperbolic, where

$$\hat{\xi}_i=2\gamma_i\prod_{j=0}^{\infty}(1-2^{-i-j-2})\ge\gamma_i.$$

It remains only to estimate γ_r, defined by the recurrent equation above, and to pass from the representation of the period n as 2^r to the usual one.

Denote ε^{m+1} by b and write a_6 as a. We have

$$\gamma_0=ab,\qquad \gamma_{r+1}=a^{2^{r+1}}b\gamma_r^{\beta}.$$

Hence

$$\begin{aligned}\gamma_1&=a^2b(ab)^{\beta}=a^{2+\beta}b^{1+\beta}\\ \gamma_2&=a^{2^2}b(a^{2+\beta}b^{1+\beta})^{\beta}=a^{2^2+2\beta+\beta^2}b^{1+\beta+\beta^2}\\ &\ \ \vdots\\ \gamma_r&=a^{2^r+2^{r-1}\beta+\dots+\beta^r}\cdot b^{1+\beta+\dots+\beta^r}.\end{aligned}$$

Since $\beta=2(m^2+mk+k-1)\ge 12$ for $m\ge 1$, $k\ge 3$, we can write the expression for γ_r as follows:

$$\gamma_r=a^{\beta^r(1+2/\beta+\dots+(2/\beta)^r)}b^{\beta^r(1+1/\beta+\dots+(1/\beta)^r)}\ge a_7^{\beta^r}b_1^{\beta^r},$$

where $a_7=a_6^{\frac{6}{5}}$, $b_1=b^{\frac{12}{11}}=\varepsilon^{\frac{12}{11}(m+1)}$.

Now for any natural n each $(n,\gamma_{[\log_2 n]+1})$-periodic point of f' is $(n,\gamma_{[\log_2 n]+1})$-hyperbolic, and we obtain:

$$\gamma_{[\log_2 n]+1}\ge(a_7b_1)^{\beta^{[\log_2 n]+1}}\ge((a_7b_1)^{\beta})^{2^{\log_2 n\,\log_2\beta}}=(a(\varepsilon))^{n^{\alpha}},$$

where $\alpha=\log_2\beta=\log_2(m^2+mk+k-1)+1=\alpha(m,k)$, and $a(\varepsilon)=(a_7b_1)^{\beta}=a_0\cdot\varepsilon^{\frac{24}{11}(m+1)(m^2+mk+k-1)}$, with $a_0=a_7^{\beta}=a_6^{\frac{12}{5}(m^2+mk+k-1)}$.

Theorem 1.6 is proved. □

Proof of lemma (5.1). We apply lemma 4.4 in the case $N=1$. Clearly, each point $x\in X$ is $(1,\eta)$-simple for any $\eta>0$, so we fix the maximal possible value of the parameter $\eta=S$ and put $\rho=\rho_0=\eta/100S=\frac{1}{100}$.

Now we choose the value of the parameter ν. The first restriction is given by lemma 4.4: $\nu\le a_3\rho_0$. Another restriction is given by the condition $d_k(f_0,f)\le\varepsilon/2$. If we want this condition to be satisfied for any $f_t^{\rho,\nu}$, $t\in T_\rho$, then, by lemma 2.1, we must have

$$K_1\nu(1/\rho)^{k-1}\le\varepsilon/2\quad\text{or}\quad \nu\le(1/2K_1)\rho_0^{k-1}\varepsilon.$$

Since by assumption $\varepsilon\le\varepsilon_0$ and $k\ge 3$, this last inequality is stronger than the first one, so we put

$$\nu_0=(1/2K_1)\rho_0^{k-1}\varepsilon.$$

By lemma 4.4, there is $t_0\in T_{\rho_0}$, such that any $(1,\gamma)$-periodic point of $f_0=f_{t_0}^{\rho_0,\nu_0}$ is $(1,\gamma)$-hyperbolic, where

$$\gamma=a_4\nu_0^{m+1}\rho_0^{m^2+m}=a_4(1/2K_1)^{(k-1)(m+1)}\varepsilon^{m+1}\rho_0^{m^2+m}\ge 2a_6\varepsilon^{m+1}=2\gamma_0.$$

Lemma 5.1 is proved. □

Proof of lemma (5.2). Let the diffeomorphism $f_r \in D^k(X)$, satisfying conditions (1) and (2) of lemma 5.2, be given.

We shall find f_{r+1} in the form $f_{r+1} = (f_r)_t^{\rho,\nu}$ for some values of real parameters ρ and ν and $t \in T_\rho$. Let us describe the choice of parameters ρ and ν.

First put $\eta = 100S\ c_1^{2^{r+1}}\gamma_r^2$, where $c_1 = 1/200SK_3K_4M_1(f)$, and let $\rho = \eta/100S = c_1^{2^{r+1}}\gamma_r^2$.

Now we choose ν. The first restriction on ν is given by lemma 4.4: $\nu \le a_3^{2^{r+1}}\rho$. Another restriction is given by the condition $d_k(f_{r+1}, f_r) \le 2^{-r-2}\varepsilon$. According to lemma 2.1, this inequality is satisfied for any $(f_r)_t^{\rho,\nu}$, if

$$K_1\nu(1/\rho)^{k-1} \le 2^{-r-2}\varepsilon,$$

or

$$\nu \le (1/K_1)\rho^{k-1}2^{-r-2}\varepsilon = (1/K_1)2^{-r-2}c_1^{(k-1)2^{r+1}}\gamma_r^{2(k-1)}\varepsilon.$$

This last inequality, in turn, is satisfied, if $\nu \le c_2^{2^{r+1}}\gamma_r^{2(k-1)}\varepsilon$, where $c_2 = c_1^{k-1}(a_3/2K_1)$. Under the assumption $\varepsilon \le \varepsilon_0$ this last inequality is stronger than the first one, $\nu \le a_3^{2^{r+1}}\rho$, so we put

$$\nu = c_2^{2^{r+1}}\gamma_r^{2(k-1)}\varepsilon.$$

Now we apply lemma 4.4, with the parameters η, ρ, ν chosen as above and $N = 2^{r+1}$. Let $t_0 \in T_\rho$ be the value of the parameter t given by lemma 4.4. We put $f_{r+1} = (f_r)_{t_0}^{\rho,\nu}$.

First the condition (a) of lemma 5.2 is satisfied for f_{r+1} by the choice of ν.

By lemma 4.4, f_{r+1} has the following property: for any $n \le 2^{r+1}$, and, in particular, for any n between 2^r and 2^{r+1}, each (n, η)-simple and (n, γ)-periodic point of f_{r+1} is (n, γ)-hyperbolic, where

$$\begin{aligned}\gamma &= a_4^{2^{r+1}}\nu^{m+1}\rho^{m^2+m}\\ &= a_4^{2^{r+1}}(c_2^{m+1})^{2^{r+1}}\gamma_r^{2(m+1)(k-1)}\varepsilon^{m+1}(c_1^{m^2+m})^{2^{r+1}}\gamma_r^{2(m^2+m)}\\ &\ge 2a_6^{2^{r+1}}\varepsilon^{m+1}\gamma_r^{2(m^2+mk+k-1)}\\ &= 2a_6^{2^{r+1}}\varepsilon^{m+1}\gamma_r^{\beta} = 2\gamma_{r+1},\end{aligned}$$

where $\beta = 2(m^2+mk+k-1)$, $a_6 = \frac{1}{2}a_4c_2^{m+1}c_1^{m^2+m}$.

Thus, we have already checked the required hyperbolicity for the almost periodic points of f_{r+1}, namely, for the $(n, 2\gamma_{r+1})$-periodic points with $2^r < n \le 2^{r+1}$, which are (n, η)-simple.

Now let us show that the hyperbolicity of almost periodic points of f_r with periods $\le 2^r$ was not destroyed by our perturbation.

Indeed, by lemma 2.1, for any $x \in X$,

$$\hat{\delta}(f_r^n(x), f_{r+1}^n(x)) \le K_1^{2^r}\cdot\nu\rho \le K_1^{2^r}c_2^{2^{r+1}}\gamma_r^{2(k-1)}\varepsilon\rho \le 2^{-r-2}\gamma_r,$$

by the choice of coefficients and since we can assume $\gamma_r < 1$.

Hence if the point $x \in X$ is $(n, (1-2^{-r-2})\xi_i)$-periodic for f_{r+1}, with some $\xi_i \ge \gamma_i \ge \gamma_r$, where $0 \le i \le r$, $2^{i-1} < n \le 2^i$, this point is also (n, ξ_i)-periodic for f_r.

By condition (2) of lemma 5.2, x is an (n, ξ_i)-hyperbolic point for f_r. Now, by lemma 2.1, in any coordinate neighbourhood containing both $f_r^n(x)$ and $f_{r+1}^n(x)$,

$$\begin{aligned}\|df_r^n(x) - df_{r+1}^n(x)\| &\le K_1^{2^r}\nu = K_1^{2^r}c_2^{2^{r+1}}\gamma_r^{2(k-1)}\varepsilon\\ &\le 2^{-r-4}(1/2M_1(f))^{2^r}\gamma_r^{2(k-1)},\end{aligned}$$

by the choice of the constants c_2 and c_1.

Since $\|df_r^n(x)\| \le (2M_1(f))^{2^r}$, we obtain, by lemma 2.4:

$$\gamma(df_{r+1}^n(x)) \ge \xi_i - (4/\xi_i)(2M_1(f))^{2^r} \cdot 2^{-r-4}(1/2M_1(f))^{2^r}\gamma_r^{2(k-1)}.$$

Now $\xi_i \ge \gamma_i \ge \gamma_r$, and, by assumption, $k \ge 3$; therefore we have:

$$\gamma(df_{r+1}^n(x)) \ge \xi_i - 2^{-r-2}\xi_i = (1-2^{-r-2})\xi_i.$$

Thus for $i = 0, \dots, r$ and for any n, $2^{i-1} < n \le 2^i$, each $(n, (1-2^{-r-2})\xi_i)$-periodic point of f_{r+1} is $(n, (1-2^{-r-2})\xi_i)$-hyperbolic. This proves the conclusion (b) of lemma 5.2.

It remains to check the conclusion (c) for the $(n, 2\gamma_{r+1})$-periodic points of f_{r+1}, which are not (n, η)-simple, with n between 2^r and 2^{r+1}.

Let $x \in X$ be such a point. Since, by construction, $\eta \ge 2\gamma_{r+1}$, we are in the situation of lemma 3.1, namely, of the case (b) of this lemma. We conclude that there is $l < n$, dividing n, such that x is an $(l, K_4^n\eta)$-periodic point of f_{r+1}, and for any j, $0 \le j \le n$, $j = ql + s$, $s < l$,

$$\hat\delta(f_{r+1}^j(x), f_{r+1}^s(x)) \le K_4^n\eta.$$

Find i such that $2^{i-1} < l \le 2^i$. Since $l < n$ and l divides n, we have $l \le n/2$, and hence $i \le r$.

Now, by the choice of η,

$$K_4^n\eta \le K_4^{2^{r+1}}\eta \le (1-2^{-r-2})\gamma_r \le (1-2^{-r-2})\xi_i.$$

Therefore the point x is $(l, (1-2^{-r-2})\xi_i)$-periodic for f_{r+1}, and by conclusion (b) of lemma 5.2, x is $(l, (1-2^{-r-2})\xi_i)$-hyperbolic for f_{r+1}.

Now we apply lemma 2.6. By the choice of η,

$$\delta = K_4^n\eta \le K_4^{2^{r+1}}\eta \le 2^{-r-2}(1-2^{-r-2})^2(1/K_3)^{2^{r+1}}\gamma_r^2$$
$$\le 2^{-r-2}[(1-2^{-r-2})\xi_i]^2/K_3^n.$$

Hence, by lemma 2.6, x is an $(n, \bar\xi)$-hyperbolic point of f_{r+1}, where $\bar\xi = (1-2^{-r-2})^2\xi_i \ge (1-2^{-r-2})^2\gamma_r > 2\gamma_{r+1}$.

Lemma 5.2 is proved. □

Proof of theorem (1.4). Theorem 1.4 follows immediately from theorem 1.6 if $k \ge 3$. For $k < 3$ the space $D^3(X) \subset D^k(X)$ is dense in $D^k(X)$ in d_k-metric. Hence the set $W_k = W_3 \subset D^3(X) \subset D^k(X)$ is dense in $D^k(X)$ and has the property required in theorem 1.4. □

Proof of corollary (1.5). We shall prove a somewhat more precise statement:

PROPOSITION (5.3). *Let $f \in W_k \subset D^k(X)$. There are constants $b_1 > 0$, $b_2 > 0$, depending on f, with the following property*:

For any two periodic points $x_1 \ne x_2$ of f with the shortest periods n_1 and n_2, respectively, $n_1 < n_2$,

$$\hat\delta(x_1, x_2) \ge b_1^{n_1^\alpha} b_2^{n_1 n_2},$$

where $\alpha = \alpha(m, k)$.

Proof. Denote f^{n_1} by $\bar f$. By theorem 1.4, x_1 is a hyperbolic fixed point of $\bar f$ with $\gamma(d\bar f(x_1)) \ge a^{n_1^\alpha}$.

Since the first and second derivatives of $\bar{f}$ are bounded by C^{n_1}, we can find a neighbourhood U of x_1, of $\hat{\delta}$-radius $c^{n_1^\alpha}$, in which $\bar{f}$ is topologically conjugated to a linear hyperbolic mapping. Consider the neighbourhood U' of x_1 of $\hat{\delta}$-radius $c^{n_1^\alpha}/C^{n_1 n_2}$. Then $\bar{f}^j(U') \subset U$ for $j = 1, 2, \ldots, n_2$. But since $\bar{f}$ is topologically a hyperbolic linear mapping in U, this implies that the only fixed point of $\bar{f}^{n_2}$ in U' is x_1.

Now x_2 is a fixed point of $\bar{f}^{n_2}$, and therefore $x_2 \notin U'$, or

$$\hat{\delta}(x_1, x_2) \geq c^{n_1^\alpha}/C^{n_1 n_2} = b_1^{n_1^\alpha} b_2^{n_1 n_2},$$

with $b_1 = c$, $b_2 = 1/C$. The proposition is proved. □

Now if $n_1, n_2 \leq n$, we obtain $\hat{\delta}(x_1, x_2) \leq b_1^{n^\alpha} b_2^{n^2} \leq (b_1 b_2)^{n^\alpha} = b^{n^\alpha}$ since $\alpha = \alpha(m, k) \geq 2$ for $m \geq 1$, $k \geq 1$. This proves that the distance between any two periodic points $x_1 \neq x_2$ of f with periods $\leq n$ is at least b^{n^α}.

Since the manifold X is compact, this implies immediately that the number of periodic points of f with periods $\leq n$ does not exceed C^{n^α}, where $C = (K/b)^m$, with K depending only on X. Corollary 1.5 is proved. □

Using deeper properties of hyperbolicity one can improve the result of proposition 5.3 and obtain additional information on the geometry of periodic trajectories of $f \in W_k$. E.g. one has the following alternative: any closed trajectory of $f \in W_k$ of period n is either iterated or (n, η)-simple, with $\eta = c^{n^\alpha}$. We do not touch on these questions here.

6. *The case of flows*

In this section we formulate the quantitative Kupka-Smale theorem and its main consequences in the case of flows and sketch the necessary alterations in proofs.

Let X be a compact m-dimensional smooth manifold, and let $V^k(X)$, $k = 1, \ldots,$ be the space of k-times continuously differentiable tangent vector fields on X.

As above, we assume that some Riemannian metric and some finite atlas on X are fixed, and we denote by $\hat{\delta}$ and d_k the distance in X and the C^k-norm in $V^k(X)$ induced by this metric and atlas.

For $v \in V^k(X)$ we denote by $\varphi_{v,t} : X \to X$ the flow generated by the vector field v.

For the sake of simplicity we state our results only for exactly closed trajectories, although the proofs necessarily involve consideration of almost-closed trajectories and provide their hyperbolicity, as in the case of discrete time, considered above.

Definition (6.1). Let $v \in V^k(X)$ be given. For any $x \in X$ such that $v(x) = 0$, the *hyperbolicity* $\gamma(x)$ of v at x is defined as $\gamma(x) = \gamma(d\varphi_{v,1}(x))$.

Let ω be a closed trajectory of a period $T > 0$ of v. The *hyperbolicity* $\gamma(\omega)$ of v on ω is defined as

$$\gamma(\omega) = \gamma(d\Psi_\omega(0)),$$

where $\Psi_\omega : \mathbb{R}^{m-1} \to \mathbb{R}^{m-1}$ is (the germ of) the Poincare mapping associated with the closed trajectory ω of v.

THEOREM (6.2). *In each space $V^k(X)$, $k = 1, 2, \ldots,$ there is a dense subset W'_k, such that vector fields $v \in W'_k$ have the following property: for some constant $a > 0$, depending*

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on v:

(1) *for each zero x of v, $\gamma(x) \geq a$;*

(2) *for each closed trajectory ω of v, of period $T>0$, $\gamma(\omega) \geq a^{T^\alpha}$, where $\alpha = \alpha'(m, k) = \log_{3/2}(2m(m+k'-2))$, $k' = \max(k, 3)$.*

COROLLARY (6.3). *For any $v \in W'_k$ there are constants $b>0$ and C, depending on v, such that*:

(1) *For any two closed orbits $\omega_1 \neq \omega_2$ of v, with periods $\leq T$, the distance between ω_1 and ω_2 (which is $\min \hat{\delta}(x_1, x_2)$, $x_1 \in \omega_1$, $x_2 \in \omega_2$), is at least b^{T^α};*

(2) *the number of closed orbits of v with periods $\leq T$ does not exceed C^{T^α}.*
Here α, as above, is equal to $\alpha'(m, k)$.

As in the case of diffeomorphisms, theorem 6.2 is implied by the following more precise statement:

THEOREM (6.4). *Let $v \in V^k(X)$, $k \geq 3$. There are constants $\varepsilon_0 > 0$, $a_1 > 0$, $a_2 > 0$, depending on v, such that for any $\varepsilon > 0$, $\varepsilon \leq \varepsilon_0$, one can find $v' \in V^k(X)$, $d_k(v', v) \leq \varepsilon$, with the following properties*:

(1) *for any zero x of v', $\gamma(x) \geq a_1(\varepsilon)$, where $a_1(\varepsilon) = a_1 \varepsilon^{m+1}$;*

(2) *for any closed orbit ω of v' of period $T>0$, $\gamma(\omega) \geq a_2(\varepsilon)^{T^\alpha}$, where $a_2(\varepsilon) = a_2 \varepsilon^{\frac{4}{11}m^2(m+k-2)}$, $\alpha = \alpha'(m, k)$.*

As in the case of diffeomorphisms, overexponentiality in our bounds appears as the result of the difficulty in controlling the behaviour of recurrent trajectories under perturbation.

In the case of flows on compact orientable surfaces this difficulty can be settled, and we obtain the following result, parallel to theorem 1.7 in the case of diffeomorphisms:

THEOREM (6.5). *Let X be a compact orientable surface. In each $V^k(X)$, $k = 1, 2, \ldots$, there is a dense subset W''_k, such that vector fields $v \in W''_k$ have the following property*:

For some constant $a>0$, depending on v, any zero of v is a-hyperbolic and any closed trajectory ω of v of period T is a^T-hyperbolic.

The proof of theorem 6.4 goes as follows: first considering an appropriate space of perturbations of vector fields of X and applying quantitative transversality theorem, we obtain at once a new vector field v_0, $d_k(v_0, v) \leq \varepsilon$, with all its zeros having the required hyperbolicity.

For this new field v_0 one can easily prove that any non-constant closed trajectory of v_0 has length at least c, where $c>0$ is some constant depending only on v.

We can also find a finite number of smoothly embedded $m-1$ dimensional disks $D_i \subset X$, such that any non-constant trajectory of v_0 intersects transversally at least one of the disks D_i.

We can assume also that each disk D_i has a neighbourhood U_i in X, diffeomorphic to $D_i \times [-1, 1]$ and v_0 under this diffeomorphism corresponds to the standard field $\partial/\partial t$ on $D_i \times [-1, 1]$.

Now for any sufficiently small $\rho > 0$ and for $\nu > 0$, $\nu \leq 1$, we build, as in § 2, the diffeomorphisms $h_{i,t}^{\rho,\nu}$ of the disks D_i into themselves, where $t \in T_\rho$.

By the standard construction, using the product structure of v_0 near D_i, we can define the corresponding perturbations $v_{0,t}^{\rho,\nu}$ of the vector field v_0, which 'move' the trajectory of v_0 by $h_{i,t}^{\rho,\nu}$ along the disks D_i.

Now for any vector field w sufficiently close to v_0, we define the mapping $f_{w,n}$ from the disks D_i to themselves (the 'succession function' of the field w) as follows: let $x \in D_i$ and let $\varphi_{w,t}(x) \in D_i$ for some t, $(n-1)c < t \leq nc$. Then we put $f_{w,n}(x) = \varphi_{w,t}(x) \in D_i$.

The mappings $f_{w,n}$ are not everywhere defined on D_i, and $f_{w,np}$ is not exactly the iteration $(f_{w,n})^p$. But to any closed trajectory ω of the vector field w (of length T, $(n-1)c < T \leq nc$), there corresponds the fixed point x of $f_{w,n}$, belonging to one of the disks D_i, and the hyperbolicity of ω is equal to the hyperbolicity of x.

Hence it is sufficient to prove the existence of v', with $d_k(v', v_0) \leq \varepsilon/2$, for which all the fixed points of $f_{v',n}$, $n = 1, 2, \ldots$, have the required hyperbolicity.

But this proof goes exactly as in the case of diffeomorphisms. Indeed, our perturbations $w_t^{\rho,\nu}$ of the vector field w, by construction, act on $f_{w,n}$ exactly as the perturbations $f_t^{\rho,\nu}$ of § 2 act on a diffeomorphism f. Hence all the estimates of § 2 remain valid. Lemma 3.1 on iterated almost closed trajectories also remains valid with minor modifications.

The application of quantitative transversality and the Peixoto induction, completing the proof, actually go through without changes. The only difference is that here we subdivide all the lengths of the periods into parts lying between $(3/2)^r$ and $(3/2)^{r+1}$ (and not between 2^r and 2^{r+1}, as in the case of diffeomorphisms), to avoid the influence of non-integral lengths of considered almost closed trajectories. As a result of this alteration, and since the dimension of the disks D_i is $m-1$, the new value $\alpha'(m, k)$ of the overexponentiality index α appears.

Theorem 6.2 and corollary 6.3 follow from theorem 6.4 exactly as in the case of discrete time.

7. *Addendum*

Here we prove theorem 4.2.

Let $Q \subset \mathbb{R}^m$ be a closed domain, such that any $x_1, x_2 \in Q$ can be joined in Q by a curve of length $\leq S_1\|x_2 - x_1\|$, and let $F: Q \times B^q \to \mathbb{R}^q$ be the C^1 mapping with $\|d_x F(x, t)\| \leq R_1$ for any $(x, t) \in Q \times B^q$, such that for any $x \in Q$, $F(x, \cdot): B^q \to \mathbb{R}^q$ is one to one with

$$\|t_2 - t_1\| \leq R_2\|F(x, t_2) - F(x, t_1)\|, \qquad t_1, t_2 \in B^q.$$

We fix $A \subset Q$ and $A' \subset \mathbb{R}^q$ and recall that $\Delta_F(A, A') \subset B^q$ is the set of all $t \in B^q$ such that $F(x, t) \in A'$ for some $x \in A$.

We want to give an upper bound for the number of balls of a given radius covering $\Delta_F(A, A')$.

Consider in $Q \times B^q$ the set $\Sigma = \{(x, t), x \in A, F(x, t) \in A'\}$. Then $\Delta_F(A, A') = \pi(\Sigma)$, where $\pi: Q \times B^q \to B^q$ is the projection on the second factor. Since projection does not increase distances, for any $\xi' > 0$,

$$M(\xi', \Delta_F(A, A')) \le M(\xi', \Sigma).$$

Hence it is sufficient to estimate the number of balls of a given radius covering Σ.

Let $\xi > 0$ be given. We fix some coverings of A and A' by balls B_i, $i = 1, 2, \ldots, M(\xi, A)$ and B'_j, $j = 1, 2, \ldots, M(\xi, A')$, of radius ξ.

LEMMA (7.1). *For any $i = 1, \ldots, M(\xi, A)$, $j = 1, \ldots, M(\xi, A')$, the set*

$$\Sigma_{i,j} = \{(x, t): x \in B_i, F(x, t) \in B'_j\},$$

is contained in some ball of radius ξ' in $Q \times B^q$, where $\xi' = 2(S_1 R_2(1 + R_1) + 1)\xi$.

Proof. Fix some point $(x_0, t_0) \in \Sigma_{i,j}$ and let (x, t) be some other point in $\Sigma_{i,j}$.

First, $\|x - x_0\| \le 2\xi$, since $x, x_0 \in B_i$. By the conditions, we can join x and x_0 by some curve s in Q of length $\le S_1\|x - x_0\| \le 2S_1\xi$.

Integrating along s and using the inequality $\|d_x F\| \le R_1$, we obtain:

$$\|F(x, t_0) - F(x_0, t_0)\| \le 2S_1 R_1 \xi.$$

Hence

$$\begin{aligned}\|F(x, t) - F(x, t_0)\| &\le \|F(x, t) - F(x_0, t_0)\| + \|F(x, t_0) - F(x_0, t_0)\| \\ &\le 2\xi + 2S_1 R_1 \xi,\end{aligned}$$

since both $F(x, t)$ and $F(x_0, t_0)$ belong to B'_j.

By the conditions we obtain:

$$\|t - t_0\| \le 2R_2(1 + S_1 R_1)\xi,$$

and combining this with $\|x - x_0\| \le 2\xi$,

$$\|(x, t) - (x_0, t_0)\| \le 2R_2(1 + S_1 R_1)\xi + 2\xi \le 2(R_2 S_1(1 + R_1) + 1)\xi = \xi'.$$

The lemma is proved. □

Now the sets $\Sigma_{i,j}$, $i = 1, \ldots, M(\xi, A)$, $j = 1, \ldots, M(\xi, A')$, cover Σ, and hence

$$M(\xi', \Sigma) \le M(\xi, A) M(\xi, A').$$

Theorem 4.2 is proved. □

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