Teaching Notes

The Devil's Advocate and the binomial expansion

"In the days before calculators, and even today in examination questions, substituting numerical values into the binomial expansion $\sqrt{1 - x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \ldots$ (for $|x| < 1$) gave a way of approximating square roots. For example, substituting $x = \frac{1}{50}$ enables us to approximate $\sqrt{\frac{49}{50}} = \frac{7}{10}\sqrt{2}$ and hence $\sqrt{2}$."

Devil's Advocate (DA): "So a good value of x to use here is one for which $1 - x = 2q^2$ for rational q?"

"Yes."

DA: "And, ideally, x would be small to reduce the number of terms needed to be used in the expansion."

"Yes – the smaller the better."

DA: "So we want $2q^2 \approx 1$ or $q \approx \frac{1}{2}\sqrt{2}$?"

"Yes."

DA: "But weren't we looking to approximate $\sqrt{2}$ at the end of the calculation rather than at the beginning?"

"Yes – but using the binomial expansion will further improve the accuracy of the approximation. For example, if $x = 1 - 2q^2$ is small, then the first two terms of the binomial expansion give

$$q\sqrt{2} \approx 1 - \tfrac{1}{2}\left(1 - 2q^2\right) = \tfrac{1}{2}\left(1 + 2q^2\right)$$

so that $\sqrt{2} \approx \frac{1}{2}\left(\frac{1}{q} + 2q\right)$."

DA: "I see: $\frac{1}{q} \approx \sqrt{2}$ and $\frac{1}{2}\left(\frac{1}{q} + 2q\right) = \frac{1}{2}\left(\frac{1}{q} + \frac{2}{q^{-1}}\right)$ is just the Newton-Raphson iterate for $x^2 - 2 = 0$ starting from $\frac{1}{q}$. I'm not yet seeing the point of using the binomial expansion."

"I didn't quite hear that last bit. But you have got me thinking. If we use three terms of the binomial expansion,

$$q\sqrt{2} \approx 1 - \tfrac{1}{2}\left(1 - 2q^2\right) - \tfrac{1}{8}\left(1 - 2q^2\right)^2 = \tfrac{3}{8} + \tfrac{3}{2}q^2 - \tfrac{1}{2}q^4$$

we obtain $\sqrt{2} \approx \frac{3}{8q} + \frac{3}{2}q - \frac{1}{2}q^3$ which turns out to be the Newton-Raphson iterate for $\dfrac{x^2 - 2}{\sqrt[5]{5x^2 - 2}} = 0$ starting from $\frac{1}{q}$…."

So ends this deliberately whimsical dialogue. In principle, higher order approximations may also be given the Newton-Raphson treatment: $x^2 - 2$ always features as a factor, but the other terms become increasingly gruesome. We illustrate this with four terms of the binomial expansion.

From $q\sqrt{2} \approx 1 - \frac{1}{2}\left(1 - 2q^2\right) - \frac{1}{8}\left(1 - 2q^2\right)^2 - \frac{1}{16}\left(1 - 2q^2\right)^3$, we obtain $\sqrt{2} \approx \frac{5}{16q} + \frac{15}{8}q - \frac{5}{4}q^3 + \frac{1}{2}q^5$. Writing $x = \frac{1}{q}$, the latter expression will match the Newton-Raphson iterate for $f(x) = 0$ provided that

$$\frac{5x^6 + 30x^4 - 20x^2 + 8}{16x^5} = x - \frac{f(x)}{f'(x)}.$$

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This leads to

$$\frac{f'(x)}{f(x)} = \frac{16x^5}{11x^6 - 30x^4 + 20x^2 - 8} = \frac{2x}{x^2 - 2} + \frac{4x - 6x^3}{11x^4 - 8x^2 + 4},$$

which integrates to give

$$f(x) = \frac{x^2 - 2}{(11x^4 - 8x^2 + 4)^{3/22}} \cdot \exp\left[\frac{5}{11\sqrt{7}} \tan^{-1}\left(\frac{11x^2 - 4}{2\sqrt{7}}\right)\right].$$

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A cautionary tale about the pole of polar coordinates

The pole of polar coordinates, given by the point $r = 0$ is usually said to have an undefined argument (just as with the complex number $z = 0$). But, as the following cautionary example shows, this is arguably not the whole story.

The Figure shows the graphs of two curves C_1 and C_2 with respective polar equations $r_1 = 1 + \cos\theta$ and $r_1 = 1 + 2\cos\theta$ for $-\pi < \theta \leqslant \pi$.

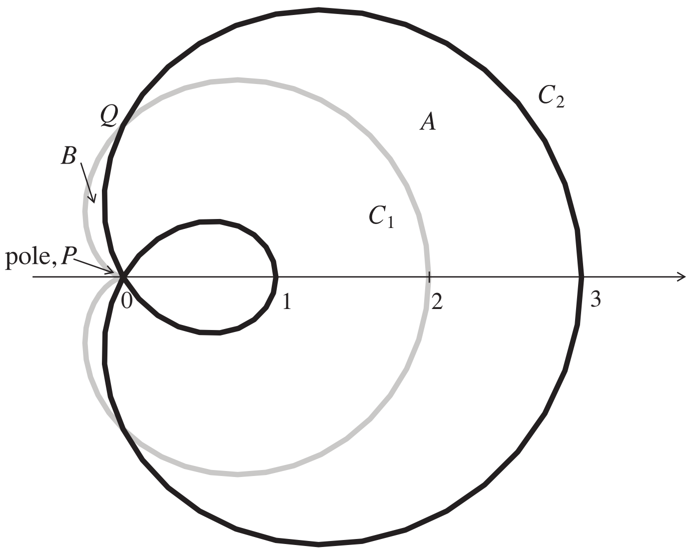

FIGURE: The inner loop of C_2 (which some authors would show dotted) is not involved in this Note.

The point Q is given by $1 + \cos\theta = 1 + 2\cos\theta$, which solves to give $\theta = \frac{1}{2}\pi$, so Q has polar coordinates $\left(1, \frac{1}{2}\pi\right)$. The area of the region labelled A is thus