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ON SHIMURA'S TRACE FORMULA

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§ 1. Introduction

In [1], G. Shimura gave a very practical formula of the traces of the Hecke operators acting on the space of cusp forms of rational weight and there he emphasized that the traces are effectively computable. We shall practice the computation in some special cases and discuss the structure of the Hecke algebra, which is not necessarily semi-simple. Though the theory of new forms is not available in the case of modular forms of half integral weight, we can clarify the whole structure of the Hecke algebra in certain cases by using [1], [2] together with our computation in this paper and the following

LEMMA 1. *$T_\kappa(4)T_\kappa(4)^* = T_\kappa(4)W_4T_\kappa(4)W_4 = 2^{(\kappa-1)-1}\cdot I + 2^{(\kappa-1)/2-1}\cdot\varepsilon\cdot T_\kappa(4)\cdot W_4$, where W_4 is the operator such that $(W_4f)(z) = f(-1/4z)(-2iz)^{-\kappa/2}$, $\varepsilon = \left(\dfrac{2}{\kappa}\right)$, $T_\kappa(4)$ is $T^4_{\kappa,1}(4)$ in [2] and $T_\kappa(4)^*$ is the adjoint operator of $T_\kappa(4)$ (see § 4).*

In [1], the trace is given in the form $\sum_C J(C)$ where C runs over Γ-conjugacy classes and $J(C)$ is a certain function of C. Most amount of this paper is devoted to the calculation of $\sum_C J(C)$ and we obtain an explicit formula of this (see § 2).

We give some definitions to explain more detail. Let $\Gamma' = \Gamma_0(2M)$, $\mathfrak{g} = \left\{\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\boldsymbol{Z})\,;\, c \equiv 0\ (2M),\ (a, 2M) = 1\right\}$ and $R = R(\Gamma', \mathfrak{g})$, the abstract Hecke algebra. Let R' be the subring generated by all $\Gamma'\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\Gamma'$, $((n, 2M) = 1)$. Define the representation $\rho_{2\lambda}\colon R \to \operatorname{End} S_{2\lambda}(\Gamma')$ by $\rho_{2\lambda}(\Gamma'\xi\Gamma') = [\Gamma'\xi\Gamma']_{2\lambda}$ as in [3]. On the other hand, another representation of R is obtained from [2]. Let namely

$$\tilde{G} = \{(\alpha, \varphi(z))\,;\, \alpha \in GL^+(2, \boldsymbol{R}), \varphi(z) = t(\det\alpha)^{-1/4}(cz + d)^{1/2}, t \in \boldsymbol{C}, |t| = 1\}$$

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and $\Delta = \{(\sigma, j(\sigma, z)) ; \sigma \in \Gamma = \Gamma_0(4M)\}$, as in [2]. Then we get the representation $\tilde{\rho}_{\kappa/2}: R \to \operatorname{End} S_{\kappa/2}(\Gamma)$ by

$$\tilde{\rho}_{\kappa/2}\left(\Gamma'\begin{pmatrix}1 & 0\\ 0 & p^k\end{pmatrix}\Gamma'\right) = \left[\Delta\left(\begin{pmatrix}1 & 0\\ 0 & p^{2k}\end{pmatrix}, p^{k/2}\right)\Delta\right]_\kappa$$

where $S_{\kappa/2}(\Gamma)$ is $S_\kappa(4M, 1)$ in [2], and $[\ ]_\kappa$ is same as in [2]. To show that $\tilde{\rho}_{\kappa/2}$ is actually a representation of R, it suffices to check the relation of $[\ ]_\kappa$. For example, let p be a prime, $(p, 2M) = 1$ and $k > 1$, then

$$\begin{aligned}\left(\Delta\left(\begin{pmatrix}1 & 0\\ 0 & p^2\end{pmatrix}, p^{1/2}\right)\Delta\right)&\cdot\left(\Delta\left(\begin{pmatrix}1 & 0\\ 0 & p^{2k}\end{pmatrix}, p^{k/2}\right)\Delta\right)\\ &= p^2\Delta\left(\begin{pmatrix}p^2 & 0\\ 0 & p^{2k}\end{pmatrix}, p^{(k-1)/2}\right)\Delta + \Delta\left(\begin{pmatrix}1 & 0\\ 0 & p^{2k+2}\end{pmatrix}, p^{(k+1)/2}\right)\Delta\\ &\quad + (p-1)/2(\Delta\xi_+\Delta + \Delta\xi_-\Delta)\ ,\end{aligned}$$

where $\xi_\pm = \left(\begin{pmatrix}p & 0\\ 0 & p^{2k+1}\end{pmatrix}, \pm\varepsilon_p p^{k/2}\right)$, and $[\Delta\xi_+\Delta]_\kappa = -[\Delta\xi_-\Delta]_\kappa$.

In [2], the following identities are noted for a prime p and for $\kappa \geqq 5$:

$$\dim S_{\kappa/2}(\Gamma_0(4p)) = \dim S_{\kappa-1}(\Gamma_0(2p))\ ,$$
$$\dim S_{\kappa/2}(\Gamma_0(4p^2)) = \dim S_{\kappa-1}(\Gamma_0(2p^2)) + (-1)^{(\kappa-1)/2}\left(p - \left(\frac{-1}{p}\right)\right)/4\ .$$

Naturally, one can expect that these identities also hold for all Hecke operators. Thus, we want to compare $\operatorname{tr}\tilde{\rho}_{\kappa/2}(\xi)$ with $\operatorname{tr}\rho_{\kappa-1}(\xi)$ for $\xi \in R$. The latter is written in [4] in a form ready to compute. But, the former is written in [1] in a form which we cannot so immediately compare with the latter. One of the aims in this paper is to give an explicit form of the former, as is discussed in § 2 and § 3. We assume $\kappa \geqq 5$ throughout this paper, and, as a special application of the explicit formula of the traces, we obtain

$$(1)\quad \begin{aligned}&\operatorname{tr}\tilde{\rho}_{\kappa/2}(\xi) = \operatorname{tr}\rho_{\kappa-1}(\xi) && \text{for } \xi \in R'\ , \text{ when } M = p\ ,\\ &\operatorname{tr}\tilde{\rho}_{\kappa/2}(\xi) = \operatorname{tr}\rho_{\kappa-1}(\xi) + \operatorname{tr} W_{p^2}\rho_{\kappa-1}(\xi) && \text{for } \xi \in R', \text{ when } M = p^2\ ,\end{aligned}$$

where W_{p^2} is a normalizer of $\Gamma_0(2p^2)$ with the determinant p^2 (see [5]). These formulas are verified in § 3. Though we assume $(n, 2M) = 1$ in the next two sections, the traces for n with $(n, 2M) \neq 1$ are easier to compute at least in the case $M = 1$. In fact, the above identity holds for R. Therefore, in view of Lemma 1, we obtain

THEOREM. *$\tilde{\rho}_{\kappa/2} \sim \rho_{\kappa-1}$, when $M = 1$.*

COROLLARY. *$\{F_{f,t};\ f \in S_{\kappa/2}(\Gamma_0(4)), t;$ square-free$\} = S_{\kappa-1}(\Gamma_0(2))$, where $F_{f,t}$ is defined in the same way as in* [2], *i.e., putting $f(z) = \sum_{n=1}^{\infty} a(n)e(nz)$ and $F_{f,t}(z) = \sum_{n=1}^{\infty} A_t(n)e(nz)$,*

$$\sum_{n=1}^{\infty} A_t(n)n^{-s} = \left(\sum_{n=1}^{\infty} a(tn^2)n^{-s}\right)\left(\sum_{n=1}^{\infty} \chi_t(n)n^{\lambda-1-s}\right).$$

We close this section by explaining the contents of the following sections. The proof of Lemma 1 will be found in § 4. In § 2, we give an explicit formula of $\sum J(C)$ under rather general assumptions. The identities in (1) are verified in § 3 by specializing the formula in § 2. Then, we are ready to give the proofs of Theorem and Corollary, which are accomplished in § 4. For the sake of accuracy, we further note that, besides the identities in (1), a little more discussions, which are themselves a part of the aim of § 4, are needed to prove Theorem and Corollary.

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General notations

$F(X) = X^2 - tX + n^2$ $\quad N = 4M$

$\nu_p = \mathrm{ord}_p N$ $\quad e(r) = \exp(2\pi i r)$

$\rho_p = \mathrm{ord}_p f$ $\quad \Gamma = \Gamma_0(N)$

$w = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \boldsymbol{Z}_+ = \{m \in \boldsymbol{Z};\ m > 0\}$

$|m|_p = p^{-\mathrm{ord}_p m}$

$Z_\Gamma(\beta)$: the centralizer of β in Γ

(a, d): the Hilbert symbol at ∞

$\{—\}$: the Eichler symbol

$J(\sigma, z) = cz + d$ for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \boldsymbol{R})$

$j(\sigma, z) = \left(\dfrac{c}{d}\right)\varepsilon_d^{-1}\sqrt{cz + d}$ (see [2])

$h(D)$: the class number of ring ideals of the order with the discriminant D in an imaginary quadratic number field $Q(\sqrt{D})$

$w(D)$: a half of the cardinality of the unit group of the above order

$\varphi(\ )$: the Euler function

Various kind of quantities are defined in [1] for an equivalence class C

in $\Gamma\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}\Gamma$, for example $J(C)$, η, λ and μ, which we do not explain completely in this paper; we refer to [1] for them. When $\beta \in \Gamma\begin{pmatrix} 1 & 0 \\ 0 & n^2 \end{pmatrix}\Gamma$ and $n^{-1}\beta$ belongs to an equivalence class C in $\Gamma\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}\Gamma$, we denote $J(C)$, η, λ and μ by $J(\beta)$, $\eta(\beta)$, $\lambda(\beta)$ and $\mu(\beta)$, respectively. We use the notations, $\Phi = \Phi\left(\Gamma\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}\Gamma/\Gamma\right)$ and $\Phi\left(\Gamma\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}\Gamma\right)$ in [1] without explanations.

§ 2. Explicit formula

First we note $\tilde{\rho}_{\kappa/2}\left(\Gamma'\begin{pmatrix} 1 & 0 \\ 0 & n^2 \end{pmatrix}\Gamma'\right) = \left[\Delta\left(\begin{pmatrix} 1 & 0 \\ 0 & n^2 \end{pmatrix}, \sqrt{n}\right)\Delta\right]_\kappa = (n^2)^{(\kappa/4)-1}\left[\Delta \cdot\left(\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}, \sqrt{n}\right)\Delta\right]_\kappa$. In [1], it was proved that $\operatorname{tr}\left[\Delta\left(\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}, \sqrt{n}\right)\Delta\right]_\kappa = \sum_{C\in\Phi} J(C)$. In this section, we shall give an explicit form of $\sum_{C\in\Phi} J(C)$. As usual, we prefer to divide it into the partial sums. Let $\tilde{p} = \sum_1 J(C)$, $\tilde{e} = \sum_2 J(C)$ and $\tilde{h} = \sum_3 J(C)$, where, in $\sum_1$, $\sum_2$ and $\sum_3$, C runs over all the parabolic, elliptic and hyperbolic equivalence classes in Φ respectively. Then, $\sum_{C\in\Phi} J(C) = \tilde{p} + \tilde{e} + \tilde{h} + \delta(n)2^{-1}((\kappa/2) - 1)\operatorname{vol}(H/\Gamma)$ where $\delta(n)$ is 0 or 1 according as $n \neq 1$ or $n = 1$. We shall discuss $\tilde{p}$, $\tilde{e}$ and $\tilde{h}$ corresponding to the three parts titled afterwards parabolic case, elliptic case and hyperbolic case respectively. The result is given as the following

PROPOSITION 1. *Let $N = 4M$, $(M, 2) = 1$ and $(n, N) = 1$, then*

$$\tilde{p} = -2^{-1}(-1)^{(\kappa-3)/2}\sqrt{n}\Big(\Big(1 + \Big\{\frac{-n}{2}\Big\}\Big)h(-n)w(-n)^{-1}$$
$$+ \Big(1 + \Big\{\frac{-4n}{2}\Big\}\Big)h(-4n)w(-4n)^{-1}\Big)\prod_{p\mid M}\tilde{c}'_p - \delta_0(\sqrt{n})\varphi(n)\prod_{p\mid M}\tilde{c}_p\,,$$
$$\tilde{e} = -n^{(4-\kappa)/2}\sum_{s,f} s^{-1}(x^{\kappa-2} - y^{\kappa-2})(x-y)^{-1}$$
$$h\big(s^2(s^2-4n)f^{-2}\big)\Big(\frac{s^2-4n}{f'}\Big)\prod_{p\mid N}\tilde{c}(t,f,p)\,,$$

and

$$\tilde{h} = -2n^{(4-\kappa)/2}\sum_{s,f} s^{-1}y^{\kappa-2}(x-y)^{-1}\cdot 2^{-1}\,\cdot$$
$$\varphi(s(s^2-4n)^{1/2}f^{-1})\prod_{p\mid N}\tilde{c}(t,f,p)\,,$$

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where the meaning of the letters is as follows:

s and f run through all natural numbers satisfying $2|s, f^2| s^2(s^2-4n)$, $(s,f,n)=1$, $s^2(s^2-4n)f^{-2}\equiv 0$ or 1 (4), and besides, $s^2-4n<0$ in $\tilde{e}$ and $(s^2-4n)^{1/2}\in \boldsymbol{Z}_+$ in $\tilde{h}$. x and y are solutions of $X^2-sX+n=0$ and $x>y$ in $\tilde{h}$. f' is such that $f=2^{\rho_2}f'f''$, $(f'f'',2)=1$, $f'|s$, $f''^2|(s^2-4n)$ and $f'>0$; t equals s^2-2n. Putting furthermore $F(X)=X^2-tX+n^2$ and $\rho_p=\operatorname{ord}_p f$, we define $\tilde{c}$ by

$$\tilde{c}(t,f,p)=\begin{cases}\sum_{\xi}\left(1+\delta_p(\xi)\left(\dfrac{s^2-2n}{p}\right)^{\nu_p}\right) & \text{for } p\neq 2\,,\\ \sum_{\xi}\left(\dfrac{2}{\xi}\right)^{\rho_2}(1+\delta_2(\xi)) & \text{for } p=2\,,\end{cases}$$

where the summations run over all the representatives ξ of $\boldsymbol{Z}/p^{\nu_p+\rho_p}\boldsymbol{Z}$ satisfying $2\xi\equiv t\ (p^{\rho_p})$, $F(\xi)\equiv 0\ (p^{\nu_p+2\rho_p})$ and besides $\xi\equiv 1\ (4)$ in case $p=2$, and $\delta_p(\xi)$ is defined by

$$\delta_p(\xi)=\begin{cases}1\,, & \text{if } (t^2-4n^2)f^{-2}\equiv 0\ (p) \text{ and } F(\xi)\equiv 0\ (p^{\nu_p+2\rho_p+1})\,,\\ 0\,, & \text{otherwise}\,,\end{cases}$$

$\tilde{c}_p$ and $\tilde{c}'_p$ are given by

$$\tilde{c}'_p=\sum_{\xi}\left(1+\delta_p(\xi)\left(\frac{-n}{p}\right)^{\nu_p}\right)$$

and

$$\tilde{c}_p=\sum_{\xi}(1+\delta_p(\xi))\,,$$

where the summations run over all the representatives ξ of $\boldsymbol{Z}/p^{\nu_p}\boldsymbol{Z}$ satisfying $\xi^2\equiv 0\ (p^{\nu_p})$, and

$$\delta_p(\xi)=\begin{cases}1\,, & \text{if } \xi^2\equiv 0\ (p^{\nu_p+1})\,,\\ 0\,, & \text{otherwise}\,.\end{cases}$$

$\delta_0(\sqrt{n})$ is 1 or 0 according as n is a square or not.

Now, we propose to rewrite $\tilde{e}$ and $\tilde{h}$ as follows.

PROPOSITION 1′. *Under the same assumptions as in Proposition* 1,

$$\tilde{e}=-n^{(4-\kappa)/2}\sum_{s,f_2,f'}s'^{\,l-1}(x^{\kappa-2}-y^{\kappa-2})(x-y)^{-1}$$

$$\frac{h(s''^2f'^{-2}D')}{w(s''^2f'^{-2}D')}\left(\frac{s^2-4n}{f'}\right)\left(1+\left\{\frac{D'}{2}\right\}\right)\prod_{p|M}\tilde{c}(t,f_2f',p)$$

and

$$\tilde{h}=-2n^{(4-\varepsilon)/2}\sum_{s,f_2,f'}s''^{-1}y^{\varepsilon-2}(x-y)^{-1}2^{-1}$$
$$\varphi(s''f'^{-1}\sqrt{D'})\cdot 2\cdot\prod_{p|M}\tilde{c}(t,f_2f',p)\;,$$

where the meaning of the letters is as follows:

s, f_2 and f' run through all natural numbers satisfying $2|s, f'|s$, $(f', s, n)=1, f_2^2|(s^2-4n), (f_2, s, n)=1, (f', 2)=1, D'=(s^2-4n)f_2^{-2}\equiv 0$ or $1\,(4)$, and besides, $s^2-4n<0$ in $\tilde{e}$ and $(s^2-4n)^{1/2}\in \boldsymbol{Z}_+$ in $\tilde{h}$; s'' equals $s\,|s|_2$. The meaning of other letters is same as in Proposition 1.

Proof. Let $f''^2|(s^2-4n), f'|s, (2, f'f'')=1, D_0=(s^2-4n)f''^{-2}, s=2^ks'', (s'', 2)=1$,

$$\tilde{E}(s, f', f'')=\sum_{e_1}\tilde{c}(t, 2^{e_1}, 2)\frac{h(D_1)}{w(D_1)}\frac{s''}{s}$$

and

$$E(s, f', f'')=\sum_{e_2}\left(1+\left\{\frac{D_2}{2}\right\}\right)\frac{h(D_2)}{w(D_2)}\;,$$

where e_1 and e_2 run satisfying $D_1=D_0s^2f'^{-2}2^{-2e_1}\equiv 0,\ 1\ (4)$ and $D_2=D_0s''^2f'^{-2}2^{-2e_2}\equiv 0,\ 1\ (4)$ respectively. Let $D_0s''^2f'^{-2}=2^{2v}d$ with $d\equiv 1\,(4)$ or $d/4\equiv -1, 2\ (4)$, and with a non negative integer v. Excluding $D_1d^{-1}=2^{2k+2v-2e_1}$ and $D_2d^{-1}=2^{2v-2e_2}$ out of $h(\ )$ in each summand of $\tilde{E}(s, f', f'')$ and $E(s, f', f'')$ respectively, we can put $\tilde{E}(s, f', f'')=\tilde{C}(s, f'')h(d)$ and $E(s, f', f'')=C(s, f'')h(d)$, and we can easily check $\tilde{C}(s, f'')=C(s, f'')$ by classifying n modulo 8 and reviewing the following

Remark 0. Let $s=2s', t'+n=2s'^2, t=2t', 2^{2e}|(t^2-4n^2), D=(t^2-4n^2)2^{-2e}, \left\{\frac{D}{2}\right\}=1$ and $\check{c}(t, e)=\left(\frac{2}{t'}\right)^e\tilde{c}(t, 2^e, 2)$. Then,

(i) For $t'\equiv -1\,(4)$, we have $\check{c}(t, e)=\begin{cases}1, & (e=0),\\ 0, & (e\geqq 1).\end{cases}$

(ii) For $t'\equiv 1\,(4)$, we have $\check{c}(t, 0)=2$ and $\check{c}(t, 1)=\begin{cases}1, & (D\equiv 0\,(32)),\\ -1, & (D\equiv 16\,(32)).\end{cases}$

(iii) For $t'\equiv 1\,(4)$ and $e\geqq 2$, we have

$$\check{c}(t,e)=\begin{cases} 2\,, & (D\equiv 1\,(8) \text{ and } e\neq 3)\,,\\ -2\,, & (D\equiv 1\,(8) \text{ and } e=3)\,,\\ 4\,, & (D\equiv 4\,(32))\,,\\ 2\,, & (D\equiv 20\,(32))\,,\\ 3\,, & (D\equiv 0,16\,(32))\,.\end{cases}$$

We can also rewrite $\tilde{h}$ in the same way as $\tilde{e}$. Thus, we have proved proposition 1′.

To verify the identities (1) in § 1, more preparations are necessary, which we leave in § 3.

Now, we give some explanations for part 1, 2 and 3 discussed in the latter part of this §.

Let $\beta'\in\Gamma\begin{pmatrix} n^{-1} & 0\\ 0 & n\end{pmatrix}\Gamma$, $\beta=n\beta'=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma\begin{pmatrix} 1 & 0\\ 0 & n^2\end{pmatrix}\Gamma$ and C be the equivalence class to which β' belongs. Then, $J(C)$ is described as in [1] using the values η,λ,μ and x for C, which we denote by $\eta(\beta)$, $\lambda(\beta)$ and $\mu(\beta)$ respectively. In order to describe $J(C)$, $\eta(\beta)$, $\lambda(\beta)$ and $\mu(\beta)$, using the entries a,b,c and d of β we give some simple remarks. Let n be an arbitraly natural number prime to N or not, $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma\begin{pmatrix} 1 & 0\\ 0 & n^2\end{pmatrix}\Gamma$ and $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\neq\begin{pmatrix} \pm n & 0\\ 0 & \pm n\end{pmatrix}$, then $(a,b,c,d)=1$, and so prime divisors of n are classified into following three types of primes p_i: (i) $p_1\nmid(a,c)$, (ii) $p_2\,|\,(a,c)$ and $p_2\nmid b$, and (iii) $p_3\,|\,(a,c,b)$. Let $\begin{pmatrix} a' & b'\\ c' & d'\end{pmatrix}=\begin{pmatrix} 1 & 0\\ Nm & 1\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} 1 & 0\\ -Nm & 1\end{pmatrix}$, then $a'=a-bNm$, and $c'=aNm+c-Nm(Nmb+d)$ are coprime if we take an integer m in such a way that $m\equiv 0 \bmod p_1$, $(m,p_2)=1$, and that $m\equiv 0 \bmod p_3$ for all primes p_i of above type. Hence we see

Remark 1. The equivalence class C always contains an element $n^{-1}\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ such that $(a,c)=1$ and that $c\neq 0$, unless $C=\pm 1$.

Now, let $\beta'=n^{-1}\beta$ be so, then there exist u, v and $w\in\boldsymbol{Z}$ such that $\beta=\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\sigma_1\begin{pmatrix} 1 & 0\\ 0 & n^2\end{pmatrix}\sigma_2$ with $\sigma_1=\begin{pmatrix} a & -v\\ c & u\end{pmatrix}\in\Gamma$ and $\sigma_2=\begin{pmatrix} 1 & w\\ 0 & 1\end{pmatrix}\in\Gamma$. Therefore, putting $\tau=\begin{pmatrix} n^{-1} & 0\\ 0 & n\end{pmatrix}$,

$$\begin{aligned}&(\sigma_1, j(\sigma_1,z))(\tau,\sqrt{n})(\sigma_2, j(\sigma_2,z))\\ &\quad=\left(\beta', \left(\frac{c}{a}\right)\varepsilon_a^{-1}\sqrt{J(\beta',z)}\right)\end{aligned}$$

$$= \left(\beta', \left(\frac{c}{a}\right)\varepsilon_a^{-1}\sqrt{J(\beta, z)}\sqrt{n}^{-1}\right).$$

Hence, we see

Remark 2. Under the same assumptions as in Remark 1,

$$\left(\beta', \left(\frac{c}{a}\right)\varepsilon_a^{-1}\sqrt{J(\beta', z)}\right) \in \Delta\left(\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}, \sqrt{n}\right)\Delta .$$

If β is an elliptic element fixing z_0 in the upper half plane, then $\eta(\beta) = \left(\frac{c}{a}\right)\varepsilon_a^{-\kappa}\sqrt{J(\beta, z_0)}^{\kappa}\sqrt{n}^{-\kappa}$. On the other hand, for parabolic or hyperbolic elements, we further note the following

Remark 3. Let $\beta' = n^{-1}\beta \neq \pm 1$, κ be a cusp fixed by $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $c \neq 0$ and $\rho = \begin{pmatrix} \kappa & \kappa - 1 \\ 1 & 1 \end{pmatrix}$, then

$$\begin{aligned} &(\rho^{-1}, \sqrt{J(\rho^{-1}, z)})(\beta', \sqrt{J(\beta', z)})(\rho, \sqrt{J(\rho, z)}) \\ &\quad = \left(\begin{pmatrix} \lambda^{-1} & \nu \\ 0 & \lambda \end{pmatrix}, (c\kappa + d, c\lambda)(-1, \lambda)\sqrt{\lambda}\right) \end{aligned}$$

where $n\nu = -c(\kappa - 1 - (a - d)/2c)^2$ and $n\lambda = a - c\kappa$.

In connection with this remark, we first note $\operatorname{sgn}\nu = -\operatorname{sgn} c$, which is used in part 1 below for determing $\mu(\beta)$.

Next, if $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is parabolic, $\operatorname{tr}\beta = 2n$, $c \neq 0$ and $(a, c) = 1$, then it follows from Remark 1 and 2 that $\eta(\beta) = \varepsilon_a^{-\kappa}\left(\frac{c}{a}\right)(c\kappa + d, c)$. Since $\kappa = (a - d)/2c$, $\eta(\beta) = \varepsilon_a^{-\kappa}\left(\frac{c}{a}\right)$.

Finally, if $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is hyperbolic, $\beta' = n^{-1}\beta$, $c \neq 0$ and $(a, c) = 1$, then $\kappa = (a - d \pm ((a + d)^2 - 4n^2)^{1/2})/2c$ and $\lambda = (a + d \mp ((a + d)^2 - 4n^2)^{1/2})/2n$ accordingly. Now, if $a + d > 2n$, then $|(a + d + ((a + d)^2 - 4n^2)^{1/2})/2n| > 1$, and if $a + d < -2n$, then $|(a + d - ((a + d)^2 - 4n^2)^{1/2})/2n| > 1$. Therefore, the upper fixed point of β defined in [1] is $\kappa_0 = (a - d - \operatorname{sgn}(a + d)\cdot((a + d)^2 - 4n^2)^{1/2})/2c$ and so $\lambda_0 = \lambda(\beta) = (a + d + \operatorname{sgn}(a + d)((a + d)^2 - 4n^2)^{1/2})/2n$. On the other hand, it follows from Remark 1 and 2 that

$\eta(\beta) = \varepsilon_a^{-\kappa}\left(\frac{c}{a}\right)(c\kappa_0 + d, c\lambda_0)(\lambda_0, -1)\sqrt{\lambda_0}^{\kappa}$. Since $\operatorname{sgn}(c\kappa_0 + d) = \operatorname{sgn}\lambda_0 = \operatorname{sgn}(a + d)$, $\eta(\beta) = \varepsilon_a^{-\kappa}\left(\frac{c}{a}\right)(a + d, c)\sqrt{\lambda(\beta)}^{\kappa}$.

By the way, we can systematically write out all the Γ-conjugacy classes as well as the entries of the representative of them by [4]. Thus, all we have to do for the computation of $\sum_C J(C)$ is to sum up $J(C)$ neatly, which is done in the following three parts.

1. Parabolic case

Let us write out all the parabolic equivalence classes in $\Phi\left(\Gamma\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}\cdot\Gamma/\Gamma\right)$. We impose the following conditions on f, ζ and S: (i) f is a representative of $(\boldsymbol{Z}/n\boldsymbol{Z})^{\times}$, (ii) ζ is a representative of $\boldsymbol{Z}/N\boldsymbol{Z}$ such that $\zeta^2 \equiv 0\,(N)$, and (iii) $S \subset S(\zeta) = \{p\,|\,N;\, \zeta^2 \equiv 0\,(p^{\nu_p+1})\}$. Let

$$\beta = \beta(f, \zeta, S) = \begin{pmatrix} n + \zeta f & f\prod_{p\in S} p^{-\nu_p}|\zeta^2|_p^{-1} \\ -f\zeta^2\prod_{p\in S} p^{\nu_p}|\zeta^2|_p & n - \zeta f \end{pmatrix}.$$

Then, $n^{-1}\beta$ forms a complete system of representatives of the parabolic equivalence classes in $\Gamma\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}\Gamma$ in the sence of [1], when f, ζ and S vary satisfying (i) $\sim$ (iii). Here, by the suitable choice of the representative, we may assume (iv) $f > 0$, (v) $4\,|\,f$, and (vi) $(\zeta, n) = 1; \zeta \neq 0$. On the other hand, $Z_\Gamma(\beta)/\{\pm 1\}$ is generated by

$$\sigma = 1 + f^{-1}(\beta - n) = \begin{pmatrix} 1 + \zeta & \prod_{p\in S} p^{-\nu_p}|\zeta^2|_p^{-1} \\ -\zeta^2\prod_{p\in S} p^{\nu_p}|\zeta^2|_p & 1 - \zeta \end{pmatrix},$$

and, because of (iv), there exists $\rho \in GL^+(2, \boldsymbol{R})$ such that $\rho^{-1}\sigma\rho = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and that $\rho^{-1}\beta\rho = \begin{pmatrix} n & f \\ 0 & n \end{pmatrix}$, in view of [4] and Remark 3. Therefore, x in [1] equals fn^{-1}. Denote $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let κ be the cusp fixed by σ and let

$\rho_1 = \begin{pmatrix} \kappa & \kappa - 1 \\ 1 & 1 \end{pmatrix}$, then $(\rho_1^{-1}, \sqrt{J(\rho_1^{-1}, z)})(\sigma, j(\sigma, z))(\rho_1, \sqrt{J(\rho_1, z)}) = \left(\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \varepsilon_a^{-1}\left(\frac{c}{a}\right)\right)$, and so

$$e(\mu(\beta)) = \varepsilon_{1+\zeta}^{-\kappa}\left(\frac{-1}{1+\zeta}\right)\prod_{p \in S}\left(\frac{p}{1+\zeta}\right)^{\nu_p}.$$

We denote $\mu(\beta)$, which depends only on ζ and S, by $\mu_{\zeta,S}$, and we put $f = 4f'$ in view of (v). Then, due to (vi), $\eta(\beta) = \varepsilon_{a'}^{-\kappa}\left(\frac{c'}{a'}\right) = \varepsilon_{n+4f'\zeta}^{-\kappa} \cdot \left(\frac{-f'}{n+4f'\zeta}\right)\prod_{p \in S}\left(\frac{p}{n+4f'\zeta}\right)^{\nu_p} = \varepsilon_n^{-\kappa}\left(\frac{-f'}{n}\right)\prod_{p \in S}\left(\frac{p}{n}\right)^{\nu_p}$ for $\beta = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. Thus, denoting f' by f,

$$\tilde{p} = \sum_{C;\ \text{parabolic}} J(C) = \sum_{\substack{f \bmod n \\ (f,n)=1 \\ f>0}} \sum_{\substack{\zeta \bmod N \\ \zeta^2 \equiv 0 (N)}} \sum_{S \subset S(\zeta)} \varepsilon_n^{\kappa}\left(\frac{-f}{n}\right) \cdot$$

$$\cdot\, e(\mu_{\zeta,S} 4fn^{-1}) \prod_{p \in S}\left(\frac{p}{n}\right)^{\nu_p} \cdot \begin{cases} (2^{-1} - \mu_{\zeta,S})\,, & (n = 1)\,, \\ (1 - e(4fn^{-1}))^{-1}\,, & (n > 1)\,. \end{cases}$$

Let $N = 4M$, $(M, 2) = 1$ and $n = 1$, then $\tilde{p}$ is the sum of

$$\tilde{p}_1 = 2 \sum_{\substack{\zeta \bmod M \\ \zeta^2 \equiv 0 (M)}} \sum_{S \subset S'(\zeta)} (-2^{-1})$$

and

$$\tilde{p}_2 = \sum_{\substack{\zeta \bmod M \\ \zeta^2 \equiv 0 (M) \\ (\zeta, 2) = 1}} \sum_{S \subset S'(\zeta)} (2^{-1} - \mu_{2\zeta,S})\,,$$

where $S'(\zeta) = \{p \,|\, M\,;\, \zeta^2 \equiv 0\ (p^{\nu_p+1})\}$. Since $e(\mu_{2\zeta,S}) = i^{-\kappa} \cdot (-1) \cdot \prod_{p \in S}\left(\frac{p}{1+2\zeta}\right)^{\nu_p} = i^{-\kappa}(-1)\prod_{p \in S}\left(\frac{-1}{p}\right)^{\nu_p}$ for odd ζ, $\mu_{2\zeta,S} = 2^{-1} - 4^{-1}\left(\frac{-1}{\kappa}\right)\prod_{p \in S}\left(\frac{-1}{p}\right)^{\nu_p}$. Therefore, $\tilde{p}_2 = 4^{-1}\left(\frac{-1}{\kappa}\right)\sum_{\zeta}\sum_{S}\prod_{p \in S}\left(\frac{-1}{p}\right)^{\nu_p} = -2^{-1} \cdot (-1)^{(\kappa-3)/2} \cdot h(-4)w(-4)^{-1}\prod_{p|M}\sum_{\xi}\left(1 + \delta_p(\xi)\left(\frac{-1}{p}\right)^{\nu_p}\right)$, where ξ runs over all the representatives of $Z/p^{\nu_p}Z$ such that $\xi^2 \equiv 0\ (p^{\nu_p})$, and

$$\delta_p(\xi) = \begin{cases} 1 & \text{for } \zeta^2 \equiv 0\ (p^{\nu_p+1}) \\ 0 & \text{otherwise}\,. \end{cases}$$

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With the same notations, $\tilde{p}_1 = -\prod_{p|M} \sum_{\xi} (1 + \delta_p(\xi))$.

Next, assume $n > 1$, then $\tilde{p} = \tilde{p}_1 + \tilde{p}_2$ with

$$\tilde{p}_1 = 2\varepsilon_n^{\kappa} \sum_{\substack{\zeta \bmod M \\ \zeta^2 \equiv 0 (M)}} \sum_{S \subset S'(\zeta)} \left(\prod_{p \in S} \left(\frac{p}{n}\right)^{\nu_p} \right) \cdot \sum_{\substack{f \bmod n \\ (f,n)=1 \\ f>0}} \left(\frac{-f}{n}\right) e(4fn^{-1})(1 - e(4fn^{-1}))^{-1}$$

and

$$\tilde{p}_2 = \varepsilon_n^{\kappa} \sum_{\substack{\zeta \bmod M \\ \zeta^2 \equiv 0 (M) \\ (\zeta,2)=1}} \sum_{S \subset S'(\zeta)} \left(\prod_{p \in S} \left(\frac{p}{n}\right)^{\nu_p} \right) \cdot \sum_{\substack{f \bmod n \\ (f,n)=1 \\ f>0}} \left(\frac{-f}{n}\right) e(\mu_{2\zeta,S} 4fn^{-1})(1 - e(4fn^{-1}))^{-1} .$$

Put $\mu_S = 4\mu_{2\zeta,S}$. Since $e(-fn^{-1})(1-e(-4fn^{-1}))^{-1} = -e(3fn^{-1})(1-e(4fn^{-1}))^{-1}$,

$$\sum_{\substack{f \bmod n \\ (f,n)=1 \\ f>0}} \left(\frac{f}{n}\right) e(\mu_S fn^{-1})(1 - e(4fn^{-1}))^{-1} = \left(-\left(\frac{-1}{n}\right)\right)^{(\mu_S+1)/2} \sum_{\substack{f \bmod n \\ (f,n)=1 \\ f>0}} \left(\frac{f}{n}\right) e(3fn^{-1})(1 - e(4fn^{-1}))^{-1} .$$

Observing $\left(\prod_{p \in S} \left(\frac{p}{n}\right)^{\nu_p}\right)\left(-\left(\frac{-1}{n}\right)\right)^{(\mu_S+1)/2} = \left(-\left(\frac{-1}{n}\right)\right)^{(\kappa-3)/2} \prod_{p \in S} \left(\frac{-n}{p}\right)^{\nu_p}$,

$$\tilde{p}_2 = (-1)^{(\kappa-3)/2} \sum_{\substack{\zeta \bmod M \\ \zeta^2 \equiv 0 (M) \\ (\zeta,2)=1}} \sum_{S \subset S'(\zeta)} \left(\prod_{p \in S} \left(\frac{-n}{p}\right)^{\nu_p} \right) \cdot \varepsilon_n^{-1} \sum_{\substack{f \bmod n \\ (f,n)=1 \\ f>0}} \left(\frac{-f}{n}\right) e(3fn^{-1})(1 - e(4fn^{-1}))^{-1} .$$

On the other hand, by the Dirichlet class number formula,

$$\varepsilon_n^{-1} \sum_{\substack{f \bmod n \\ (f,n)=1 \\ f>0}} \left(\frac{-f}{n}\right) e(3fn^{-1})(1 - e(4fn^{-1}))^{-1} = \begin{cases} -2^{-1}n^{1/2}h(-4n)w(-4n)^{-1} , & (n \equiv 1 \ (4)) , \\ -2^{-1}n^{1/2}\left(1 - \left(\frac{2}{n}\right)\right)h(-n)w(-n)^{-1} , & (n \equiv -1 \ (4)) , \end{cases}$$

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and

$$\varepsilon_n^{-1} \sum_{\substack{f \bmod n \\ (f,n)=1 \\ f>0}} \left(\frac{-f}{n}\right) e(fn^{-1})(1 - e(fn^{-1}))^{-1}$$

$$= \begin{cases} -\delta_0(\sqrt{n})2^{-1}\varphi(n)\,, & (n \equiv 1\ (4)) \\ -h(-n)w(-n)^{-1}\sqrt{n}\,, & (n \equiv -1\ (4))\,, \end{cases}$$

$$\text{where } \delta_0(\sqrt{n}) = \begin{cases} 1 & \text{for } \sqrt{n} \in \boldsymbol{Z}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Put $\tilde{c}_p = \sum_\xi \left(1 + \delta_p(\xi)\left(\frac{p}{n}\right)^{\nu_p}\right)$ and $\tilde{c}'_p = \sum_\xi \left(1 + \delta_p(\xi)\left(\frac{-n}{p}\right)^{\nu_p}\right)$, where the summations run over all the representatives ξ of $\boldsymbol{Z}/p^{\nu_p}\boldsymbol{Z}$ satisfying $\xi^2 \equiv 0\ (p^{\nu_p})$, and $\delta_p(\xi) = \begin{cases} 1 & \text{for } \xi^2 \equiv 0\ (p^{\nu_p+1}), \\ 0 & \text{otherwise.} \end{cases}$ Then $\tilde{p}_2$ is equal to $-\frac{1}{2}\sqrt{n}(-1)^{(\kappa-3)/2}\left(\prod_{p\mid M} \tilde{c}'_p\right)$ times

$$\begin{cases} h(-4n)w(-4n)^{-1}\,, & (n \equiv 1\ (4))\,, \\ \left(1 - \left(\frac{2}{n}\right)\right)h(-n)w(-n)^{-1}\,, & (n \equiv -1\ (4))\,, \end{cases}$$

and

$$\tilde{p}_1 = -\varepsilon_n^{\kappa-3}\left(\prod_{p\mid M} \tilde{c}_p\right)\begin{cases} \delta_0(\sqrt{n})\varphi(n)\,, & (n \equiv 1\ (4))\,, \\ 2h(-n)w(-n)^{-1}\sqrt{n}\,, & (n \equiv -1\ (4))\,. \end{cases}$$

If $n \equiv -1\ (4)$, then $\tilde{c}_p = \tilde{c}'_p$, $\varepsilon_n^{\kappa-3} = (-1)^{(\kappa-3)/2}$ and $\left(1 - \left(\frac{2}{n}\right)\right)h(-n)w(-n)^{-1} + 4h(-n)w(-n)^{-1} = \left(1 + \left\{\frac{-n}{2}\right\}\right)h(-n)w(-n)^{-1} + \left(1 + \left\{\frac{-4n}{2}\right\}\right)h(-4n)w(-4n)^{-1}$. Therefore, $\tilde{p} = -(-1)^{(\kappa-3)/2}\cdot 2^{-1}\cdot\sqrt{n}\left(\left(1 + \left\{\frac{-n}{2}\right\}\right)h(-n)w(-n)^{-1} + \left(1 + \left\{\frac{-4n}{2}\right\}\right)h(-4n)w(-4n)^{-1}\right)\prod_{p\mid M} \tilde{c}'_p$. If $n \equiv 1\ (4)$, then $\tilde{p}_1 = -\delta_0(\sqrt{n})\varphi(n)\prod_{p\mid M} \tilde{c}_p$ and $\tilde{p}_2 = -\frac{1}{2}\sqrt{n}(-1)^{(\kappa-3)/2}\cdot\left(1 + \left\{\frac{-4n}{2}\right\}\right)h(-4n)w(-4n)\prod_{p\mid M} \tilde{c}'_p$. These expressions also fit for the case $n = 1$. Note besides that $\sqrt{n} \in \boldsymbol{Z}$ implies $c_p = \sum_\xi (1 + \delta_p(\xi))$.

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2. Elliptic case

Let $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an elliptic element in $\Gamma\begin{pmatrix} 1 & 0 \\ 0 & n^2 \end{pmatrix}\Gamma$ such that $(a, c) = 1$ and that $c \neq 0$. Let z_0 be the fixed point of β in the upper half plane. Then, $\eta(\beta) = \left(\frac{c}{a}\right)\sqrt{J(\beta, z_0)}^{\kappa}\sqrt{n}^{-\kappa} = \chi(\beta)\zeta(\beta)^{\kappa}$ with $\chi(\beta) = \left(\frac{c}{a}\right)$ and $\zeta(\beta) = (2\sqrt{n}\varepsilon_a)^{-1}(\sqrt{2n + a + d} + i\,(\operatorname{sgn} c)\sqrt{2n - a - d})$, and so $J(\beta) = 2^{-1}\chi(\beta)\zeta(\beta)^{-\kappa}/(1 - \zeta(\beta)^{-4})$. Thus, it is easy to see that $J(-\beta') = J(\beta)$ and $J(w\beta w) = \overline{J(\beta)}$. Let $\Gamma^* = \Gamma \cup \Gamma w$ and $W(\beta) = [Z_{\Gamma^*}(\beta) : Z_\Gamma(\beta)]$, then $\tilde{e} = \sum_{C;\ \text{elliptic}} J(C) = \frac{1}{2}\sum_1 (J(\beta) + J(w\beta w)) = \sum_2 W(\beta)^{-1}(J(\beta) + J(w\beta w)) = 2\sum_3 (J(\beta) + \overline{J(\beta)})W(\beta)^{-1}$, where β in $\sum_1$ runs over all the representatives of elliptic Γ-conjugacy classes, β in $\sum_2$ runs over those of elliptic Γ^*-conjugacy classes, and β in $\sum_3$ runs over those of the elliptic Γ^*-conjugacy classes congruent to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ modulo 4. Thus, $\tilde{e} = \sum_3 \chi(\beta)\Xi(\beta)W(\beta)^{-1}$ with $\Xi(\beta) = (\zeta(\beta)^{-\kappa+2} - \zeta(\beta)^{\kappa-2})/(\zeta(\beta)^2 - \zeta(\beta)^{-2})$ which depends only on $\operatorname{tr}\beta$. Now, we want to write out all the representatives, congruent to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ modulo 4, of the elliptic Γ^*-conjugacy classes. For this, let (i) $t^2 - 4n^2 < 0$, (ii) $f^2 \mid (t^2 - 4n^2)$, (iii) $(f, t, n) = 1$, and (iv) ξ be a representatives of $\boldsymbol{Z}/Nf\boldsymbol{Z}$ such that $F(\xi) \equiv 0\ (Nf^2)$ with $F(X) = X^2 - tX + n^2$, $2\xi \equiv t\ (f)$, and that $(\xi, n) = 1$. Let (v) $S \subset S(\xi) = \{p \mid N\,;\ (t^2 - 4n^2)f^{-2} \equiv 0\ (p),\ F(\xi) \equiv 0\ (p^{\nu_p + 2\rho_p + 1})\}$ with $\rho_p = \operatorname{ord}_p f$. Let (vi) $\mathfrak{p}$ be a prime ideal of Λ with $(\mathfrak{p}, nN(t^2 - 4n^2)) = 1$ representing a ring ideal class of the order Λ, where $\Lambda = I \cap (\boldsymbol{Q} + \boldsymbol{Q}\varphi)$ with $I = \begin{pmatrix} \boldsymbol{Z} & \boldsymbol{Z} \\ N\boldsymbol{Z} & \boldsymbol{Z} \end{pmatrix}$ and with

$$\begin{aligned} \varphi &= \varphi(t, f, \xi, S) \\ &= \begin{pmatrix} \xi & -f \prod_{p \in S} p^{-\nu_p} |f^{-2}F(\xi)|_p^{-1} \\ f^{-1}F(\xi) \prod_{p \in S} p^{\nu_p} |F(\xi)f^{-2}|_p & t - \xi \end{pmatrix}. \end{aligned}$$

(Note that each ideal class contains such $\mathfrak{p}$ i.e., $(\mathfrak{p}, nN\cdot(t^2 - 4n^2)) = 1$. This is obvious when Λ is maximal and for this when Λ is not, it is sufficient to refer to Cor. 1 of Prop. 1 in [9].) We denote by $h(\Lambda)$ (respectively $w(\Lambda)$) the class number of ring ideals of the order Λ (respectively $[\Lambda^\times : \boldsymbol{Z}^\times]$). Let $M_\mathfrak{p}$ be such an element in $GL^+(2, \boldsymbol{Q})$ that $IM_\mathfrak{p} = I\mathfrak{p}$. Then, $\beta = M_\mathfrak{p}\varphi M_\mathfrak{p}^{-1}$ forms a complete system of representatives of the elliptic Γ^*-conjugacy classes when t, f, ξ, S and $\mathfrak{p}$ vary under the above conditions

(i) $\sim$ (vi). Moreover β satisfies the conditions in Remark 1. Note that the solution ξ does not exist unless $\left\{\frac{t^2-4n^2}{2}\right\}=1$, so that $t\equiv 2\,(4)$. Furthermore, note that all the β are congruent to $\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}$ or $\begin{pmatrix}-1 & *\\ 0 & -1\end{pmatrix}$ modulo 4 alltogether according as $\xi\equiv 1\,(4)$ or $\xi\equiv -1\,(4)$. Therefore, $\tilde{e}=\sum_t\sum_f\sum_\xi\sum_S\sum_{\mathfrak{p}}\chi(\beta)\Xi(\beta)W(\beta)^{-1}$ where t, f, ξ, S and $\mathfrak{p}$ run under the conditions (i) $\sim$ (vi) and (vii) $\xi\equiv 1\,(4)$. Since $W(\beta)=w(\Lambda)=[\Lambda^\times; \boldsymbol{Z}^\times]=w((t^2-4n^2)f^{-2})$ and $\Xi(\beta)=\Xi_t=(\zeta_t^{-\kappa+2}-\zeta_t^{\kappa-2})/(\zeta_t^2-\zeta_t^{-2})$ with $\zeta_t=(2\sqrt{n})^{-1}\cdot(\sqrt{2n+t}+i\sqrt{2n-t})$, we see that $\tilde{e}=\sum_t\Xi_t\sum_f\sum_\xi\sum_S w(\Lambda)\sum_{\mathfrak{p}}\chi(\beta)$. To see $\sum_{\mathfrak{p}}\chi(\beta)$, let $\varphi=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $\beta=M_{\mathfrak{p}}\varphi M_{\mathfrak{p}}^{-1}=\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}$. If $\mathfrak{p}$ is not principal, then the cardinality $\#(\Lambda/\mathfrak{p})$ of the residue field $\Lambda/\mathfrak{p}$ is a prime p, and so we can take $\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}$ or $\begin{pmatrix}1 & j\\ 0 & p\end{pmatrix}$ for $M_{\mathfrak{p}}$ such that $I\mathfrak{p}=IM_{\mathfrak{p}}$. When $M_{\mathfrak{p}}=\begin{pmatrix}1 & j\\ 0 & p\end{pmatrix}$, it follows from

$$\begin{aligned}\varphi=\begin{pmatrix}a & b\\ c & d\end{pmatrix}&=\begin{pmatrix}1 & -jp^{-1}\\ 0 & p^{-1}\end{pmatrix}\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}\begin{pmatrix}1 & j\\ 0 & p\end{pmatrix}\\ &=\begin{pmatrix}a'-jp^{-1}c' & ja'-j^2p^{-1}c'+pb'+d'j\\ c'p^{-1} & c'jp^{-1}+d'\end{pmatrix},\end{aligned}$$

$\varphi\in M_2(\boldsymbol{Z})$ and $\beta\in M_2(\boldsymbol{Z})$, that $p\,|\,c'$ and $(a', p)=1$ in view of (vi), and so

$$\chi(\varphi)=\left(\frac{c}{a}\right)=\left(\frac{c'p^{-1}}{a'-jp^{-1}c'}\right)=\left(\frac{c'p^{-1}}{a'}\right)=\left(\frac{c'}{a'}\right)\left(\frac{p}{a'}\right)=\left(\frac{p}{a'}\right)\chi(\beta)\,.$$

Since $a\equiv a'\equiv 1\,(4)$ and $a+d=a'+d'=t$,

$$\begin{aligned}\left(\frac{p}{a'}\right)=\left(\frac{a'}{p}\right),\quad \left(\frac{a'}{p}\right)\left(\frac{a'+d'+2n}{p}\right)&=\left(\frac{a'^2+a'd'+2na'}{p}\right)\\ &=\left(\frac{a'^2+n^2+b'c'+2na'}{p}\right)=\left(\frac{a'^2+n^2+2a'n}{p}\right)\\ &=\left(\frac{(a'+n)^2}{p}\right)=1\end{aligned}$$

and so $\left(\frac{p}{a'}\right)=\left(\frac{t+2n}{p}\right)$. When $M_{\mathfrak{p}}=\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}$, $\chi(\beta)=\left(\frac{t+2n}{p}\right)\chi(\varphi)$ in the same way. Let $K_\varphi=\boldsymbol{Q}((2n+t)^{1/2}, (t-2n)^{1/2})$, then the conductor of the extention $K_\varphi/\boldsymbol{Q}((t^2-4n^2)^{1/2})$ is equal to (d_+, d_-) with the conductors $d_\pm$

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of the extentions $\boldsymbol{Q}((t \pm 2n)^{1/2})/\boldsymbol{Q}$. Reviewing that $(t, f, n) = 1$ and that $\xi \equiv 1\,(4)$, we can easily check that (d_+, d_-) divides the conductor of the order Λ. So, the character χ_φ of $K_\varphi/\boldsymbol{Q}((t^2 - 4n^2)^{1/2})$ is welldefined on the ring ideal class group of Λ. Since $\chi_\varphi(\mathfrak{p}) = \left(\frac{d_+}{p}\right)$ for $\mathfrak{p}$ such that $\sharp(\Lambda/\mathfrak{p})$ is a prime p not dividing the conductor of Λ, we have $\chi(\beta) = \chi_\varphi(\mathfrak{p})\chi(\varphi)$. Hence, $\sum_{\mathfrak{p}} \chi(\beta) = \chi(\varphi) \sum_{\mathfrak{p}} \chi_\varphi(\mathfrak{p})$, and so $\sum_{\mathfrak{p}} \chi_\varphi(\mathfrak{p})$ is either $h(\Lambda) = h((t^2 - 4n^2)f^{-2})$ or 0 according as $t + 2n$ is a square or not. Thus, $\tilde{e} = \sum_t \Xi_t \sum_f h((t^2 - 4n^2)f^{-2})w((t^2 - 4n^2)f^{-2})^{-1} \sum_\xi \sum_S \chi(\varphi)$, where t, f, ξ and S run under the conditions (i) $\sim$ (v), (vii) and that $t + 2n$ is a square of a non zero integer. Note that if $t + 2n = s^2$ and $s > 0$, then $t^2 - 4n^2 = s^2(s^2 - 4n)$ and $\Xi_t = -s^{-1}n^{(4-\kappa)/2}(x^{\kappa-2} - y^{\kappa-2})(x - y)^{-1}$ with the solutions x and y of $X^2 - sX + n = 0$. Thus, $\tilde{e} = -n^{(4-\kappa)/2} \sum_s s^{-1}(x^{\kappa-2} - y^{\kappa-2})(x - y)^{-1} \sum_f h(s^2(s^2 - 4n)f^{-2})w(s^2(s^2 - 4n)f^{-2})^{-1} \sum_\xi \sum_S \chi(\varphi)$, where s runs under the conditions that $s > 0$ and that $s^2 - 4n < 0$, and f, ξ and S run under the conditions (ii) $\sim$ (v) and (vii) with $t = s^2 - 2n$. Now, we want to find the value of $\sum_\xi \sum_S \chi(\varphi)$, where $\chi(\varphi) = \left(\frac{c}{a}\right) = \left(\frac{f}{\xi}\right) \prod_{p \in S} \left(\frac{p^{\nu_p}|F(\xi)|_p^{-1}}{\xi}\right)$, and ξ and S run satisfying the conditions (i) $\sim$ (v) and (vii) with $t = s^2 - 2n$. Put $t = 2t'$ and $f = 2^{\rho_2}f'f''$ with $f' \mid s$, $f''^2 \mid s^2 - 4n$ and $(f'f'', 2) = 1$. Then,

$$\left(\frac{f}{\xi}\right) = \left(\frac{2}{\xi}\right)^{\rho_2}\left(\frac{\xi}{f'}\right)\left(\frac{\xi}{f''}\right) = \left(\frac{2}{\xi}\right)^{\rho_2}\left(\frac{t'}{f'}\right)\left(\frac{t'}{f''}\right) = \left(\frac{2}{\xi}\right)^{\rho_2}\left(\frac{s^2 - 4n}{f'}\right).$$

Therefore, it is easy to see

$$\sum_\xi \sum_S \chi(\varphi) = \left(\frac{s^2 - 4n}{f'}\right) \sum_\xi \sum_S \prod_{p \in S} \left(\frac{p^{\nu_p}|F(\xi)|_p^{-1}}{\xi}\right) = \left(\frac{s^2 - 4n}{f'}\right) \prod_{p \mid N} \tilde{c}(t, f, p)$$

with

$$\tilde{c}(t, f, p) = \begin{cases} \sum_\xi \left(1 + \delta_p(\xi)\left(\frac{p^{\nu_p}|F(\xi)|_p^{-1}}{\xi}\right)\right), & (p \neq 2), \\ \sum_\xi \left(\frac{2}{\xi}\right)^{\rho_2}\left(1 + \delta_2(\xi)\left(\frac{|F(\xi)|_2^{-1}}{\xi}\right)\right), & (p = 2), \end{cases}$$

where the summations run over all the representatives ξ of $\boldsymbol{Z}/p^{\nu_p+\rho_p}\boldsymbol{Z}$ such that $F(\xi) \equiv 0\,(p^{\nu_p+2\rho_p})$, $2\xi \equiv t\,(p^{\rho_p})$, $(\xi, n) = 1$ and that $\xi \equiv 1\,(4)$, and

$$\delta_p(\xi) = \begin{cases} 1 & \text{if } p \mid D = (t^2 - 4n^2)f^{-2} \text{ and } F(\xi) \equiv 0\ (p^{\nu_p+2\rho_p+1}), \\ 0 & \text{otherwise}. \end{cases}$$

Let $p \neq 2$ and $\delta_p(\xi) = 1$, then $\xi \equiv t'\ (p)$ and $\left(\frac{p}{\xi}\right) = \left(\frac{\xi}{p}\right) = \left(\frac{t'}{p}\right) = \left(\frac{2s'^2 - n}{p}\right)$ with $s = 2s'$. Since $D \equiv 0\ (p^2)$, we have $p \mid sf'^{-1}$ or $p^2 \mid (s^2 - 4n)f''^{-2}$. Let $p \mid s$ and $\left(\frac{s'^2 - n}{p}\right) = -1$, then $\mathrm{ord}_p F(\xi) \equiv 0\ (2)$. Therefore, $\left(\frac{|F(\xi)|_p^{-1}}{\xi}\right) = 1$ if $\delta_p(\xi) = 1$. Hence, $\tilde{c}(t, f, p) = \sum_{\xi} \left(1 + \delta_p(\xi)\left(\frac{2s'^2 - n}{p}\right)^{\nu_p}\right)$, and $\tilde{c}(t, f, 2) = \sum_{\xi} \left(\frac{2}{\xi}\right)^{\rho_2} \cdot (1 + \delta_2(\xi))$ in the same way.

3. Hyperbolic case

Let $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a hyperbolic element in $\Gamma\begin{pmatrix} 1 & 0 \\ 0 & n^2 \end{pmatrix}\Gamma$ such that $(a, c) = 1$ and that $c \neq 0$; then $\lambda(\beta) = (2n)^{-1}(a + d + \mathrm{sgn}\,(a + d)((a + d)^2 - 4n^2)^{1/2})$ and $\eta(\beta) = \left(\frac{c}{a}\right)\varepsilon_a^{-\kappa}(a + d, c)\sqrt{\lambda(\beta)}^{\kappa}$. Therefore, $J(-\beta) = J(\beta)$ and $J(w\beta w) = \left(\frac{-1}{a}\right)(a + d, -1)J(\beta)$ due to $\mathrm{sgn}\,(a + d) = \mathrm{sgn}\,\lambda(\beta)$. Hence $\tilde{h} = \sum_1 J(\beta) = \frac{1}{2}\sum_1 (J(\beta) + J(w\beta w)) = \frac{1}{2}\sum_2 (J(\beta) + J(w\beta w)) = \sum_3 (J(\beta) + J(w\beta w)) = 2\sum_4 (J(\beta) + J(w\beta w))W(\beta)^{-1} = 4\sum_4 J(\beta)W(\beta)^{-1}$, where β in $\sum_1$ runs over all the representatives of the hyperbolic Γ-conjugacy classes, β in $\sum_2$ runs over those such that either $a + d > 0, a \equiv 1\ (4)$ or $a + d < 0, a \equiv -1\ (4)$, β in $\sum_3$ runs over those such that $a + d > 0$ and that $a \equiv 1\ (4)$, and β in $\sum_4$ runs over those, such that $a + d > 0$ and that $a \equiv 1\ (4)$, of the hyperbolic Γ^*-conjugacy classes. Let $t^2 - 4n^2$ be the square of a non zero integer. Let f, ξ, φ, etc. be same as in the elliptic case. Then

$$\tilde{h} = -2\sum_{t>0}\sum_{f}\sum_{\xi\equiv 1\ (4)}\sum_{S}\sum_{\mathfrak{p}} \chi(M_{\mathfrak{p}}\varphi M_{\mathfrak{p}}^{-1})w(\Lambda)^{-1}\lambda(\varphi)^{-\kappa/2}(1 - \lambda(\varphi)^{-2})^{-1},$$

and, in the same way as in the elliptic case, $\tilde{h} = -2\sum_{t>0}\sum_{f}\sum_{\xi\equiv 1\ (4)}\sum_{S} h(\Lambda)w(\Lambda)^{-1} \cdot \chi(\varphi)\lambda(\varphi)^{-\kappa/2}(1 - \lambda(\varphi)^{-2})^{-1}$, where t runs under the condition that $t + 2n$ is the square of a positive integer. Thus, in the same way as in the elliptic case,

$$\tilde{h} = -2\sqrt{n}^{4-\kappa}\sum_{s>0}\sum_{f} s^{-1}y^{\kappa-2}(x - y)^{-1}\tfrac{1}{2}\varphi(s(s^2 - 4n)^{1/2}f^{-1})\prod_{p\mid N}\tilde{c}(t, f, p),$$

where s runs under the condition that $s^2 - 4n$ is the square of a non zero integer, x and y are the solutions of $X^2 - sX + n = 0$ such that

$x > y$, and other notations are same as in the elliptic case.

§ 3. Explicit formula (special cases)

The aim of this section is to derive the identities (1) of § 1 from Proposition 1′. First, we assume $M = 1$. Then, observing

$$s''^{-1} \sum_{f'} \frac{h(s''^2 f'^{-2} D')}{w(s''^2 f'^{-2} D')} \left(\frac{s^2 - 4n}{f'} \right) = \frac{h(D')}{w(D')}$$

in $\tilde{e}$ of Proposition 1′, we obtain

$$\tilde{e} = -n^{(4-\kappa)/2} \sum_{s, f_2} (x^{\kappa-2} - y^{\kappa-2})(x - y)^{-1} \cdot \frac{h(D')}{w(D')} \left(1 + \left\{ \frac{D'}{2} \right\} \right),$$

where s and f_2 run in such a way that $s^2 - 4n < 0$, $s > 0$, $f_2 > 0$, $f_2^2 \,|\, (s^2 - 4n)$, $D' = (s^2 - 4n) f_2^{-2} \equiv 0$ or $1\,(4)$ and $(f_2, n) = 1$.

Since of course $\tilde{e}$ depends on a given n, we denote $\tilde{e}$ by $\tilde{e}_n$ in the indexed form. We should further note that $T'(n) = \sum_{f_1^2 | n} \Gamma' \begin{pmatrix} f_1 & 0 \\ 0 & n f_1^{-1} \end{pmatrix} \Gamma'$ and $T_{\kappa-1}(n) = \rho_{\kappa-1}(T'(n))$ imply the fact that $T(n)$ in [4] equals $n^{1-(\kappa-1)/2} T_{\kappa-1}(n)$. Observing now

$$\tilde{\rho}_{\kappa/2}(T'(n)) = \sum_{f_1^2 | n} f_1 n^{\kappa/2-2} \left[\Delta \left(\begin{pmatrix} n^{-1} f_1^2 & 0 \\ 0 & n f_1^{-2} \end{pmatrix}, \quad \sqrt{n} f_1^{-1} \right) \Delta \right]_\kappa,$$

put $\check{e} = \sum_{f_1^2 | n} f_1 n^{-1/2} \tilde{e}_{n/f_1^2}$. Then it is easy to see $\check{e}$ coincides with the part of (e) in [4] such that $s \neq 0$ with the notation s in [4]. Define $\check{h}$ in the same way, then $\check{h}$ equals (h) in [4]. Let $\tilde{p}_n^{(2)}$ (respectively $\tilde{p}_n^{(1)}$) be the first (respectively second) term in $\tilde{p}$ of Proposition 1. Put $\check{p}_i = \sum_{f_1^2 | n} f_1 n^{-1/2} \tilde{p}_{n/f_1^2}^{(i)}$, $(i = 1, 2)$, then $\check{p}_2$ equals the part of (e) in [4] such that $s = 0$ with the notation s in [4] and $\check{p}_1$ equals (p) in [4]. Therefore, we have verified (1).

Next we assume $M = p$ or p^2. Then, to verify (1) in this case, it suffices to note the following obvious

LEMMA 2. *Let* $p \neq 2$, $f' | s$, $f''^2 | (s^2 - 4n)$, $2 | s$, $(f'f'', 2) = 1$ *and* $t = s^2 - 2n$.

(i) *If* $\nu_p = 1$, *then* $\tilde{c}(t, f'f'', p) = 1 + \left\{ \frac{(s^2 - 4n) f''^{-2}}{p} \right\}$.

(ii) *If* $\nu_p = 2$, *then*

$$\tilde{c}(t, f'f'', p) = \begin{cases} c(s, f'', p)\ , & (p \nmid sf'^{-1})\ , \\ p - \left(\dfrac{s^2 - 4n}{p}\right) + c(s, f'', p)\ , & (p^2 \mid sf'^{-1})\ , \\ p + c(s, f'', p)\ , & (p \,\|\, sf'^{-1})\ , \end{cases}$$

where $c(s, f'', p)$ *is same as defined in* [4] *for* $\Phi(X) = X^2 - sX + n$ *and for* $\nu = 2$ *with the notations* Φ, s *and* ν *in* [4].

Thus, we have completed all necessary steps.

Remark 4. The same type of equalities as in (1) hold for an arbitrary cubefree (level) N.

§ 4. Proof of Theorem and Corollary

In this section, we assume $M = 1$ and put $T_\kappa(p^2) = \tilde{\rho}_{\kappa/2}\left(\Gamma'\begin{pmatrix}1 & 0\\ 0 & p^2\end{pmatrix}\Gamma'\right)$ which is same as $T^N_{\kappa,1}(p^2)$ in [2].

First, we prove Lemma 1. In order to make sure, we repeat the explanation for some of the notations. Namely, let

$$\Delta = \{(\sigma, j(\sigma, z))\,;\, \sigma \in \Gamma_0(4)\}\ ,$$

$$\alpha = \begin{pmatrix}1 & 0\\ 0 & 4\end{pmatrix}, \qquad \xi = (\alpha, 4^{1/4})\ ,$$

$$\beta = \begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix}, \qquad \eta = (\beta, 4^{-1/4})\ ,$$

$$\omega = \begin{pmatrix}0 & -1\\ 4 & 0\end{pmatrix}, \qquad \tau = (\omega, 4^{1/4}(-iz)^{1/2})\,;$$

then $T_\kappa(4) = [\Delta\xi\Delta]_\kappa$, $T_\kappa(4)^* = [\Delta\eta\Delta]_\kappa$ and $W_4 = 4^{-\kappa/4+1}[\Delta\tau\Delta]_\kappa$. Now, it is easy to see that

$$(\Delta\xi\Delta)\cdot(\Delta\eta\Delta) = 4\Delta\left(\begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, 1\right) + \Delta\left(\begin{pmatrix}4 & 1\\ 0 & 4\end{pmatrix}, 1\right)\Delta + \Delta\left(\begin{pmatrix}4 & -1\\ 0 & 4\end{pmatrix}, 1\right)\Delta + \Xi$$

with $\Xi = \Delta\left(\begin{pmatrix}4 & 2\\ 0 & 4\end{pmatrix}, 1\right)\Delta$ and that $[\Xi]_\kappa = 0$. Accordingly, $\sigma_1\left(\begin{pmatrix}4 & \pm 1\\ 0 & 4\end{pmatrix}, 1\right)\tau\sigma_2 = \left(\begin{pmatrix}4 & 0\\ 0 & 16\end{pmatrix}, 4^{1/4}(\mp i)^{1/2}\right)$ with $\sigma_1 = \left(\begin{pmatrix}1 & 0\\ \mp 4 & 1\end{pmatrix}, (\mp 4z + 1)^{1/2}\right) \in \Delta$ and with $\sigma_2 = \left(\begin{pmatrix}\pm 1 & 1\\ 0 & \pm 1\end{pmatrix}, 1\right) \in \Delta$. Thus $\left[\Delta\left(\begin{pmatrix}4 & \pm 1\\ 0 & 4\end{pmatrix}, 1\right)\Delta\right]_\kappa W_4 = 4^{-\kappa/4+1}\left[\Delta\left(\begin{pmatrix}4 & \pm 1\\ 0 & 4\end{pmatrix}, 1\right)\tau\Delta\right]_\kappa$

$= 4^{\kappa/4-1}e^{\pm 2\pi i\kappa/8}T_\kappa(4)$. Therefore, $\left[\Delta\left(\begin{pmatrix}4 & 1\\ 0 & 4\end{pmatrix}, 1\right)\Delta\right]_\kappa + \left[\Delta\left(\begin{pmatrix}4 & -1\\ 0 & 4\end{pmatrix}, 1\right)\Delta\right]_\kappa =$ $2^{(\kappa-1)/2-1}\varepsilon T_\kappa(4)W_4$. Hence, Lemma 1 is proved.

Now, it follows from Lemma 1 that the eigen value of the self-adjoint operator $T_\kappa(4)W_4$ is either $\varepsilon 2^{(\kappa-1)/2}$ or $-\varepsilon 2^{(\kappa-1)/2-1}$ and therefore that the eigen value of $T_\kappa(4)T_\kappa(4)^*$ is either $2^{\kappa-1}$ or $2^{(\kappa-1)-2}$. By the way, as stated in §1, we can easily show that the identities (1) holds for R in case $M = 1$, i.e.,

$$\text{(1')} \qquad \mathrm{tr}\, \tilde{\rho}_{\kappa/2}(\xi) = \mathrm{tr}\, \rho_{\kappa-1}(\xi) \qquad \text{for } \xi \in R\,,$$

but we will skip its proof.

By these remarks, we are ready to prove Theorem and Corollary. We recall that both representations $\tilde{\rho}_{\kappa/2}$ and $\rho_{\kappa-1}$ are not necessarily semi-simple even in the case $M = 1$ which we treat in this section. Of course their restrictions to R' are semi-simple. Choose the basis $\{g_i\}$ of $S_{\kappa-1}(SL(2, \boldsymbol{Z}))$ which are common eigen functions of all the Hecke operators for $SL(2, \boldsymbol{Z})$. Let $T_{\kappa-1}(p)g_i = \omega_p^{(i)} g_i$ for $p \neq 2$; then, it follows from (1) that

$$S_{\kappa/2}(\Gamma) = \Big(\bigoplus_i \tilde{O}_i\Big) \oplus \Big(\bigoplus_j \tilde{N}_j\Big)\,,$$

where $\dim \tilde{N}_j = 1$, $\dim \tilde{O}_i = 2$ and $T_\kappa(p^2)f = \omega_p^{(i)} f$ for any $p\ (\neq 2)$ and for any $f \in \tilde{O}_i$. Moreover, it follows from (1′) that

$$T_\kappa(4)|_{\tilde{O}_i} \sim \begin{pmatrix} a & b\\ 0 & d\end{pmatrix} \cdots \text{(i)}\,,$$

$$T_{\kappa-1}(2)|_{O_i} \sim \begin{pmatrix} a & b'\\ 0 & d\end{pmatrix} \cdots \text{(ii)}\,,$$

where $O_i = \boldsymbol{C}g_i(z) + \boldsymbol{C}g_i(2z)$. Now, we have nothing to say for O_i in which $T_{\kappa-1}(2)$ is diagonalizable. So, we assume $a = d$. Then, $a = d = \pm 2^{(\kappa-1)/2-1/2}$ and $b' \neq 0$ in (ii). If further $b = 0$, we would have

$$T_\kappa(4)T_\kappa(4)^*|_{\tilde{O}_i} \sim \begin{pmatrix} 2^{(\kappa-1)-1} & 0\\ 0 & 2^{(\kappa-1)-1}\end{pmatrix}$$

which contradicts Lemma 1. Therefore, $b \neq 0$ if $a = d$, which proves Theorem. Next, in order to prove Corollary, let $f_\iota(z) = \sum_{n=1}^{\infty} a_\iota(n)e(nz) \in \tilde{O}_i$, $\iota = 1, 2$,

$$T_\varepsilon(4)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \omega_2 & 1 \\ 0 & \omega_2 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

and let $F_{\iota,t}(z) = F_{f_\iota,t}(z) = \sum_{n=1}^{\infty} A_{\iota,t}(n)e(nz)$. Then $a_1(4tn^2) = \omega_2 a_1(tn^2) + a_2(tn^2)$ and $a_2(4tn^2) = \omega_2 a_2(tn^2)$. Therefore, $A_{1,t}(2n) = \omega_2 A_{1,t}(n) + A_{2,t}(n)$ and $A_{2,t}(2n) = \omega_2 A_{2,t}(n)$. It follows from this that $F_{\iota,t} \in O_i$. Moreover, we see that $F_{1,t}$ and $F_{2,t}$ are linearly independent for t such that $a_2(t) \neq 0$. Thus, we have proved Corollary.

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