GREEN'S FORMS AND MEROMORPHIC FUNCTIONS ON COMPACT ANALYTIC VARIETIES

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1. Introduction. Let $\mathfrak{M}$ be a compact complex analytic variety of the complex dimension n with a positive definite Kählerian metric [4] ; the local analytic coordinates on $\mathfrak{M}$ will be denoted by $z = (z^1, z^2, \ldots, z^n)$. Now, suppose a meromorphic function $f(z)$ defined on $\mathfrak{M}$ as given. Then the poles and zero-points of $f(z)$ constitute an analytic surface[1] in $\mathfrak{M}$ consisting of a finite number of irreducible closed analytic surfaces $\Gamma_1, \Gamma_2, \ldots, \Gamma_\kappa$, each of which is a polar or a zero-point variety of $f(z)$. The formal sum $D = \Sigma m_k \Gamma_k$ of these varieties multiplied respectively by the multiplicity m_k of Γ_k is called the *divisor* of $f(z)$, where the multiplicities of the polar varieties are to be associated with the negative sign. The divisor D of $f(z)$ can be also defined in case $f(z)$ is a many-valued meromorphic function, if the absolute value $|f(z)|$ is one-valued. Such a function $f(z)$ will be called *multiplicative*, since, if one prolongs $f(z)$ analytically along a closed continuous curve ζ, then $f(z)$ is multiplied by a constant factor $\chi(\zeta)$ of modulus 1 depending only on the homology class of ζ on $\mathfrak{M}$. From the topological viewpoint, the divisor D is a $(2n - 2)$-cycle on $\mathfrak{M}$. It can be readily verified that the divisor D of an arbitrary multiplicative meromorphic function must be a bounding cycle.[2] Obviously a multiplicative meromorphic function is determined by its divisor uniquely up to a non-vanishing multiplicative constant. Now, assume conversely that a bounding $(2n - 2)$-cycle $D = \Sigma m_k \Gamma_k$ consisting of a finite number of irreducible closed analytic surfaces $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ is given. Then, *does a multiplicative meromorphic function having D as its divisor exist?* This question of fundamental importance was solved affirmatively by A. Weil in a more general form.[3] His skilful method of proof of the existence is based on the theory of harmonic integrals but entirely differs from the classical potential theoretical treatment of the problem in the case of Riemann surfaces.

In the present paper we shall prove the existence of the multiplicative meromorphic function with the given divisor by a *potential-theoretical method* and give simultaneously *an explicit expression of the multiplicative meromorphic function in terms of the integral of the Green's form extending over the given divisor* [10, pp. 140-143]. In order to explain our main idea more explicitly, let us first

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[1]By an analytic surface we shall mean a $(n - 1)$-dimensional analytic subvariety of $\mathfrak{M}$.

[2]By a bounding cycle we shall mean a cycle which is the boundary of a chain with real coefficients.

[3]Weil [12]. As to the special case $n = 2$, this question was solved also by the author in Japan independently of the results of A. Weil; see Kodaira [6].

consider a compact Riemann surface $\mathfrak{R}$ with the metric $ds^2 = g^{\frac{1}{2}} |dz|^2$, where $z = x + iy$ denote local uniformization variables on $\mathfrak{R}$. Choose two different points $\mathfrak{p}$, $\mathfrak{q}$ on $\mathfrak{R}$ arbitrarily and suppose that $\mathfrak{p}$ or $\mathfrak{q}$ carries respectively unit "charge" of positive or negative sign. The "electric potential" $\gamma(z, \mathfrak{p}, \mathfrak{q})$ produced by these two charges is a harmonic function of z having logarithmic singularities at $\mathfrak{p}$ and $\mathfrak{q}$, whose existence is proved by the Dirichlet principle [13]. Now, let a divisor $\mathfrak{d} = \Sigma m_k \mathfrak{p}_k$ of total order 0 be given, and consider the sum

$$\gamma[\mathfrak{d}](z) = \Sigma m_k \gamma(z, \mathfrak{p}_k, \mathfrak{q})$$

of the potential $\gamma(z, \mathfrak{p}, \mathfrak{q})$ extending over the divisor $\mathfrak{d}$. Then, denoting by $*d\gamma$ the dual form $\gamma_y dx - \gamma_x dy$ of $d\gamma = \gamma_x dx + \gamma_y dy$, the multiplicative meromorphic function with the divisor $\mathfrak{d}$ is given by the formula

$$f(z) = c. \exp\left\{ -\gamma[\mathfrak{d}](z) + i \int^z *d\gamma[\mathfrak{d}](z) \right\}, \tag{1.1}$$

where c means a non-vanishing constant [13, §17]. In the above construction of $f(z)$, the negative charge at $\mathfrak{q}$ plays merely the role of *compensating term* in order to make the total sum of the charges over $\mathfrak{R}$ equal to zero and is cancelled in making the sum $\gamma[\mathfrak{d}](z)$. Hence, replacing it by the charge of the total magnitude -1 distributed uniformly over the whole space $\mathfrak{R}$, we can eliminate the auxiliary point $\mathfrak{q}$ from our whole construction. This leads to replacing $\gamma(z, \mathfrak{p}, \mathfrak{q})$ by its *mean*

$$\gamma(z, \mathfrak{p}) = \sigma^{-1} \iint \gamma(z, \mathfrak{p}, \mathfrak{q}) d\sigma_{\mathfrak{q}},$$

where $d\sigma_{\mathfrak{q}}$ denotes the surface element $g^{\frac{1}{2}} dxdy$ and σ is the total area of $\mathfrak{R}$. $\gamma(z, \mathfrak{p})$ thus obtained will be called *the Green's function for the compact Riemann surface $\mathfrak{R}$. The Green's function is not harmonic but satisfies the inhomogeneous Laplace equation*

$$\Delta\gamma(z, \mathfrak{p}) = 2\pi g^{\frac{1}{2}} \sigma^{-1}$$

and has the typical singularity $-\log|z - z_{\mathfrak{p}}|$ *at* $\mathfrak{p}$. *By using the Green's function, the sum* $\gamma[\mathfrak{d}](z)$ *is represented as*

$$\gamma[\mathfrak{d}](z) = \Sigma m_k \gamma(z, \mathfrak{p}_k), \tag{1.2}$$

and thus the auxiliary point $\mathfrak{q}$ has been eliminated.

Now we turn to the compact analytic variety $\mathfrak{M}$ of an arbitrary dimension n with a positive definite Kählerian metric. Considering $\mathfrak{M}$ as a Riemannian variety, we have, in $\mathfrak{M}$, the *Green's form* $\gamma^\rho(z, \mathfrak{p})$ of any rank $\rho, 0 \leqslant \rho \leqslant 2n$, introduced by G. de Rahm [10, pp. 140-143], which can be considered as a generalization of the Green's function mentioned above. Indeed, the Green's form $\gamma^\rho(z, \mathfrak{p})$ is defined as a solution of the inhomogeneous Laplace equation having the typical singularity at $\mathfrak{p}$, and, in the simplest case of compact Riemann surfaces, the Green's form $\gamma^0(z, \mathfrak{p})$ of the rank 0 coincides with the Green's function $\gamma(z, \mathfrak{p})$ defined above. We may expect therefore that the Green's form $\gamma^{2n-2}(z, \mathfrak{p})$ of the rank $2n - 2$ acts in $\mathfrak{M}$ just as the Green's function in the case of Riemann surfaces so that *the multiplicative meromorphic function with the divisor D is given by such a formula as* (1.1) *in terms of the integral*

$$\gamma[D](z) = \int_D \gamma^{2n-2}(z, \quad)$$

of $\gamma^{2n-2}(z, \mathfrak{p})$ extending over the divisor D, which corresponds to the sum $\gamma[\mathfrak{d}](z)$ defined in (1.2). According to this idea, we shall first examine, in §**2**, the properties of the integral $\gamma[Z](z)$ of the Green's form extending over an arbitrary cycle Z, and then, in §**3**, we shall deduce a formula representing the Picard integral of the third kind with the logarithmic polar cycle D in terms of the integral $\gamma[D](z)$. The formula representing the multiplicative meromorphic function with the given divisor D in terms of $\gamma[D](z)$ will be obtained in the last §**4**. Finally we shall prove a theorem concerning the necessary and sufficient condition for D in order that D is the divisor of a one-valued meromorphic function, which can be considered as a generalization of Abel's theorem in the classical theory of algebraic functions [13, pp. 126-127].

2. Green's forms. Let $\mathfrak{M}$ be a n-dimensional (topologically $2n$-dimensional) compact analytic variety with a positive definite Kählerian metric

$$ds^2 = 2g_{\alpha\dot{\beta}}dz^\alpha d\bar{z}^\beta$$

where $z^1, z^2, \ldots, z^n$ denote local analytic coordinates on $\mathfrak{M}$. Putting

$$z^\alpha = x^\alpha + ix^{n+\alpha} \qquad (\alpha = 1, 2, \ldots, n);$$

we introduce the real coordinates $x^1, x^2, \ldots, x^{2n}$ on $\mathfrak{M}$; then $\mathfrak{M}$ becomes a $2n$-dimensional compact orientable Riemannian variety with the positive definite metric

$$ds^2 = 2g_{\alpha\dot{\beta}}dz^\alpha d\bar{z}^\beta = g_{jk}dx^j dx^k$$

(in what follows Latin subscripts j, k, etc., take values ranging from 1 to $2n$ and Greek subscripts α, β denote $1, 2, \ldots, n - 1$ or n). Now we shall consider differential forms

$$\psi = \psi^\rho = \left(\frac{1}{\rho!}\right) \psi_{jk\ldots l}[dx^j dx^k \ldots dx^l]$$

defined on $\mathfrak{M}$, where ρ denotes the rank of ψ. A differential form of rank ρ will be called simply a *ρ-form*. A ρ-form ψ is said to be *measurable*, to *have continuous derivatives* or to be *regular*, if the coefficients $\psi_{jk\ldots l}$ are measurable, have continuous derivatives or are regular analytic functions of the real local coordinates $x^1, x^2, \ldots, x^{2n}$. We denote the derived form of ψ by $d\psi$ and the dual form of ψ by $*\psi$ or ψ^*; as is well known, they are defined respectively by

$$d\psi^\rho = \left(\frac{1}{\rho!}\right)[d\psi_{jk\ldots l}dx^j dx^k \ldots dx^l],$$

$$*\psi^\rho = \frac{g_{ip}g_{jq}\ldots g_{kr}}{g^{\frac{1}{2}}\rho!(2n-\rho)!}\operatorname{sgn}\begin{pmatrix} i\,j \ldots k\,l \ldots m \\ 1\,2 \ldots\ldots\ldots 2n\end{pmatrix}\psi^{\rho}{}_{l\ldots m}[dx^p dx^q \ldots dx^r],$$

where $g = |g_{jk}|$. The dual derivation δ and the Laplacian Δ are defined as[4]

$$\delta = {*d*},$$

$$\Delta = -d\delta - \delta d.$$

Again, we introduce the "contraction product"

$$\psi^\rho \cdot \phi^\sigma = \phi^\sigma \cdot \psi^\rho = \left(\frac{1}{(\rho-\sigma)!\,\sigma!}\right)\psi_{ij\ldots kl\ldots m}\phi^{l\ldots m}[dx^i dx^j \ldots dx^k] \quad (\sigma \leqslant \rho),$$

and the "inner product"

$$(\psi^\rho, \phi^\rho)_G = \int_G \psi^\rho \cdot \phi^\rho g^{\frac{1}{2}} dG \qquad (dG = dx^1 dx^2 \ldots dx^{2n}),$$

where G means an arbitrary subdomain of $\mathfrak{M}$; especially in case $G = \mathfrak{M}$, we write[5] (ψ, ϕ) for $(\psi, \phi)_{\mathfrak{M}}$. Then we have the Green's formula

$$(d\phi^\rho, \psi^{\rho+1})_G - (\phi^\rho, \delta\psi^{\rho+1})_G = \int_B (\phi\cdot\psi)^j g^{\frac{1}{2}} do_j, \tag{2.1}$$

where B is the boundary of the domain G and do_j denotes the surface element $(-1)^{j-1}[dx^1 \ldots dx^{j-1}dx^{j+1}\ldots dx^{2n}]$. We introduce furthermore the "absolute value"

$$|\psi(\mathfrak{p})| = |\psi(\mathfrak{p}).\overline{\psi(\mathfrak{p})}|^{\frac{1}{2}} = \left|\left(\frac{1}{\rho!}\right)\psi_{jk\ldots l}(\mathfrak{p})\bar{\psi}^{jk\ldots l}(\mathfrak{p})\right|^{\frac{1}{2}}$$

of ψ at a point $\mathfrak{p}$ in $\mathfrak{M}$ and, by its means, define the norm $||\psi||_G$ as

$$||\psi||_G = \int_G |\psi(\mathfrak{p})|\, g^{\frac{1}{2}} dG_{\mathfrak{p}}.$$

In case $G = \mathfrak{M}$, we write $||\psi||$ for $||\psi||_{\mathfrak{M}}$. Incidentally, by a chain or a cycle will be meant a chain or a cycle with real coefficients; the boundary of a chain C will be denoted by ∂C. Again, if Z is a bounding cycle in $\mathfrak{M}$, we write $Z \approx 0$.

A form ψ is said to be *regular harmonic* in a subdomain G of $\mathfrak{M}$, if ψ is regular and satisfies the differential equations $d\psi = 0$, $\delta\psi = 0$ everywhere in G. By a *harmonic form* in $\mathfrak{M}$ we shall mean a form ψ which is regular harmonic in $\mathfrak{M}$ except for a nowhere dense compact subset S of $\mathfrak{M}$; then ψ is said to be *regular*

[4]Cf. de Rahm [10], Kodaira [5]. Our use of the notations d, δ coincides with that of de Rham [10], while our $*\psi^\rho$ $\Delta\psi^\rho$ correspond to $(-1)^\rho *\psi^\rho$, $-\Delta\psi^\rho$ of de Rham [10]. In Kodaira [5], we write $\mathfrak{r}^*$, $\mathfrak{r}$, $\mathfrak{d}$ for d, δ, $*$, respectively.

[5]In the present paper we do not use the well known "outer product".

in $\mathfrak{M} - S$ and to be *singular* on S. In case ψ is regular harmonic everywhere in $\mathfrak{M}$, ψ is called a *harmonic form of the first kind.* The linear space consisting of all real harmonic ρ-forms of the first kind will be denoted by $\mathfrak{E}^\rho$. Then $\mathfrak{E}^\rho$ constitutes a finite dimensional Euclidean vector space with respect to the inner product introduced above.[6] Choose a normalized orthogonal base $\{e_1{}^\rho, e_2{}^\rho, \ldots, e_b{}^\rho\}$ of the space $\mathfrak{E}^n$ arbitrarily and put

$$w^\rho(x, \xi) = \Sigma_\nu e_\nu{}^\rho(x) e_\nu{}^\rho(\xi).$$

The double harmonic ρ-form $w^\rho(x, \xi)$ thus defined is obviously independent of the choice of the base $\{e_\nu{}^\rho\}$ and therefore determined uniquely by $\mathfrak{M}$. Now we associate with every differentiable ρ-cycle Z on $\mathfrak{M}$ the harmonic ρ-form.

$$w[Z](x) = \int_Z w^\rho(x, \quad).$$

Then a famous theorem of W. V. D. Hodge can be stated as follows:

THEOREM 1 (HODGE).[7] *The mapping $Z \rightarrow w[Z]$ gives an isomorphism between the ρ-Betti group (over the real field) of $\mathfrak{M}$ and the space $\mathfrak{E}^\rho$ of all real harmonic ρ-forms of the first kind attached to $\mathfrak{M}$.*

In what follows this theorem will be cited as *Hodge's Theorem.* The character of the isomorphism $Z \rightarrow w[Z]$ will become more clear if we notice that the relation[8]

$$\int_\zeta w^*[Z] = I(\zeta, Z) \tag{2.2}$$

holds for an arbitrary $(2n - \rho)$-cycle ζ, where $I(\zeta, Z)$ means the intersection number of ζ and Z.

Now we put

$$w[\phi^\rho](\xi) = (w^\rho(\quad , \xi), \phi^\rho) = \Sigma_\nu(\phi^\rho, e_\nu{}^\rho) \cdot e_\nu{}^\rho(\xi)$$

for an arbitrary ρ-form ϕ^ρ with $(\phi^\rho, \bar{\phi}^\rho) < + \infty$. Then we have

THEOREM 2 (DE RHAM [10 pp. 140-143]). *For each ρ, $0 \leqslant \rho \leqslant 2n$, there exists on $\mathfrak{M}$ one and only one real double ρ-form*

$$\gamma^\rho(x, \xi) = \left(\frac{1}{\rho!}\right)^2 \gamma_{jk \ldots l\lambda\mu \ldots \nu}(x, \xi)[dx^j dx^k \ldots dx^l][d\xi^\lambda d\xi^\mu \ldots d\xi^\nu]$$

satisfying the following three conditions:

(i) *for every fixed ξ on $\mathfrak{M}$, $\gamma^\rho(x, \xi)$ is regular with respect to $x^1, x^2, \ldots, x^{2n}$ except for $x = \xi$ and satisfies*

$$\Delta_x \gamma^\rho(x, \xi) = w^\rho(x \;\; \xi); \tag{2.3}$$

[6]The fact that the space $\mathfrak{E}^\rho$ has a finite dimension can be proved independently of a famous theorem of Hodge. See de Rham [10, p. 138].

[7]Hodge [3, chap. III], Weyl [15]; see also Kodaira [5, §5].

[8]de Rham [10, p. 147]; see also Kodaira [5, p. 640, Theorem 17].

(ii) *in some neighbourhood* $N(\xi)$ *of* ξ, $\gamma^\rho(x, \xi)$ *is represented as*

$$\gamma^\rho(x, \xi) = \mathrm{E}^\rho(x, \xi) + \mu_\xi(x), \tag{2.4}$$

where μ_ξ *is regular in* $N(\xi)$, *and* $\mathrm{E}^\rho(x, \xi)$ *means an elementary solution* [5, §7] *of Laplace's equation* $\Delta \mathrm{E} = 0$ *having typical singularities at* ξ (*therefore we have* $||\gamma^\rho(\ , \xi)|| < +\infty$);

(iii) $\gamma^\rho(\ , \xi)$ *is orthogonal to all harmonic* ρ*-forms of the first kind, i.e. we have*

$$(\gamma^\rho(\ , \xi), e) = 0, \qquad \textit{for all } e \in \mathfrak{E}^\rho. \tag{2.5}$$

The double ρ*-form* $\gamma^\rho(x, \xi)$ *is symmetric*:

$$\gamma^\rho(x, \xi) = \gamma^\rho(\xi, x); \tag{2.6}$$

thus, for arbitrary fixed x, $\gamma^\rho(x, \xi)$ *is regular with respect to* $\xi^1, \ldots, \xi^{2n}$ *except for* $\xi = x$. *We have the identities*

$$\delta_x \gamma^\rho(x, \xi) = d_\xi \gamma^{\rho-1}(x, \xi), \tag{2.7}$$

$$*_x *_\xi \gamma^\rho(x, \xi) = \gamma^{2n-\rho}(x, \xi). \tag{2.8}$$

As a function of $4n$ *variables* $x^1\ x^2, \ldots, x^{2n}, \xi^1, \ldots, \xi^{2n}, \gamma^\rho(x, \xi)$ *admits continuous derivatives of arbitrary orders except for* $x = \xi$. *Furthermore the norms* $||d\gamma^\rho(\ , \xi)||$, $||\delta\gamma^\rho(\ , \xi)||$ *of the derived forms* $d_x\gamma^\rho(x, \xi)$, $\delta_x\gamma^\rho(x, \xi)$ *are uniformly bounded with respect to* ξ *and, for an arbitrary form* ψ^ρ *with continuous first derivatives, the identity*[9]

$$\psi^\rho(\xi) - w[\psi^\rho](\xi) = (d\gamma^\rho(\ , \xi), d\psi^\rho) + (\delta\gamma^\rho(\ , \xi), \delta\psi^\rho) \tag{2.9}$$

holds.

The double ρ-form $\gamma^\rho(x, \xi)$ is called the *Green's form* of rank ρ.

Incidentally, we shall mean by $\psi \subset G$ that the closure of the subset $\{\mathfrak{p};\ |\psi(\mathfrak{p})| \neq 0\}$ of $\mathfrak{M}$ is contained in G, G being an arbitrary subdomain of $\mathfrak{M}$. Then the principle of the method of orthogonal projections can be stated as follows:

THEOREM 3 (PRINCIPLE OF ORTHOGONAL PROJECTIONS).[10] *Let* G *be an open subset of* $\mathfrak{M}$ *and* ψ *be a measurable* ρ*-form defined in* G *with* $||\psi||_G < +\infty$. *Then, if* ψ *satisfies the integral equation*

$$(\psi, \Delta\eta)_G = 0$$

for all ρ*-forms* $\eta \subset G$ *having continuous third derivatives,* ψ *is regular in* G *and satisfies* $\Delta\psi = 0$. *Again, if* ψ *satisfies the integral equations*

$$(\psi, d\lambda)_G = 0, \qquad (\psi, \delta\eta)_G = 0$$

[9]This formula is an immediate consequence of the formula (4.5) in de Rham [10].

[10]Kodaira [5, pp. 608-609]. The method of orthogonal projections was first introduced by H. Weyl [14].

for arbitrary $(\rho - 1)$- *and* $(\rho + 1)$-*forms* $\lambda, \eta \subset G$ *with continuous second derivatives, then* ψ *is regular harmonic in* G.

Now we introduce for an arbitrary differentiable ρ-cycle Z, $0 \leqslant \rho \leqslant 2n - 1$, the ρ-form

$$\gamma[Z](x) = \int_Z \gamma^\rho(x, \ ), \tag{2.10}$$

which will play a fundamental role in our theory. Then, denoting the support[11] of Z by $|Z|$, we have

THEOREM 4. *$\gamma[Z]$ is regular in $\mathfrak{M} - |Z|$ and satisfies the differential equations*

$$\delta\gamma[Z](x) = 0, \tag{2.11}$$

$$\delta d\gamma[Z](x) = - w[Z](x). \tag{2.12}$$

The derived form $d\gamma[Z]$ of $\gamma[Z]$ is regular harmonic in $\mathfrak{M} - |Z|$ if and only if Z is a bounding cycle on $\mathfrak{M}$. Furthermore $d\gamma[Z]$ has the finite norm: $||d\gamma[Z]|| < + \infty$ and satisfies the integral equations

$$(d\gamma[Z], d\psi) = \int_Z \{\psi - w[\psi]\}, \tag{2.13}$$

$$(d\gamma[Z], \tau) = 0 \qquad (\delta\tau = 0), \tag{2.14}$$

where ψ means an arbitrary ρ-form having continuous first derivatives and τ an arbitrary $(\rho + 1)$-form with continuous first derivatives satisfying $\delta\tau = 0$.

Proof. It is obvious that $\gamma[Z]$ is regular in $\mathfrak{M} - |Z|$. Now, using (2.7), we get

$$\delta\gamma[Z](x) = \int_Z \delta_x \gamma^\rho(x, \xi) = \int_Z d_\xi \gamma^{\rho-1}(x, \xi) = \int_{\partial Z} \gamma^{\rho-1}(x, \xi) = 0,$$

proving (2.11). Again, we obtain, using (2.3) and (2.11),

$$\begin{aligned} \delta d\gamma[Z](x) &= \int_Z \delta_x d_x \gamma^\rho(x, \xi) = - \int_Z \{d_x \delta_x \gamma^\rho(x, \xi) + w^\rho(x, \xi)\} \\ &= - d\delta\gamma[Z](x) - w[Z](x) = - w[Z](x), \end{aligned}$$

proving (2.12). Combined with the trivial relation $dd\gamma[Z](x) = 0$, (2.12) shows that $d\gamma[Z]$ is regular harmonic in $\mathfrak{M} - |Z|$ if and only if $Z \approx 0$ since, by virtue of Hodge's Theorem, $w[Z](x)$ vanishes identically if and only if $Z \approx 0$. The inequality $||d\gamma[Z]|| < + \infty$ follows immediately from the fact that the norm $||d\gamma^\rho(\ , \xi)||$ is uniformly bounded with respect to ξ. Now, integrating the identity (2.9) over the cycle Z and using (2.11), we get immediately (2.13), while we have

$$(d\gamma[Z], \tau) = (\gamma[Z], \delta\tau) = 0,$$

proving (2.14).

[11] Z is a formal sum $\Sigma m_k T_k$ of a finite number of differentiable simplexes T_k lying in $\mathfrak{M}$ associated with real coefficients $m_k \neq 0$; then the support $|Z|$ is, by definition, the set theoretical sum ΣT_k of these simplexes.

In case Z is a bounding cycle: $Z = \partial C$, C being a differentiable $(\rho + 1)$-chain in $\mathfrak{M}$, we have the formula [10, p. 143]

$$\int_{\zeta} *d\gamma[\partial C] = I(\zeta, C) + (-1)^{\rho}\int_{C} w^*[\zeta], \tag{2.15}$$

where $I(\zeta, C)$ denotes the intersection number of ζ and C.

REMARK. The Green's forms $\gamma^{\rho}(x, \xi)$ are closely related to the double harmonic forms $e(x, \xi)$, $e^{**}(x, \xi)$ introduced in a recent paper of the author [5, §§14, 17]. Indeed, we have the relations

$$e(x, \xi) = - \delta_x d_x \gamma^{\rho}(x, \xi) = - \delta_x \delta_{\xi} \gamma^{\rho+1}(x, \xi),$$

$$e^{**}(x, \xi) = - d_x \delta_x \gamma^{\rho}(x, \xi) = - d_x d_{\xi} \gamma^{\rho-1}(x, \xi).$$

3. Picard integrals of the third kind. Now we introduce $2n$ formally independent variables $z^1, z^2, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n$ instead of $x^1, \ldots, x^n, x^{n+1}, \ldots, x^{2n}$ and rewrite ρ-forms ψ as

$$\psi^{\rho} = \sum_{\sigma+\tau=\rho} \left(\frac{1}{\sigma!\tau!}\right) \psi_{\alpha\beta \ldots \gamma\dot{\delta} \ldots \dot{\epsilon}} [\overbrace{dz^{\alpha}dz^{\beta} \ldots dz^{\gamma}}^{\sigma} \; \overbrace{d\bar{z}^{\delta} \ldots d\bar{z}^{\epsilon}}^{\tau}].$$

Again, we introduce as usual the formal partial differentiation operators

$$\frac{\partial}{\partial z^{\alpha}} = \frac{1}{2}\left(\frac{\partial}{\partial x^{\alpha}} - i\,\frac{\partial}{\partial x^{n+\alpha}}\right),$$

$$\frac{\partial}{\partial \bar{z}^{\alpha}} = \frac{1}{2}\left(\frac{\partial}{\partial x^{\alpha}} + i\,\frac{\partial}{\partial x^{n+\alpha}}\right).$$

As is well known, a function $f(x^1, x^2, \ldots, x^{2n})$ of real variables $x^1, x^2, \ldots, x^{2n}$ with continuous first derivatives is regular analytic with respect to complex variables $z^1, z^2, \ldots, z^n$ if and only if $\partial f/\partial \bar{z}^1, \partial f/\partial \bar{z}^2, \ldots, \partial f/\partial \bar{z}^n$ vanish identically. Incidentally, a ρ-form ψ of the type

$$\psi = \left(\frac{1}{\rho!}\right) \psi_{\alpha\beta \ldots \gamma} [dz^{\alpha}dz^{\alpha} \ldots dz^{\gamma}], \qquad (0 \leqslant \rho \leqslant n)$$

will be called *regular analytic* in a domain G, if the coefficients $\psi_{\alpha\beta \ldots \gamma}$ are regular analytic functions of *complex variables* $z^1, z^2, \ldots, z^n$ (whereas by a *regular* ρ-form we mean a ρ-form with coefficients which are regular analytic functions of *real coordinates* $x^1, x^2, \ldots, x^{2n}$).

Since, by hypothesis, the metric $ds^2 = 2g_{\alpha\dot{\beta}}dz^{\alpha}d\bar{z}^{\beta}$ is Kählerian, the 2-form

$$\omega = g_{\alpha\dot{\beta}}[dz^{\alpha}d\bar{z}^{\beta}]$$

is a harmonic form of the first kind [3, pp. 168-171]. Now we define two linear operators $\mathfrak{C}$, Λ acting on differential forms as follows[12]: $\mathfrak{C}$ is the operator which

[12]These two operators were first introduced by W. V. D. Hodge [3, p. 171]. The simple definitions of $\mathfrak{C}$, Λ employed here are due to Weil [12].

transforms dz^α, $d\bar{z}^\alpha$ into $i\,dz^\alpha$, $-\,i\,d\bar{z}^\alpha$, respectively; Λ is the operator which transforms ψ into $i\omega\cdot\psi$, where "$\cdot$" means the contraction product introduced in §2. In tensor notations, $\mathfrak{C}$, Λ are therefore defined as

$$(\mathfrak{C}\psi)_{\overbrace{\alpha\ldots\beta}^{\sigma}\overbrace{\dot\gamma\ldots\dot\delta}^{\tau}} = i^{\sigma-\tau}\psi_{\alpha\ldots\beta\dot\gamma\ldots\dot\delta},$$

$$(\Lambda\psi)_{\alpha\ldots\beta\dot\gamma\ldots\dot\delta} = i\,g^{\lambda\dot\mu}\psi_{\lambda\dot\mu\alpha\ldots\beta\dot\gamma\ldots\dot\delta}.$$

Then we have

$$\Delta\mathfrak{C} = \mathfrak{C}\Delta, \tag{3.1}$$

$$\Delta\Lambda = \Lambda\Delta, \tag{3.2}$$

$$\delta\Lambda = \Lambda\delta, \tag{3.3}$$

$$\Lambda\mathfrak{C} = \mathfrak{C}\Lambda, \tag{3.4}$$

$$\Lambda d - d\Lambda = \mathfrak{C}^{-1}\delta\mathfrak{C}, \tag{3.5}$$

$$\mathfrak{C}\mathfrak{C}\psi^p = (-\,i)^p\psi^p, \tag{3.6}$$

and

$$\Lambda^{n-1}{}_*\psi^1 = -(-\,1)^{n(n+1)/2}(n-1)!\,\mathfrak{C}\psi^1. \tag{3.7}$$

The formulae (3.1), (3.2), (3.3), (3.6) were proved by W. V. D. Hodge [3, pp. 165-168, 171] by straightforward calculations, while (3.4) can be readily verified. The formula (3.5) is due to A. Weil [12]. The formula (3.7) can be proved also by a mere calculation, so the proof might be omitted. But, because of the importance of that formula, we shall give here a brief sketch of the calculation. By the definition of the operator Λ, we have

$$\begin{aligned}
(\Lambda^{n-1}{}_*\psi^1)^{\lambda} &= (-i)^{n-1}\overbrace{g_{\alpha\dot\kappa}g_{\beta\dot\tau}\cdots g_{\gamma\dot\sigma}}^{n-1}({}_*\psi^1)^{\gamma\dot\sigma\cdots\beta\dot\tau\alpha\dot\kappa\lambda} \\
&= (-\,1)^{(n-1)n/2}i^{n-1}g_{\alpha\dot\kappa}g_{\beta\dot\tau}\cdots g_{\gamma\dot\sigma}({}_*\psi^1)^{\lambda\alpha\beta\cdots\gamma\dot\kappa\dot\tau\cdots\dot\sigma} \\
&= (-\,1)^{(n-1)n/2}i^{n-1}(n-1)!\,\Sigma_\mu(-\,1)^{\mu-1}G_{\lambda\dot\mu}({}_*\psi^1)^{1\cdots n\dot1\cdots\dot\mu-1\,\dot\mu+1\cdots\dot n}
\end{aligned}$$

$$\begin{aligned}
(\Lambda^{n-1}{}_*\psi^1)^{\dot\mu} &= (-i)^{n-1}\overbrace{g_{\alpha\dot\kappa}g_{\beta\dot\tau}\cdots g_{\gamma\dot\sigma}}^{n-1}({}_*\psi^1)^{\gamma\dot\sigma\cdots\beta\dot\tau\alpha\dot\kappa\dot\mu} \\
&= (-\,1)^{n(n-1)/2}(-i)^{n-1}g_{\alpha\dot\kappa}g_{\beta\dot\tau}\cdots g_{\gamma\dot\sigma}({}_*\psi^1)^{\alpha\beta\cdots\gamma\dot\mu\dot\kappa\dot\tau\cdots\dot\sigma} \\
&= (-\,1)^{n(n-1)/2}(-i)^{n-1}(n-1)!\,\Sigma_\lambda(-\,1)^{\lambda-1}G_{\lambda\dot\mu}({}_*\psi^1)^{1\cdots\lambda-1\,\lambda+1\cdots n\dot1\cdots\dot n},
\end{aligned}$$

where these $G_{\lambda\dot\mu}$ mean minor determinants $|g_{\alpha\dot\beta}|$, while ${}_*\psi^1$ is given by

$$\begin{cases}
({}_*\psi^1)^{1\cdots n\dot1\cdots\dot\mu-1\,\dot\mu+1\cdots\dot n} = g^{-\frac12}(-\,1)^{1+\mu}(2/i)^n\psi^1{}_{\dot\mu}, \\
({}_*\psi^1)^{1\cdots\lambda-1\,\lambda+1\cdots n\dot1\cdots\dot n} = g^{-\frac12}(-\,1)^{\lambda+n-1}(2/i)^n\psi^1{}_{\lambda}.
\end{cases}$$

Using the relations

$$\Sigma_\lambda g_{\lambda\dot\mu}G_{\lambda\dot\nu} = \delta_{\mu\nu}|g_{\alpha\dot\beta}|, \quad \Sigma_\mu g_{\lambda\dot\mu}G_{\nu\dot\mu} = \delta_{\lambda\nu}|g_{\alpha\dot\beta}|, \quad g^{\frac12} = 2^n|g_{\alpha\dot\beta}|,$$

we get therefore

$$(\Lambda^{n-1} * \psi^1)_{\dot{\mu}} = (-1)^{n(n+1)/2}(n-1)!\, i\, \psi^1{}_{\dot{\mu}},$$

$$(\Lambda^{n-1} * \psi^1)_\lambda = -(-1)^{n(n+1)/2}(n-1)!\, i\, \psi^1{}_\lambda,$$

proving (3.7). Furthermore we have

$$\overline{\mathfrak{C}\,\psi} = \mathfrak{C}\,\bar{\psi}, \tag{3.8}$$

$$(\mathfrak{C}\,\phi,\ \overline{\mathfrak{C}\,\psi}) = (\phi,\ \bar{\psi}). \tag{3.9}$$

As is well known, from the formula

$$-\ (\Delta\psi,\ \bar{\psi}) = (d\psi,\ \overline{d\psi}) + (\delta\psi,\ \overline{\delta\psi})$$

it follows that *a form ψ with continuous second derivatives is a harmonic form of the first kind if and only if ψ satisfies $\Delta\psi = 0$ everywhere in $\mathfrak{M}$*. Combined with this fact, (3.1) shows that, if e is a harmonic form of the first kind, $\mathfrak{C}e$ is also a harmonic form of the first kind. Whence we conclude, using (3.8) and (3.9), that $\mathfrak{C}e_1{}^p, \mathfrak{C}e_2{}^p, \ldots, \mathfrak{C}e_b{}^p$ *constitute a normalized orthogonal base of the space* $\mathfrak{E}^p$, where $\{e_1{}^p, e_2{}^p, \ldots, e_b{}^p\}$ is a normalized orthogonal base of $\mathfrak{E}^p$ introduced in §2. We get therefore

$$\Sigma_\nu \mathfrak{C}e_\nu{}^p(x)\mathfrak{C}e_\nu{}^p(\xi) = \Sigma_\nu e_\nu{}^p(x)e_\nu{}^p(\xi),$$

or

$$\mathfrak{C}_x\mathfrak{C}_\xi w^p(x,\ \xi) = w^p(x,\ \xi). \tag{3.10}$$

LEMMA. *We have*

$$\mathfrak{C}_x\mathfrak{C}_\xi \gamma^p(x,\ \xi) = \gamma^p(x,\ \xi). \tag{3.11}$$

Proof. Put, for simplicity's sake, $\tilde{\gamma}(x, \xi) = \mathfrak{C}_x\mathfrak{C}_\xi\gamma^p(x, \xi)$, $\tilde{\mathrm{E}}(x, \xi) = \mathfrak{C}_x\mathfrak{C}_\xi \mathrm{E}(x, \xi)$. We fix the point ξ and consider γ^p, $\tilde{\gamma}$, E^p, $\tilde{\mathrm{E}}$ as functions of x. The "highest term" of the elementary solution $\mathrm{E}^p(x, \xi)$ is given by

$$[(2n-2)\ \Omega_{2n}]^{-1} r(x,\ \xi)^{2-2n} \times$$

$$\sum_{\sigma+\tau=p} \left(\frac{1}{\sigma!\tau!}\right) \underbrace{g_{\alpha\dot{\kappa}} \ldots g_{\beta\dot{\lambda}}}_{\sigma}\ \underbrace{g_{\mu\dot{\gamma}} \ldots g_{\nu\dot{\delta}}}_{\tau}\ [dz^\alpha \ldots dz^\beta\, d\bar{z}^\gamma \ldots d\bar{z}^\delta][d\bar{\zeta}^\kappa \ldots d\bar{\zeta}^\lambda d\zeta^\mu \ldots d\zeta^\nu],$$

where $g_{\alpha\dot{\lambda}} = g_{\alpha\dot{\lambda}}(\xi)$, $z^\alpha = x^\alpha + ix^{n+\alpha}$, $\zeta^\lambda = \xi^\lambda + i\xi^{n+\lambda}$, Ω_{2n} is the surface area of a $2n$-dimensional unit sphere, and $r(x, \xi)$ means the geodesic distance from x to ξ [5 §7]. It is obvious that this highest term remains unchanged by simultaneous application of the operators $\mathfrak{C}_x$, $\mathfrak{C}_\xi$. Hence we have

$$\tilde{\mathrm{E}}(x,\ \xi) - \mathrm{E}^p(x,\ \xi) = r^{2-2n}O(r) + \log\,(1/r).\,O(1) \qquad (r = r(x,\ \xi)),$$

where $O(r)$, $O(1)$ mean holomorphic functions of $x^1 - \xi^1, \ldots, x^{2n} - \xi^{2n}$ having the orders $O(r)$, $O(1)$, respectively. On the other hand, it is obvious by (3.1) that $\tilde{\mathrm{E}}(x, \xi)$ satisfies also $\Delta\tilde{\mathrm{E}}(x, \xi) = 0$. Let $N(\xi)$ be a sufficiently small neighbourhood of ξ. Then we have therefore

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$$(\tilde{\mathrm{E}}(\ ,\xi) - \mathrm{E}^{\rho}(\ ,\xi),\ \Delta\eta)_{N(\xi)} = (\Delta\tilde{\mathrm{E}} - \Delta\mathrm{E},\ \eta)_{N(\xi)} = 0$$

for an arbitrary ρ-form $\eta \subset N(\xi)$ with continuous second derivatives, and

$$\|\tilde{\mathrm{E}}(\ ,\xi) - \mathrm{E}^{\rho}(\ ,\xi)\|_{N(\xi)} < +\infty.$$

Hence, by virtue of the principle of orthogonal projections, $\tilde{\mathrm{E}}(\ ,\xi) - \mathrm{E}^{\rho}(\ ,\xi)$ is regular everywhere in $\mathrm{N}(\xi)$. Now, using (3.1) and (3.10) we get from (2.3)

$$\Delta_x\tilde{\gamma}(x,\xi) = w^{\rho}(x,\xi), \qquad (x \neq \xi) \tag{3.12}$$

while from (2.4) follows

$$\tilde{\gamma}(x,\xi) = \tilde{\mathrm{E}}(x,\xi) + \tilde{\mu}(x), \qquad \tilde{\mu} \text{ is regular in } \mathrm{N}(\xi). \tag{3.13}$$

Comparing (3.12), (3.13) with (2.3), (2.4), we infer from the above result that $\tilde{\gamma}(\ ,\xi) - \gamma^{\rho}(\ ,\xi)$ is regular everywhere in $\mathfrak{M}$ and satisfies

$$\Delta\{\tilde{\gamma}(\ ,\xi) - \gamma^{\rho}(\ ,\xi)\} = 0;$$

hence $\tilde{\gamma}(\ ,\xi) - \gamma^{\rho}(\ ,\xi)$ is a harmonic form of the first kind. On the other hand, from (2.5) and (3.9) follows

$$(\tilde{\gamma}(\ ,\xi),\ \mathfrak{C}e) = 0, \qquad \text{for all } e \in \mathfrak{E}^{\rho},$$

while the mapping $e \to \mathfrak{C}e$ maps $\mathfrak{E}^{\rho}$ isometrically *on* itself. Hence $\tilde{\gamma}(\ ,\xi) - \gamma^{\rho}(\ ,\xi)$ is orthogonal to all harmonic ρ-forms of the first kind and therefore $\tilde{\gamma}(\ ,\xi) - \gamma^{\rho}(\ ,\xi) = 0$, q.e.d.

A compact subset Γ of $\mathfrak{M}$ will be called a *closed analytic surface*, if, for every point $\mathfrak{p} \in \mathfrak{M}$, there exists a regular analytic function $f_{\mathfrak{p}}(z)$ of complex coordinates $z^1, z^2, \ldots, z^n$ defined in a neighbourhood $N(\mathfrak{p})$ of $\mathfrak{p}$ such that Γ coincides in $N(\mathfrak{p})$ with the zero-point variety of $f_{\mathfrak{p}}(z)$; then

$$f_{\mathfrak{p}}(z) = 0$$

will be called a local equation of Γ at $\mathfrak{p}$. Choose the local coordinates $z^1, z^2, \ldots, z^n$ so that $\mathfrak{p}$ coincides with the origin $(0, 0, \ldots. 0)$. Then the set of all regular analytic functions

$$h(z) = c_0 + \Sigma c_\alpha z^\alpha + \frac{1}{2!}\Sigma c_{\alpha\beta}z^\alpha z^\beta + \frac{1}{3!}\Sigma c_{\alpha\beta\gamma}z^\alpha z^\beta z^\gamma + \ldots$$

defined in some (not fixed) neighbourhoods of $\mathfrak{p} = (0, 0, \ldots, 0)$ constitutes a ring $\mathfrak{o}_{\mathfrak{p}}$ without null divisor, in which every $h(z)$ with $h(0) \neq 0$ is considered as a unit. As an element of $\mathfrak{o}_{\mathfrak{p}}$, $f_{\mathfrak{p}}(z)$ can be decomposed into the power product

$$f_{\mathfrak{p}}(z) = U(z).\ \Pi_j\{f_{\mathfrak{p}}^{(j)}(z)\}^{m_j} \qquad (U(0) \neq 0)$$

of irreducible factors $f_{\mathfrak{p}}^{(j)}(z)$, where the decomposition is unique up to the unit[13] $U(z)$. In accordance with this Γ is decomposed in a neighbourhood of $\mathfrak{p}$ into the sum

[13]Cf. Bochner and Martin [1, chap. IX].

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$$\Gamma'_{\mathfrak{p}} \cup \Gamma''_{\mathfrak{p}} \cup \ldots \cup \Gamma_{\mathfrak{p}}^{(j)} \cup \ldots$$

of the *branches* $\Gamma_{\mathfrak{p}}^{(j)}$, each of which is the zero-point variety of the corresponding factor $f_{\mathfrak{p}}^{(j)}(z)$. Obviously the local equation $f_{\mathfrak{p}}(z) = 0$ of Γ at $\mathfrak{p}$ is equivalent to

$$f_{\mathfrak{p}}^*(z) = \Pi_j f_{\mathfrak{p}}^{(j)}(z) = 0,$$

which will be called a *minimal local equation* of Γ at $\mathfrak{p}$. The minimal local equation is characterized by the following property: *If $f_{\mathfrak{p}}^*(z)$ is represented as the product*

$$f_{\mathfrak{p}}^*(z) = q(z)h(z)$$

of two functions in $\mathfrak{o}_{\mathfrak{p}}$ and $q(z)$ vanishes on Γ in some neighbourhood $N(\mathfrak{p})$ of $\mathfrak{p}$, then $h(z)$ is a unit in $\mathfrak{o}_{\mathfrak{p}}$, i.e., $h(0) \neq 0$. In case, in a sufficiently small neighbourhood $N(\mathfrak{p})$ of $\mathfrak{p}$, Γ consists of a single branch $\Gamma'_{\mathfrak{p}}$ and (at least) one of the partial derivatives $\partial f'_{\mathfrak{p}}/\partial z^{\alpha}$ ($\alpha = 1, 2, \ldots, n$) does not vanish at $\mathfrak{p}$, then $\mathfrak{p}$ is called a *simple point* of Γ; otherwise $\mathfrak{p}$ is a *singular point*. *If $\mathfrak{p}$ is a simple point of Γ, we can choose the system of local coordinates with the origin $\mathfrak{p}$ so that the minimal local equation of Γ at $\mathfrak{p}$ becomes the simple form* $z^1 = 0$. The set S of all singular points of Γ is an analytic subvariety of Γ of the complex dimension $\leqslant n - 2$, which is called the *singular locus* of Γ. In case $\Gamma - S$ is a connected set, Γ is called *irreducible*; otherwise Γ is said to be *reducible*. A reducible closed analytic surface can be decomposed uniquely into the sum of a finite number of irreducible ones. From the topological viewpoint, an irreducible closed analytic surface Γ is an orientable $(2n - 2)$*-pseudo-manifold* with respect to the "natural" orientation induced by local analytic parameter systems on $\Gamma - S$, and thus Γ *is a* $(2n - 2)$*-cycle.*

Now, suppose a bounding $(2n - 2)$*-cycle* $D = \Sigma m_k \Gamma_k$ on $\mathfrak{M}$ with real coefficients m_k consisting of a finite number of irreducible closed analytic surfaces Γ_k as given and consider the $(2n - 2)$-form

$$\gamma[D](x) = \int_D \gamma^{2n-a}(x, \ ) \tag{3.14}$$

associated with D. It is to be noted here that the cycle D is not necessarily differentiable in the usual sense. We assume, as usual, that *the variety $\mathfrak{M}$ is triangulated with its subvarieties Γ_k into "analytic simplexes"* so that $\mathfrak{M}$ becomes a finite simplicial complex K containing Γ_k as its subcomplexes.[14] Then, on each ρ-simplex T of K, we can choose a real parameter system $\{t^1, t^2, \ldots, t^\rho\}$ describing every point $\mathfrak{p}$ on T as $\mathfrak{p} = \mathfrak{p}\ (t^1, t^2, \ldots, t^\rho)$ so that the coordinates $x^j(\mathfrak{p}(t))$ of $\mathfrak{p}(t)$ have continuous first derivatives with respect to $t^1, t^2, \ldots, t^\rho$ in every *inner* point $\mathfrak{p}$ of T, but the differentiability of $x^j(\mathfrak{p}(t))$ might break down on the *boundary* of T; while, in the usual definition of the "differentiable simplex," we request the existence of a parameter system $\{t^1, \ldots, t^\rho\}$ such that $x^j(p(t))$ admit continuous derivatives with respect to $t^1, \ldots t^\rho$ everywhere

[14]As to the possibility of triangulation, see Koopman and Brown [7], Lefschetz and Whitehead [9]; cf. also van der Waerden [11], Lefschetz [8, pp. 362-369].

in the simplex *including the boundary*. Thus, the cycle D might not be differentiable. But, as one readily infers, each ρ-simplex T of K has a finite (ρ-dimensional) area, and therefore *the integral*

$$\int_T \phi^\rho = \frac{1}{\rho!}\int_T \phi_{jk\ldots l}(x)\,\frac{\partial(x^j x^k\ldots x^l)}{\partial(t^1 t^2\ldots t^\rho)}\,dt^1 dt^2\ldots dt^\rho$$

converges absolutely for an arbitrary continuous ρ-form ϕ defined in a neighbourhood of T. This asserts the absolute convergence of the integral in (3.14). Furthermore, if we take for granted, as usual, the validity of the Green-Stokes' formula

$$\int_C d\psi = \int_{\partial C}\psi$$

for an arbitrary chain C of K, the arguments expounded in §**2** can be applied to arbitrary cycles of K, and thus Theorem 4 is valid also for $\gamma[D]$. By virtue of Theorem 4, $d\gamma[D]$ is a real harmonic $(2n-1)$-form. Hence, putting

$$4\pi i \,{}_*d\gamma[D](x) = \phi_\alpha dz^\alpha - \bar{\phi}_\alpha d\bar{z}^\alpha \tag{3.15}$$

we can introduce a 1-form $\phi_\alpha dz^\alpha$. Now we shall prove that $\int \phi_\alpha dz^\alpha$ *is the Picard integral of the third kind having D as its "logarithmic polar cycle."*[15] Since D consists of analytic surfaces, the "surface elements" of the types

$$[dz^1 dz^2\ldots dz^n\,\overbrace{d\bar{z}^\lambda\ldots d\bar{z}^\nu}^{n-2}],$$

$$[\overbrace{dz^\beta\ldots dz^\gamma}^{n-2}\,d\bar{z}^1 d\bar{z}^2\ldots d\bar{z}^n]$$

vanish identically on D; consequently $\gamma[D](x)$ has the form

$$\gamma[D](x) = \frac{1}{(2n-2)!}\int_D \gamma_{\alpha\beta\ldots\gamma\dot{\lambda}\dot{\mu}\ldots\dot{\nu}}(x,\xi)\,[\overbrace{d\zeta^\alpha d\zeta^\beta\ldots d\zeta^\gamma}^{n-1}\,\overbrace{d\bar{\zeta}^\lambda d\bar{\zeta}^\nu\ldots d\bar{\zeta}^\mu}^{n-1}],$$

where $\zeta^\alpha = \xi^\alpha + i\,\xi^{n+\alpha}$. Hence we get, using (3.6) and (3.11),

$$\mathfrak{C}\gamma[D](x) = \int_D \mathfrak{C}_x\gamma(x,\xi) = \int_D \mathfrak{C}_x\mathfrak{C}_x\mathfrak{C}_\xi\gamma(x,\xi) = \int_D \mathfrak{C}_\xi\gamma(x,\xi) = \int_D \gamma(x,\xi),$$

or

$$\mathfrak{C}\gamma[D](x) = \gamma[D](x). \tag{3.16}$$

Keeping the relations $\mathfrak{C}\Lambda = \Lambda\mathfrak{C}$, $\delta\Lambda = \Lambda\delta$ in mind, we can readily deduce from (3.16) and (3.5) the formula

$$\Lambda^m d\gamma[\mathrm{D}](x) = d\Lambda^m\gamma[D](x) \quad (m = 1, 2, \ldots, n-1). \tag{3.17}$$

Indeed, if one assumes that (3.17) were already proved for $m = k$, then we get

[15]Cf. Weil [12].

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$$\Lambda^{k+1}d\gamma[\mathrm{D}] - d\Lambda^{k+1}\gamma[D] = (\Lambda d - d\Lambda)\Lambda^k\gamma[D]$$
$$= \mathfrak{C}^{-1}\delta\mathfrak{C}\Lambda^k\gamma[D] = \mathfrak{C}^{-1}\Lambda^k\delta\mathfrak{C}\gamma[D] = \mathfrak{C}^{-1}\Lambda^k\delta\gamma[D] = 0,$$

since, by (2.11), $\delta\gamma[D](x) = 0$. Thus (3.17) is proved by induction on m.

Now, using (3.7) and (3.17), we get from (3.15)

$$\phi_a dz^a + \bar{\phi}_a d\bar{z}^a = d\left\{\frac{4\pi(-1)^{n(n+1)/2}}{(n-1)!}\Lambda^{n-1}\gamma[D](x)\right\};$$

hence we obtain

$$\phi_a dz^a = d\left\{\frac{2\pi(-1)^{n(n+1)/2}}{(n-1)!}\Lambda^{n-1}\gamma[D](x)\right\} + 2\pi i \, {}_*d\gamma[D](x). \tag{3.18}$$

From (3.18) follows first that the coefficients ϕ_a are regular with respect to $x^1, x^2, \ldots, x^{2n}$ in $\mathfrak{M} - |D|$, where $|D|$ means the support of D, i.e. $|D| = \bigcup \Gamma_k$. Again, since $d\gamma[D](x)$ is regular harmonic in $\mathfrak{M} - |D|$, (3.18) yields immediately

$$d(\phi_a dz_a) = 0, \tag{3.19}$$

which implies $\partial\phi^a/\partial\bar{z}^\beta = 0$ $(\alpha, \beta = 1, 2, \ldots, n)$. Hence ϕ_a *are regular analytic with respect to complex coordinates* $z^1, z^2, \ldots, z^n$ in $\mathfrak{M} - |D|$. Moreover (3.19) shows that the integral $\int^z \phi_a dz^a$ is locally univalent in $\mathfrak{M} - |D|$. Thus, *putting*

$$\Phi(z) = \int^z \phi_a dz^a + \text{const.},$$

we obtain a many valued analytic function $\Phi(z)$ *on* $\mathfrak{M}$ *which is regular in* $\mathfrak{M} - |D|$. Furthermore, since, as one readily infers, the formula

$$\overline{\Lambda\psi} = \Lambda\bar{\psi}$$

holds, the 0-form $\Lambda^{n-1}\gamma[D](x)$ is real and therefore *the real part of* $\Phi(z)$ *is given by*

$$\Re\Phi(z) = 2\pi\{(-1)^{n(n+1)/2}/(n-1)!\}\Lambda^{n-1}\gamma[D](x) + \text{const.}$$

Thus $\Re\Phi(z)$ *is one-valued on* $\mathfrak{M}$.

Our next task is to determine the singularities of $\Phi(z)$ on $|D|$. For that purpose, denoting the minimal local equation of each irreducible component Γ_k of $|D|$ at $\mathfrak{p} \in |D|$ by $f_{k\mathfrak{p}}(z) = 0$, we shall compare $\Phi(z)$ with $\Sigma_k m_k \log f_{k\mathfrak{p}}(z)$ in a sufficiently small neighbourhood $N(\mathfrak{p})$ of $\mathfrak{p}$ and show that *the difference*

$$\Phi(z) - \Sigma_k m_k \log f_{k\mathfrak{p}}(z)$$

is regular analytic in $N(\mathfrak{p})$. Assume first $\mathfrak{p}$ to be a simple point of[16] $|D|$. Then, in $N(\mathfrak{p})$, $|D|$ consists of a single component, say Γ_1, and, after a suitable choice of the local coordinates with the origin $\mathfrak{p}$, the minimal local equation of Γ_1 at $\mathfrak{p}$ has the simple form

[16]Remember that $|D|$ is a closed analytic surface consisting of irreducible components $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$.

$$f_{1\mathfrak{p}}(z) \equiv z^1 = 0.$$

Now it can be readily verified that the 1-form $\mathfrak{J}d \log f_{1\mathfrak{p}}$ satisfies the integral equations

$$(3.20) \qquad (\mathfrak{J}d \log f_{1\mathfrak{p}},\ \delta\eta)_N = 2\pi \int_{\Gamma_1} \eta^*,$$

$$(3.21) \qquad (\mathfrak{J}d \log f_{1\mathfrak{p}},\ d\lambda)_N = 0,$$

where $N = N(\mathfrak{p})$ and η or λ means respectively an arbitrary 2- or 0-form $\subset N(\mathfrak{p})$ having continuous first derivatives. In fact, denoting by $G(\epsilon)$ the cylindrical domain $|z^1| < \epsilon$, by $T(\epsilon)$ the cylindrical surface $|z^1| = \epsilon$, and putting

$$z^1 = x^1 + i\, x^{n+1} = qe,^{i\theta} \qquad (q = |z^1|),$$

we have

$$\mathfrak{J}\ d \log f_{1\mathfrak{p}} = d\theta = q^{-2}(x^1 dx^{n+1} - x^{n+1} dx^1)$$

and therefore, by virtue of the Green's formula (2.1),

$$(\mathfrak{J}d \log f_{1\mathfrak{p}},\ \delta\eta)_N = \lim_{\epsilon\to 0}\ (d\theta,\ \delta\eta)_{N-G(\epsilon)} = \lim_{\epsilon\to 0} \int_{T(\epsilon)} (d\theta\ .\ \eta)^j g^{\frac{1}{2}} do_j$$

$$= \lim_{\epsilon\to 0} \int_{T(\epsilon)} q^{-2}(x^1\eta^{j\ n+1} - x^{n+1}\eta^{j1}) g^{\frac{1}{2}} do_j = \lim_{\epsilon\to 0} \int_{T(\epsilon)} q^{-2}\eta^{1\ n+1} g^{\frac{1}{2}}(x^1 do_1 + x^{n+1} do_{n+1}),$$

since $\int_{T(\epsilon)} x^j do_k$ vanishes for $j \neq k$. Whence we get

$$(\mathfrak{J}d \log f_{1\mathfrak{p}},\ \delta\eta)_N = \lim_{\epsilon\to 0}\ (-1)^{n-1} \int_{T(\epsilon)} \eta^{1\ n+1} g^{\frac{1}{2}} [d\theta dx^2 \ldots dx^n dx^{n+2} \ldots dx^{2n}]$$

$$= (-1)^{n-1} 2\pi \int_{\Gamma_1} \eta^{1\ n+1} g^{\frac{1}{2}} [dx^2 \ldots dx^n dx^{n+2} \ldots dx^{2n}] = 2\pi \int_{\Gamma_1} \eta^*,$$

proving (3.20). Again we have

$$(\mathfrak{J}d \log f_{1\mathfrak{p}},\ d\lambda)_N = \lim_{\epsilon\to 0}\ (d\theta,\ d\lambda)_{N-G(\epsilon)} = -\lim_{\epsilon\to 0} \int_{T(\epsilon)} (\lambda d\theta)^j g^{\frac{1}{2}} do_j$$

$$= \lim_{\epsilon\to 0} \int_{T(\epsilon)} q^{-2}(x^{n+1} g^{j1} - x^1 g^{j\ n+1}) \lambda g^{\frac{1}{2}} do_j = \lim_{\epsilon\to 0} \int_{T(\epsilon)} q^{-2} g^{1\ n+1} \lambda g^{\frac{1}{2}} (x^{n+1} do_{n+1} - x^1 do_1)$$

$$= \lim_{\epsilon\to 0}\ (-1)^n \int_{T(\epsilon)} g^{1\ n+1} \lambda g^{\frac{1}{2}}\ (\cos^2\theta - \sin^2\theta) [d\theta dx^2 \ldots dx^n dx^{n+2} \ldots dx^{2n}]$$

proving (3.21). On the other hand, we infer from (2.13) and (2.14) that the 1-form $_*d\gamma[D]$ satisfies

$$({}_*d\gamma[D],\ \delta\eta) = \int_D \eta^* = m_1 \int_{\Gamma_1} \eta^*, \qquad ({}_*d\gamma[D],\ d\lambda) = 0.$$

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Combined with (3.20), (3.21), these formulae yield

$$(2\pi {*d\gamma}[D] - \Im m_1 d \log f_{1\mathfrak{p}},\ \delta\eta)_N = 0,$$

$$(2\pi {*d\gamma}[D] - \Im m_1 d \log f_{1\mathfrak{p}},\ d\lambda)_N = 0,$$

while we have

$$\| 2\pi {*d\gamma}[D] - \Im m_1 d \log f_{1\mathfrak{p}} \|_N \leqslant 2\pi \| d\gamma[D] \| + |m_1| \, \| d\theta \|_N < +\infty.$$

Hence by virtue of the principle of orthogonal projections, the difference

$$2\pi {*d\gamma}[D] - \Im m_1 d \log f_{1\mathfrak{p}} = \Im \{ \phi_\alpha dz^\alpha - m_1 d \log f_{1\mathfrak{p}} \}$$

is regular harmonic in $N = N(\mathfrak{p})$. This implies that the difference $\phi_\alpha dz^\alpha - m_1 d \log f_{1\mathfrak{p}}$ is regular with respect to *real* coordinates $x^1, x^2, \ldots, x^n, x^{n+1}, \ldots, x^{2n}$ everywhere in $N(\mathfrak{p})$, while, in $N(\mathfrak{p}) - \Gamma_1$, $\phi_\alpha dz^\alpha$ and $d \log f_{1\mathfrak{p}}$ are regular analytic with respect to *complex* coordinates $z^1, z^2, \ldots, z^n$. Hence $\phi_\alpha dz^\alpha - m_1 d \log f_{1\mathfrak{p}}$ is regular analytic with respect to complex variables $z^1, z^2, \ldots, z^n$ everywhere in $N(\mathfrak{p})$, and therefore *the difference*

$$\phi_\alpha dz^\alpha - \Sigma m_k d \log f_{k\mathfrak{p}}$$

is regular analytic with respect to complex coordinates $z^1, \ldots, z^n$ *everywhere in* $N(\mathfrak{p})$, since, by hypothesis, $f_{k\mathfrak{p}}(z)$ $(k = 2, 3, \ldots, \kappa)$ do not vanish in $N(\mathfrak{p})$.

Consider now the case that $\mathfrak{p}$ is a singular point of $|D|$. Denoting the singular locus of $|D|$ by S, we infer from the above result that $\phi_\alpha dz^\alpha - \Sigma m_k d \log f_{k\mathfrak{p}}$ is regular analytic with respect to complex coordinates $z^1, \ldots, z^n$ in every point $\mathfrak{q} \in N(\mathfrak{p}) - S$, $N(\mathfrak{p})$ being a sufficiently small neighbourhood of $\mathfrak{p}$ such that the local equations $f_{k\mathfrak{p}}(z) = 0$ of Γ_k are available in the whole of $N(\mathfrak{p})$. Indeed, denoting for each k the minimal local equation of Γ_k at $\mathfrak{q}$ by $f_{k\mathfrak{p}}(z) = 0$, $f_{k\mathfrak{q}}(z)$ can be written, in a sufficiently small neighbourhood $N(\mathfrak{q})$ of $\mathfrak{q}$, as

$$f_{k\mathfrak{p}}(z) = U_k(z) f_{k\mathfrak{q}}(z),$$

where $U_k(z)$ is a non-vanishing regular analytic function in $N(\mathfrak{q})$; hence we get

$$\phi_\alpha dz^\alpha - \Sigma m_k d \log f_{k\mathfrak{p}} = \phi_\alpha dz^\alpha - \Sigma m_k d \log f_{k\mathfrak{q}} - \Sigma m_k d \log U_k$$

proving that $\phi_\alpha dz^\alpha - \Sigma m_k d \log f_{k\mathfrak{p}}$ is regular analytic in $N(\mathfrak{q})$. Thus $\phi_\alpha dz^\alpha - \Sigma m_k d \log f_{k\mathfrak{p}}$ is regular analytic in $N(\mathfrak{p})$ excepting at most the analytic subvariety $S \cap N(\mathfrak{p})$ of the complex dimension $\leqslant n - 2$, while, by a theorem of Hartogs [2], an analytic variety containing all singular points of an analytic function of n complex variables must have the complex dimension $n - 1$. Hence $\phi_\alpha dz^\alpha - \Sigma m_k d \log f_{k\mathfrak{p}}$ is regular analytic everywhere in $N(\mathfrak{p})$. Thus we see that, for every point $\mathfrak{p} \in |D|$, *the integral* $\Phi(z) = \int^z \phi_\alpha dz^\alpha$ *can be written as*

$$\Phi(z) = \Sigma_k m_k \log f_{k\mathfrak{p}}(z) + \textit{regular part, in } N(\mathfrak{p}).$$

Finally we shall evaluate the period

$$[\Phi]_\zeta = \int_\zeta \phi_\alpha dz^\alpha$$

of the integral $\Phi(z)$ on an arbitrary 1-cycle ζ. Denote by C a $(2n-1)$-chain such that $\partial C = D$. Then, using (2.15), we get from (3.18) the formula

$$[\Phi]_\zeta = 2\pi i \left\{ I(\zeta, C) + \int_C w^*[\zeta] \right\}. \tag{3.22}$$

Thus we have proved the following

THEOREM 5. *Let $D = \Sigma_k m_k \Gamma_k$ be a bounding $(2n-2)$-cycle on $\mathfrak{M}$ consisting of a finite number of closed analytic surfaces Γ_k with minimal local equations $f_{k\mathfrak{p}}(z) = 0$ $(k = 1, 2, \ldots, \kappa)$. Then the integral*

$$\Phi_D(z) = \frac{2\pi(-1)^{n(n+1)/2}}{(n-1)!} \Lambda^{n-1}\gamma[D](x) + 2\pi i \int^z *d\gamma[D](x) + \text{const.}$$

is the Picard integral of the third kind with the logarithmic polar cycle D, i.e. $\Phi_D(z)$ is a many valued analytic function on $\mathfrak{M}$ which is regular in $\mathfrak{M} - |D|$ and, for every $\mathfrak{p} \in |D|$, $\Phi_D(z)$ has, in a sufficiently small neighbourhood $N(\mathfrak{p})$ of $\mathfrak{p}$, the form

$$\Phi_D(z) = \Sigma m_k \log f_{k\mathfrak{p}}(z) + \textit{regular part},$$

where $\gamma[D](x)$ means the integral of the Green's form $\gamma^{2n-2}(x, \xi)$ over the cycle D:

$$\gamma[D](x) = \int_D \gamma^{2n-2}(x, \ ).$$

The real part

$$\Re\Phi_D(z) = 2\pi \left\{ (-1)^{n(n+1)/2}/(n-1)! \right\} \Lambda^{n-1}\gamma[D](x)$$

of $\Phi_D(z)$ is one-valued on $\mathfrak{M}$. The period $[\Phi_D]_\zeta$ of the integral $\Phi_D(z)$ over an arbitrary 1-cycle ζ is given by

$$[\Phi_D]_\zeta = 2\pi i \left\{ I(\zeta, C) + \int_C w^*[\zeta] \right\} \qquad (\partial C = D),$$

where C is a $(2n-1)$-chain with $\partial C = D$ and $w\,[\zeta]$ means the harmonic 1-form of the first kind associated with the 1-cycle ζ in the sense of Hodge's Theorem.

4. Multiplicative meromorphic functions. Now it is obvious how to construct a multiplicative meromorphic function having the given divisor D. Assume that a bounding $(2n-2)$-cycle $D = \Sigma m_k \Gamma_k$ on $\mathfrak{M}$ with *integral* coefficients m_k consisting of a finite number of closed analytic surfaces Γ_k is given. Then, constructing the Picard integral $\Phi_D(z)$ of the third kind as above and putting

$$F_D(z) = \exp\left\{ \Phi_D(z) \right\},$$

we get the multiplicative meromorphic function $F_D(z)$ having D as its divisor. Thus we obtain the following

THEOREM 6. *Let $D = \Sigma m_k \Gamma_k$ be a bounding $(2n-2)$-cycle on $\mathfrak{M}$ with integral coefficients m_k consisting of a finite number of closed analytic surfaces Γ_k with minimal local equations $f_{k\mathfrak{p}}(z) = 0$ $(k = 1, 2, \ldots, \kappa)$. Then*

$$F_D(z) = c.\exp 2\pi\left\{\frac{(-1)^{n(n+1)/2}}{(n-1)!}\Lambda^{n-1}\gamma[D](x) + i\int^z *d\gamma[D](x)\right\} \qquad (c \neq 0)$$

is a meromorphic function having D as its divisor, i.e. $F_D(z)$ is a many valued analytic function on $\mathfrak{M}$ which is regular in $\mathfrak{M} - |D|$ and, for every point $\mathfrak{p} \in |D|$, $F_D(z)$ is represented in a sufficiently small neighbourhood $N(\mathfrak{p})$ of $\mathfrak{p}$ as

$$F_D(z) = U(z)\Pi_k\{f_{k\mathfrak{p}}(z)\}^{m_k} \qquad (U(z) \neq 0),$$

where $U(z)$ is a non-vanishing regular analytic function defined in $N(\mathfrak{p})$. The absolute value

$$|F_D(z)| = |c| \exp 2\pi\{(-1)^{n(n+1)/2}/(n-1)!\}\Lambda^{n-1}\gamma[D](x)$$

of $F_D(z)$ is one-valued on $\mathfrak{M}$; thus $F_D(z)$ is multiplicative. If one prolongs $F_D(z)$ along an arbitrary closed curve ζ, then $F_D(z)$ is multiplied by the factor

$$\chi_D(\zeta) = \exp 2\pi i\{I(\zeta, C) + \textstyle\int_C w^*[\zeta]\} \qquad (\partial C = D), \tag{4.1}$$

where C is a $(2n-1)$-chain with $\partial C = D$ and $w[\zeta]$ means the harmonic 1-form of the first kind associated with the 1-cycle ζ in the sense of Hodge's Theorem.

As was already mentioned in §**1**, a multiplicative meromorphic function is determined by its divisor uniquely up to a multiplicative constant. Hence we get, as a corollary of the above theorem, the following

THEOREM 7. *A meromorphic function $F(z)$ with the divisor $D = \partial C$ is one-valued if and only if the congruence*

$$I(\zeta, C) + \int_C w^*[\zeta] \equiv 0 \pmod 1$$

holds for every 1-cycle ζ with integral coefficients.

This theorem can be considered as a generalization of Abel's Theorem [13, pp. 126-127] in the classical theory of Riemann surfaces.

Considered as a functional of 1-cycles ζ with integral coefficients, $\chi_D = \chi_D(\zeta)$ is a character of the 1-homology group $H^1(\mathfrak{M})$ of $\mathfrak{M}$ over the additive group of all integers. Then χ_D will be called the *integral character* [13, p. 125] associated with the divisor D.

As a simplest example, let us consider a "torus" $\mathfrak{T}$ obtained from the n-dimensional complex vector space $\{z;\ z = (z^1, z^2, \ldots, z^n)\}$ by identifying points which are congruent to each other with respect to the discrete subgroup generated by $2n$ linearly independent vectors $\pi_k = (\pi_k{}^\alpha)$ $(k = 1, 2, \ldots, 2n)$. The Kählerian metric

$$ds^2 = \Sigma_{\alpha=1}^n |dz^\alpha|^2$$

defined in the vector space is invariant under every translation

$$T_v:\ z^\alpha \rightarrow z^\alpha + v^\alpha;$$

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hence it can be considered as a Kählerian metric attached to $\mathfrak{T}$. Putting

$$\pi_k^{\alpha} = \pi_k^{j} + i\pi_k^{n+j} \qquad (j = \alpha;\ \alpha = 1, 2, \ldots n),$$

$$\pi_{k\alpha} = \tfrac{1}{2}\,\bar{\pi}_k^{\alpha} = \tfrac{1}{2}\,(\pi_k^{j} - i\,\pi_k^{n+j}) \qquad (j = \alpha),$$

we introduce real components π_k^j and covariant components $\pi_{k\alpha}$ of π_k. Again, for simplicity's sake, we assume that $|\pi_k^j| = 1$, so that the volume of $\mathfrak{T}$ with respect to the metric ds^2 is equal to 1. The point $t\pi_k$ describes a closed curve in $\mathfrak{T}$ when the real parameter t moves from 0 to 1 which will be denoted by ζ_k. Then $\{\zeta_1, \zeta_2, \ldots, \zeta_{2n}\}$ constitutes a base of the 1-homology group $H^1(\mathfrak{T})$ of $\mathfrak{T}$. A base of the space $\mathfrak{F}^1$ of all harmonic 1-forms of the first kind is given simply by $\{dx^1, dx^2, \ldots\ dx^{2n}\}$; hence we have

$$w^1(x, \xi) = \Sigma_{k=1}^{2n}\, dx^k d\xi^k.$$

The harmonic 1-form $w[\zeta_k]$ associated with ζ_k is therefore represented as

$$w[\zeta_k](x) = \pi_{k\alpha} dz^{\alpha} + \bar{\pi}_{k\alpha} d\bar{z}^{\alpha}.$$

Now, suppose an irreducible closed analytic surface Γ in $\mathfrak{T}$ to be given and consider the $(2n - 2)$-cycle

$$D = m\Gamma_v - m\Gamma, \quad \Gamma_v = T_v\Gamma,$$

m being a positive integer. Obviously D is homologous to zero; moreover, constructing the "cylinder"

$$C(v, \Gamma) = \{tv + z;\ 0 \leqslant t \leqslant 1,\ z \in \Gamma\}$$

over the surface Γ, we have

$$D = m\partial C(v, \Gamma).$$

From (4.1) follows therefore

$$\chi_D(\zeta) = \exp 2\pi i \left\{ m \int_{C(v,\,\Gamma)} w^*[\zeta] \right\}.$$

By a simple calculation, we obtain

$$\int_{C(v,\,\Gamma)} w^*[\zeta_k] = i \sum_{\alpha,\beta=1}^{n} A_{\alpha\beta}(\bar{\pi}_{k\alpha}\bar{v}^{\beta} - \pi_{k\beta}v^{\alpha}), \tag{4.2}$$

where

$$A_{\alpha\beta} = \left(\frac{i}{2}\right)^{n-1} \int_{\Gamma} (-1)^{n+\alpha+\beta}[dz^1 \ldots dz^{\alpha-1}dz^{\alpha+1} \ldots dz^n d\bar{z}^1 \ldots d\bar{z}^{\beta-1}d\bar{z}^{\beta+1} \ldots d\bar{z}^n].$$

$C(\pi_j, \Gamma)$ is a $(2n-1)$-cycle which is homologous to the direct product $\zeta_j \times \Gamma$. Hence, putting

$$Q_{jk} = i \sum_{\alpha,\beta=1}^{n} A_{\alpha\beta}(\pi_{k\beta}\pi_j{}^{\alpha} - \bar{\pi}_{k\alpha}\bar{\pi}_j{}^{\beta}), \tag{4.3}$$

we get by (4.2)

$$Q_{jk} = \int_{C(\pi_j, \Gamma)} w^*[\zeta_k] = I(\zeta_j \times \Gamma, \zeta_k) = I(\Gamma, \zeta_j \times \zeta_k);$$

thus Q_{jk} are integers. Finally, putting

$$v^a = \Sigma_{j=1}^{2n} \delta^j \pi_j{}^a,$$

δ^j being real numbers, we get from (4.2) and (4.3)

$$\int_{C(v, \Gamma)} w^*[\zeta_k] = \Sigma_j Q_{jk} \delta^j$$

and thus obtain the *formula*

$$\chi_D(\zeta_k) = \exp 2\pi i \{\Sigma_j Q_{jk} m \delta^j\}, \tag{4.4}$$

where $Q_{jk} = I(\Gamma, \zeta_j \times \zeta_k)$.

From this formula we can deduce several conclusions concerning meromorphic functions on $\mathfrak{T}$. First, since Q_{jk} are integers, we can choose, for given Γ, the vector v and the integer m so that $\Gamma_v \neq \Gamma$ and $\Sigma_j Q_{jk} m \delta^j \equiv 0 \pmod 1$. Hence *if $\mathfrak{T}$ contains an irreducible closed analytic surface Γ then there exists on $\mathfrak{T}$ a one-valued meromorphic function with the divisor of the type $D, = m\Gamma_v - m\Gamma$.* As is well known, we can choose the periods π_k so that there exists on $\mathfrak{T}$ no one-valued meromorphic function other than constants. *Such $\mathfrak{T}$ contains therefore no closed analytic surface.* On the other hand, in case Γ is not a "cylindrical surface,"[17] we have

$$|Q_{jk}| = |A_{\alpha\beta}|^2 \neq 0,$$

and therefore we can choose v and m so that $\Gamma_v \neq \Gamma$ and that χ_D coincides with an arbitrarily preassigned character χ of $H^1(\mathfrak{T})$. Thus, *if $\mathfrak{T}$ contains an irreducible closed analytic surface Γ which is not cylindrical, then there exists on $\mathfrak{T}$ a multiplicative meromorphic function with the divisor of the type $D = m\Gamma_v - m\Gamma$ whose "multiplier" χ_D coincides with an arbitrarily preassigned χ of $H^1(\mathfrak{T})$.*

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[17] Γ is called a cylindrical surface if all tangential planes of Γ contain one and the same direction.

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