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A FAILURE OF STABILITY UNDER COMPLEX INTERPOLATION

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Let $\mathscr{L}_\alpha^q = \mathscr{L}_\alpha^q(R)$ denote the spaces of Bessel potentials as defined and discussed in [**6**]. When $1 < q < \infty$ and α is an integer $\mathscr{L}_\alpha^q = L_\alpha^q$, the Sobolev space which consists of functions F in L^q with α derivatives in L^q and with norm $\sum_{j=0}^{\alpha}(\int |F^{(j)}(x)|^q dx)^{1/q}$.

If now we make the change of variables $x = e^y$, $f(x) = F(\ln x)$ it is easily seen that the ratio

$$\sum_{j=0}^{\alpha}\left(\int |F^{(j)}(y)|^q dy\right)^{1/q} \Big/ \sum_{j=0}^{\alpha}\left(\int |x^j f^{(j)}(x)|^q x^{-1} dx\right)^{1/q}$$

is bounded above and below by positive constants. Localized Sobolev spaces can be defined by the norm

$$\sup\left\{\sum_{j=0}^{\alpha}\left(\int_k^{k+1} |F^{(j)}(y)|^q dy\right)^{1/q} : k = 0, \pm 1, \pm 2, \ldots\right\}$$

the ratio of which to

$$||f||_{q,\alpha} = \sup\left\{\sum_{j=0}^{\alpha}\left(\int_{2^k}^{2^{k+1}} |x^j f^{(j)}(x)|^q x^{-1} dx\right)^{1/q} : k = 0, \pm 1, \pm 2, \ldots\right\}$$

is bounded above and below by positive constants. The space defined by the norm $||\cdot||_{q,\alpha}$ will be called $S(q, \alpha)$; these spaces can be constructed for all (q, α) $1 < q < \infty$, $0 \leqq \alpha < \infty$ by means of the Bessel potential spaces [**3**], and for $q = 1$ or ∞ and α integral, by means of the Sobolev spaces. The spaces $S(q, \alpha)$ play an important role in Fourier analysis since the classical multiplier theorems—e.g. those of Marcinkiewicz, Michlin, Hörmander, etc., can be expressed in terms of these spaces. For example, the theorem of Hörmander can be stated by saying that the mapping $T : S(2, \alpha) \oplus L^p(R) \rightarrow L^p(R)$ is continuous for $\alpha \geqq 1$, $1 < p < \infty$, where $T(m, f) = (m\hat{f})^\vee$, $m \in S(2, \alpha)$, $f \in L^p$.

Let $[A, B]_s$ denote the intermediate space between A and B obtained by Calderón's first method of complex interpolation [**1**]. Let $1 < q_0, q_1 < \infty$, $0 < \alpha_0, \alpha_1 < \infty$, $0 \leqq s \leqq 1$, and

$$(1/q, \alpha) = (1 - s)(1/q_0, \alpha_0) + s(1/q_1, \alpha_1).$$

Then it can be shown that the Bessel Potential spaces have the property of stability, i.e. $[\mathscr{L}_{\alpha_0}^{q_0}, \mathscr{L}_{\alpha_1}^{q_1}]_s = \mathscr{L}_\alpha^q$. The close connection between the spaces

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$\mathscr{L}^q_\alpha$ and $S(q, \alpha)$ suggests that at least $S(q, \alpha)$ should be equivalent to $[S(q_0, \alpha_0), S(q_1, \alpha_1)]_s$. Unfortunately this is false.

THEOREM 1. *Suppose α_0, α_1, and α are integers and $1 < q_0, q_1 < \infty$. If q is between q_0 and q_1, α is between α_0 and α_1, and $\alpha q > 1$ then if $0 < s < 1$ the Banach spaces $[S(q_0, \alpha_0), S(q_1, \alpha_1)]_s$ and $S(q, \alpha)$ are not equivalent unless $q_1 = q_0$ and $\alpha_1 = \alpha_0$.*

Proof. Assume $\alpha_1 \geqq \alpha_0$ and if $\alpha_1 = \alpha_0$ that $q_1 > q_0$. Then $S(q_1, \alpha_1) \subset S(q_0, \alpha_0)$ (see [**3**]); now according to a theorem of Calderón $A \cap B$ must be dense in $[A, B]_s$, $0 \leqq s \leqq 1$, so it will suffice to show that $S(q_1, \alpha_1) = S(q_0, \alpha_0) \cap S(q_1, \alpha_1)$ is not dense in $S(q, \alpha)$.

Let

$$I(q, \alpha, k, f) = \left\{ \int_{2^k}^{2^{k+1}} |x^\alpha f^{(\alpha)}(x)|^q x^{-1} dx \right\}^{1/q}$$

and first consider the case $\alpha_0 = \alpha_1$, $q_0 < q_1$. Let $f(y) = c \sin^\alpha \pi y$ if $0 \leqq y \leqq 1$ and $f(y) = 0$ otherwise, where c is chosen so that if

$$g(x) = \sum_{k=0}^{\infty} 2^{k(1/q - \alpha)} f(x - 2^k) \quad (0 \leqq x < \infty),$$

then $I(q, \alpha, k, g) > 1$ ($k = 1, 2, 3, \ldots$). The quantities $I(q, j, k, g)$ ($j = 0, 1, \ldots \alpha$) all have bounds independent of k so $g \in S(q, \alpha)$; moreover $g \in L^\infty$. We will show that the ball of radius $\sqrt{2}/2$ about g in $S(q, \alpha)$ does not intersect $S(q_1, \alpha_1)$. To see this assume $\|g - h\|_{q,\alpha} < \sqrt{2}/2$. It then follows that

$$2^{-q/2} > I(q, \alpha, k, g - h)^q = \int_{2^k}^{2^{k+1}} |x^{2\alpha}[g^{(\alpha)}(x) - h^{(\alpha)}(x)]^2|^{q/2} x^{-1} dx$$

so that if $q \geqq 2$

$$\frac{1}{2} > \left\{ \int_{2^k}^{2^{k+1}} |x^{2\alpha}[g^{(\alpha)}(x)]^2 + x^{2\alpha}[h^{(\alpha)}(x)]^2|^{q/2} x^{-1} dx \right\}^{2/q} - \left\{ \int_{2^k}^{2^{k+1}} |2x^{2\alpha} g^{(\alpha)}(x) h^{(\alpha)}(x)|^{q/2} x^{-1} dx \right\}^{2/q}.$$

and the first integral is bounded below by $I(q, \alpha, k, g)^2$ which is at least unity. Thus

$$(1) \quad \left\{ \int_{2^k}^{2^{k+1}} |x^\alpha g^{(\alpha)}(x)|^{q/2} |x^\alpha h^{(\alpha)}(x)|^{q/2} x^{-1} dx \right\}^{2/q} > B > 0$$

with $B = 1/4$. If $1 \leqq q \leqq 2$ a similar argument based on [**8**, p. 19 (9.13)] yields the same result with a different B. Now if $q_1 > q$ an application of Hölder's inequality with exponents $2q_1/q$ and r where $1/r + q/(2q_1) = 1$ shows that $I(q_1, \alpha, k, h) I(qr/2, \alpha, k, g) > B$. It follows from the definition of g and r that $I(qr/2, \alpha, k, g) < A 2^{-k\epsilon}$ where A is independent of k and $\epsilon = 1/q - 1/q_1$. So $I(q_1, \alpha, k, h) > 2^{k\epsilon} B/A$ and $h \notin S(q_1, \alpha_1)$.

If $\alpha_1 > \alpha_0$ let g and h be as above and apply the Schwarz inequality to (1). If it is kept in mind that the integrand in (1) is zero except on the interval $[2^k, 2^k + 1]$, it follows that there is a positive constant A such that

$$\int_{2^k}^{2^k+1} |x^\alpha h^{(\alpha)}(x)|^q x^{-1} dx \geqq A.$$

Now if $h \in S(q_1, \alpha_1)$ it follows that $|x^\alpha h^{(\alpha)}(x)|$ must be bounded by some constant A_0 ([**2**] contains a discrete analogue of this fact with $q = 2$; that argument can be adapted to the present case), so

$$\int_{2^k}^{2^k+1} |x^\alpha h^{(\alpha)}(x)|^q x^{-1} dx \leqq 2^{-k} A_0{}^q$$

which contradicts (1).

Analogous counterexamples can be constructed for spaces of sequences, and localized versions of the Taibelson spaces Λ_α^{qq}. For these and other generalizations of the above methods see [**3**].

Basically, this lack of density makes it impossible for the $S(q, \alpha)$ spaces to be stable with respect to either the first method of interpolation of Calderón or even the real method of interpolation (this in spite of the results claimed in [**4**; **5**, and **7**]). However, the referee has suggested to us that this lack of density would not be an impediment to the second method of complex interpolation of Calderón, the "upper-s" method. Thus the following

THEOREM 2. *If* $1 < q_0, q_1 < \infty$, $0 \leqq \alpha_0, \alpha_1$, *and* α, q *as before, then* $[S(q_0, \alpha_0), S(q_1, \alpha_1)]^s = S(q, \alpha)$.

One way of proving this is to employ 13.6(ii) of [**1**] with $B_0 = \mathscr{L}_{\alpha_0}^{q_0}$, $B_1 = \mathscr{L}_{\alpha_1}^{q_1}$, and $X_0 = X_1 = l^\infty$, the Banach space of bounded sequences to show

$$[l^\infty(\mathscr{L}_{\alpha_0}^{q_0}), l^\infty(\mathscr{L}_{\alpha_1}^{q_1})]^s = l^\infty(\mathscr{L}_\alpha^q)$$

which is equivalent to the theorem.

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