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ON CERTAIN SUBLATTICES OF THE LATTICE OF SUBGROUPS GENERATED BY THE PREFRATTINI SUBGROUPS, THE INJECTORS AND THE FORMATION SUBGROUPS

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1. Introduction. Various characteristic conjugacy classes of subgroups having covering/avoidance properties with respect to chief factors have recently played a major role in the study of finite soluble groups. Apart from the subgroups which are now called Hall subgroups, P. Hall [**7**] also considered the system normalizers of a finite soluble group and showed that these form a characteristic conjugacy class, cover the central chief factors and avoid the rest. The system normalizers were later shown by Carter and Hawkes [**1**] to be the simplest example of a wealth of characteristic conjugacy classes of subgroups of finite soluble groups which arise naturally as a consequence of the theory of formations. They show that a finite soluble group has, corresponding to each saturated formation $\mathfrak{X}$ containing the class $\mathfrak{N}$ of all finite nilpotent groups, a characteristic conjugacy class of subgroups called the $\mathfrak{X}$-normalizers which have properties closely analogous to the system normalizers of P. Hall and coincide with the latter when $\mathfrak{X} = \mathfrak{N}$. This part of the theory of formations has been extended by Wright [**12**] for the case when a saturated formation does not necessarily contain $\mathfrak{N}$.

Dual to the concept of formations is the concept of Fitting classes introduced by Fischer [**3**]. As Fischer, Gaschütz and Hartley [**4**] have shown, a finite soluble group has, corresponding to each Fitting class $\mathfrak{Y}$, a characteristic conjugacy class of subgroups called the $\mathfrak{Y}$-injectors. These either cover or avoid a chief factor of the group (see Hartley [**8**]).

On the other hand, Gaschütz [**5**] also considered what he called the Prefrattini subgroups of finite soluble groups. In particular, these cover the Frattini chief factors and avoid the complemented ones. Working in this direction, Hawkes [**9**], in turn, obtained further characteristic conjugacy classes, one class in each group corresponding to each saturated formation, of subgroups of finite soluble groups which have properties closely analogous to the Prefrattini subgroups and which coincide with the latter in the case when the saturated formation under consideration is the trivial one. Hawkes' subgroups corresponding to a saturated formation $\mathfrak{X}$ are called the $\mathfrak{X}$-Prefrattini subgroups. Besides possessing an interesting covering/avoidance property, an $\mathfrak{X}$-Prefrattini

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subgroup of a finite soluble group can be expressed as a product of a Prefrattini subgroup and an $\mathfrak{X}$-normalizer of the group.

This latter fact suggested the study in a finite soluble group of the lattice of subgroups generated by the Prefrattini subgroups of the group, the $\mathfrak{F}$-normalizers of the group corresponding to a saturated formation $\mathfrak{F}$ and the $\mathfrak{H}$-injectors of the group corresponding to a Fitting class $\mathfrak{H}$. The idea is to find within this lattice further characteristic conjugacy classes of subgroups with covering/avoidance properties.

The work in the present paper derives from our attempt to study this lattice. We had to restrict ourselves to $\mathfrak{H}$ being a Fischer class since in this case more is known about the behaviour of the Sylow subgroups of the $\mathfrak{H}$-injectors (see Hartley [**8**]), and this information is vital in our investigation. The results of our investigation may be summarized as follows. With a Sylow system Σ (see P. Hall [**6**], for the definition) in a finite soluble group G, one can naturally associate a Prefrattini subgroup W of G (see Hawkes [**9**]), an $\mathfrak{F}$-normalizer D of G (see Wright [**12**]) and an $\mathfrak{H}$-injector V of G, namely the one into which Σ reduces (see the proof of Lemma 3.2 in [**10**]). Let $\mathfrak{L}(D, W, V)$ be the lattice of subgroups of G generated by D, W and V (in the full subgroup lattice of the group). Then, we have

1.1 THEOREM. (i) *The lattice* $\mathfrak{L}(D, W, V)$ *is distributive.*

(ii) *Any two subgroups of* G *in* $\mathfrak{L}(D, W, V)$ *are permutable in* G.

(iii) *Each subgroup of* G *in* $\mathfrak{L}(D, W, V)$ *has a covering/avoidance property with respect to the chief factors of* G.

(iv) Σ *reduces into each subgroup of* G *in* $\mathfrak{L}(D, W, V)$.

(v) *If* A *is a subgroup of* G *in* $\mathfrak{L}(D, W, V)$, *then the family* $\{A^\alpha | \alpha$ *is an automorphism of* $G\}$ *of subgroups constitutes a characteristic conjugacy class of subgroups of* G.

The above theorem is proved in Section 3, where we give an example of G in which $\mathfrak{L}(D, W, V)$ is a free distributive lattice on the three generators and includes neither G nor $\{1\}$.

All groups considered in this paper are assumed to be finite and soluble.

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2. Preliminaries. In this section we collect some of the known results about certain subgroups of the lattice $\mathfrak{L}(D, W, V)$.

The following three results concern the subgroups D, W and DW in the lattice $\mathfrak{L}(D, W, V)$.

2.1 THEOREM (Carter and Hawkes [**1**], Wright [**12**]).

(i) *An $\mathfrak{F}$-normalizer of G covers the $\mathfrak{F}$-central chief factors of G and avoids the rest.*

(ii) *A Sylow system of G reduces into the corresponding $\mathfrak{F}$-normalizer of G.*

2.2. THEOREM (Gaschütz [**5**]). *A Prefrattini subgroup of G covers the Frattini chief factors of G and avoids the complemented ones.*

2.3. THEOREM (Hawkes [**9**]). (i) *An $\mathfrak{F}$-Prefrattini subgroup of G avoids the $\mathfrak{F}$-eccentric, complemented chief factors of G and covers the rest.*

(ii) *If U is the $\mathfrak{F}$-Prefrattini subgroup of G corresponding to the Sylow system Σ of G, then $U = DW$.*

In [**10**], the author studied the sublattice of $\mathfrak{L}(D, W, V)$ generated by the subgroups V and W. In order to present here the relevant results of [**10**], we need to make the following definition.

2.4. *Definition.* (i) A chief factor of a group G is said to be *partially $\mathfrak{H}$-complemented* if it is complemented in G and at least one of its complements in G contains an $\mathfrak{H}$-injector of G.

(ii) A p-chief factor H/K of G is said to be *$\mathfrak{H}$-Frattini* in G if it is $\mathfrak{H}$-covered and $H \cap U_p^G/K \cap U_p^G$ is Frattini in G, where U_p^G is the normal closure in G of a Sylow p-subgroup U_p of an $\mathfrak{H}$-injector U of G.

2.5. THEOREM [**10**]. (i) *V and W are permutable subgroups of G and, moreover, their product VW avoids the partially $\mathfrak{H}$-complemented chief factors of G and covers the rest.*

(ii) *$W \cap V$ covers the $\mathfrak{H}$-Frattini chief factors of G and avoids the rest.*

The permutability of D and V and that of DW and V, and also the covering/avoidance properties of DV and DWV have been established independently by Graham Chambers [**2**] and the author [**11**]. Chambers obtains these results as special cases of his more general results, namely, Theorems 3 and 4 of [**2**].

2.6. THEOREM (Chambers [**2**], and the author [**11**]). (i) *D and V are permutable subgroups of G, and moreover, their product DV avoids the $\mathfrak{F}$-eccentric, $\mathfrak{H}$-avoided chief factors of G and covers the rest.*

(ii) *D and WV are permutable subgroups of G, and moreover, their product $D(WV)$ avoids the $\mathfrak{F}$-eccentric, partially $\mathfrak{H}$-complemented chief factors of G and covers the rest.*

3. The lattice $\mathfrak{L}(D, W, V)$. In this section, we will prove Theorem 1.1. Apart from part (iii) of the theorem, we will obtain the rest of the theorem from a more general consideration. We begin with the following lemma.

3.1 LEMMA. *Let Λ be a chief series of a group G, let p be a prime and let A, B be subgroups of G each of which either covers or avoids each p-chief factor of G in Λ. Then the following are equivalent:*

(i) *$\langle A, B\rangle$ avoids each p-chief factor in Λ which is avoided by both A and B;*
(ii) *$|\langle A, B\rangle|_p = |AB|_p$ and $A \cap B$ covers each p-chief factor of Λ which is covered simultaneously by A and B.*

Proof. (i) $\Rightarrow$ (ii). Let α be the product of the orders of the p-chief factors in Λ which are covered by A, β the product of the orders of the p-chief factors in Λ which are covered by B and γ the product of the orders of the p-chief factors in Λ which are covered simultaneously by A and B. Then, by (i), $|\langle A, B\rangle|_p = \alpha\beta/\gamma \leqq |AB|_p$. Since $|AB|_p \leqq |\langle A, B\rangle|_p$, clearly the equality must hold, and so, also $|A \cap B|_p = \gamma$. Thus, $A \cap B$ covers each p-chief factor of Λ which is covered by both A and B.

(ii) $\Rightarrow$ (i). This is proved similarly.

Before we can state the next lemma, we need to make the following definition.

3.2. *Definition.* Let Λ be a chief series of a group G, let p be a prime and let A, B be subgroups of G. Then, A, B are said to be *p-compatible at* Λ if each of A and B either covers or avoids each p-chief factor of Λ, and, moreover, the equivalent conditions (i) and (ii) of Lemma 3.1 hold.

3.3. LEMMA. *Let $\mathscr{X}$ be a set of subgroups of a group G, and, for each prime p, let Λ_p be a chief series of G. Suppose that the members of $\mathscr{X}$ are pairwise p-compatible at Λ_p for each p. Then, for X, Y, $Z \in \mathscr{X}$, we have*
(i) *$XY = YX$;*
(ii) *XY and $X \cap Y$ are p-compatible with Z at Λ_p for each p;*
(iii) *$XY \cap Z = (X \cap Z)(Y \cap Z)$ and $(X \cap Y)Z = XZ \cap YZ$.*

Proof. (i) Since X and Y are p-compatible at Λ_p for each p, this is immediate from Lemma 3.1 (ii).

(ii) and (iii). Since X and Y are p-compatible at Λ_p, any p-chief factor in Λ_p covered simultaneously by XY and Z is covered either by X or by Y and by Z, and so, since also X, Z and Y, Z are p-compatible at Λ_p, it is covered either by $X \cap Z$ or by $Y \cap Z$. Thus, such a p-chief factor is covered by $XY \cap Z \geqq \langle X \cap Z, Y \cap Z\rangle$. Now, by (i), XY permutes with Z, and so, by Lemma 3.1, XY, Z are p-compatible at Λ_p. Furthermore, $XY \cap Z$ and $(X \cap Z)(Y \cap Z)$ cover the same p-chief factors in Λ_p. Thus,

$$|XY \cap Z|_p = |(X \cap Z)(Y \cap Z)|_p.$$

Since p was an arbitrary prime under consideration, we have shown that $XY \cap Z = (X \cap Z)(Y \cap Z)$.

Similarly, any p-chief factor in Λ_p which is avoided by $X \cap Y$ and by Z is avoided either by X and Z or by Y and Z. Therefore, either XZ avoids such a p-chief factor or YZ avoids it. Hence, $\langle X \cap Y, Z\rangle \leqq XZ \cap YZ$ avoids it, and so, by Lemma 3.1, $X \cap Y$, Z are p-compatible at Λ_p, and $|\langle X \cap Y, Z\rangle|_p = |(X \cap Y)Z|_p$. Since p was an arbitrary prime, it follows now that $(X \cap Y)Z = \langle X \cap Y, Z\rangle$. Finally, since, for each p, any p-chief factor in Λ_p avoided by $(X \cap Y)Z$ is also avoided by $XZ \cap YZ$, as seen above, we have $(X \cap Y)Z = XZ \cap YZ$.

3.4. THEOREM. *With the hypothesis of Lemma* 3.3, *the lattice* $\mathfrak{L}(\mathscr{X})$ *of subgroups generated by* $\mathscr{X}$ *is a distributive lattice of pairwise permutable subgroups. Any two members of* $\mathfrak{L}(\mathscr{X})$ *are* p*-compatible at* Λ_p *for each* p.

Proof. By repeated application of Lemma 3.3 (ii), any two members of $\mathfrak{L}(\mathscr{X})$ are p-compatible at Λ_p for each p.

The rest follows by applying Lemma 3.3 (i) and (iii) to $\mathfrak{L}(\mathscr{X})$.

In order to state an immediate corollary of Theorem 3.4 and also for later purposes we need the following definition.

3.5. *Definition.* A pair of subgroups A, B of a group G is said to be *compatible* if each of A and B has the covering/avoidance property, and moreover, each chief factor of G avoided (covered) simultaneously by A and B is avoided (covered) by $\langle A, B\rangle$ $(A \cap B)$.

3.6. COROLLARY. *Let* $\mathscr{X}$ *be a set of pairwise compatible subgroups. Then the lattice they generate is a distributive lattice of pairwise permutable compatible subgroups.*

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. For each prime p, let Σ_p be the unique Sylow p-subgroup of G contained in Σ, let $V_p = V \cap \Sigma_p$ and let Λ_p be any chief series of G through $V_p{}^G$, the normal closure in G of V_p. By Theorem 2.3 (i) of Hawkes [**9**], D and W are p-compatible at Λ_p (and indeed are compatible). Also, a p-chief factor of Λ_p avoided by both V and D is an $\mathfrak{F}$-eccentric chief factor of G above $V_p{}^G$, and so, in view of Theorem 2.1 and the homomorphism-invariance of the $\mathfrak{F}$-normalizers of G, it is avoided by $V_p{}^G D$, and hence, also by $V_p{}^G D\Sigma_{p'}$ where $\Sigma_{p'}$ is the unique Sylow p-complement of G in Σ. (Note that D permutes with $\Sigma_{p'}$ since $D\Sigma_{p'} = N_G(\Sigma_{p'} \cap G^{\mathfrak{F}(p)})$, where $\mathfrak{F}(p) \in \{\mathfrak{F}(q)\}$, the family of full, integrated formations which define $\mathfrak{F}$ locally.) The latter subgroup evidently contains $\langle V, D\rangle$. Thus, V, D are p-compatible at Λ_p.

Similarly, a p-chief factor of Λ_p avoided by both V and W is a complemented chief factor of G above $V_p{}^G$, and so, it is avoided by $\langle V, W\rangle$ (see the proof of the main theorem in [**10**]). Hence, V, W are p-compatible at Λ_p, and therefore, V, W and D are pairwise p-compatible at Λ_p for each p. Now, it follows, by Theorem 3.4, that $\mathfrak{L}(D, W, V)$ is a distributive lattice of pairwise permutable subgroups, and so, parts (i) and (ii) of Theorem 1.1 are proved.

In order to show (iv), we observe that if a Hall π-subgroup P of G, where π is a set of primes, reduces into a pair of permutable subgroups A and B then it reduces into AB and $A \cap B$. For,

$$|(P \cap A)(P \cap B)| \leqq |P \cap AB| \leqq |AB|_\pi = |A|_\pi \cdot |B|_\pi/|A \cap B|_\pi \leqq |P \cap A|\, |P \cap B|/|P \cap A \cap B| = |(P \cap A)(P \cap B)|.$$

Thus, $|P \cap AB| = |AB|_\pi$ and $|A \cap B|_\pi = |P \cap A \cap B|$. Hence, since Σ

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reduces into each of D, W and V (see Theorem 2.1 (ii) and Lemma 3.5 in [**10**]) and since $\mathfrak{L}(D, W, V)$ is a lattice of pairwise permutable subgroups, it follows that Σ reduces into each member of $\mathfrak{L}(D, W, V)$, and so (iv) is proved.

Also, since the stabilizer $\mathscr{B}$ of Σ in the group $\mathscr{U}$ of automorphisms of G also stabilizes each of D, W and V, $\mathscr{B}$ stabilizes every element in the lattice $\mathfrak{L}(D, W, V)$. Moreover, in view of the well-known result of P. Hall [**6**], $\mathscr{B}$ supplements in $\mathscr{U}$ the group of inner automorphisms of G; hence the statement of part (v) of Theorem 1.1.

Finally, we prove part (iii) of the theorem. The covering/avoidance properties of $W \cap V$, D, W, V, DW, DV, WV and DWV have been described in Section 2. It is clear from these that each of the pairs D, W; D, V; D, WV; VW, DW; and DV, VW is a compatible pair of subgroups. Therefore, by Corollary 3.6, each of them generates a lattice of covering and avoiding subgroups whose covering/avoidance properties can be deduced from those of the generators. Thus, in particular, $D \cap W$, $D \cap V$, $D \cap WV$, $VW \cap DW$ and $DV \cap VW$ are covering and avoiding subgroups.

Similarly, in view of the following lemma and the covering/avoidance properties of $W \cap D$, $W \cap V$ and D, each of the pairs $W \cap D$, $W \cap V$ and D, $W \cap V$ is a compatible pair of subgroups, and so, the covering/avoidance properties of $(W \cap D)(W \cap V) = W \cap DV$ and $D(W \cap V) = DW \cap DV$ can be deduced from those of $W \cap D$, $W \cap V$ and D.

3.7. Lemma. *$V \cap W \cap D$ covers the $\mathfrak{F}$-central, $\mathfrak{H}$-Frattini chief factors of G and avoids the rest.*

Proof. First of all we observe that, for each prime p, $\Sigma_p \cap (V \cap W \cap D)$ is a Sylow p-subgroup of $V_p{}^G \cap W \cap D$ as well as that of $V \cap W \cap D$. Here $V_p = V \cap \Sigma_p$. That it is a Sylow p-subgroup of $V \cap W \cap D$ is a consequence of part (iv) of the theorem. Also, for the same reason, Σ_p reduces into $W \cap D$. Since it clearly reduces into $V_p{}^G$ too, it follows that $\Sigma_p \cap V_p{}^G \cap W \cap D$ is a Sylow p-subgroup of $V_p{}^G \cap W \cap D$. But, by the Corollary to Lemma 3 of Hartley [**8**], $\Sigma_p \cap V_p{}^G = \Sigma_p \cap V$; hence our assertion.

Now, in view of Theorem 2.1 (i) and Theorem 2.2, $V \cap W \cap D$ clearly avoids the complemented, the $\mathfrak{H}$-avoided and the $\mathfrak{F}$-eccentric chief factors of G. Let H/K be a Frattini, $\mathfrak{H}$-covered and $\mathfrak{F}$-central p-chief factor of G for some p dividing $|G|$, and assume first that H/K is $\mathfrak{H}$-Frattini in G. Then, by Definition 2.4 (ii), $H \cap V_p{}^G/K \cap V_p{}^G$ is a non-trivial Frattini p-chief factor of G. Moreover, $H \cap V_p{}^G/K \cap V_p{}^G$ is $\mathfrak{F}$-central in G, being G-isomorphic to H/K. Thus, $W \cap D$, which covers each $\mathfrak{F}$-central, Frattini chief factor of G, covers $H \cap V_p{}^G/K \cap V_p{}^G$. But then

$$(H \cap V_p{}^G \cap W \cap D)K = (H \cap V_p{}^G \cap W \cap D)(K \cap V_p{}^G)K = (H \cap V_p{}^G)K = H$$

since H/K is $\mathfrak{H}$-covered in G so that, by the Corollary to Lemma 3 of Hartley

[**8**], it is covered by V_p^G. Hence $V_p^G \cap W \cap D$ covers H/K. From our initial observation it follows now that $V \cap W \cap D$, too, covers H/K.

Assume next that H/K is not $\mathfrak{H}$-Frattini in G. Then, by Definition 2.4 (ii), $H \cap V_p^G/K \cap V_p^G$ is complemented in G. Thus, by Theorem 2.2, W, and hence $W \cap D$, avoids $H \cap V_p^G/K \cap V_p^G$. In particular, $V_p^G \cap W \cap D$ avoids H/K. Hence, once again from initial observation it follows that $V \cap W \cap D$, too, avoids H/K. The proof of the lemma is complete.

Finally, in view of the covering/avoidance properties of $D \cap W \cap V$, $V \cap W$, $D \cap V$ and $D \cap WV$, we have that each of the pairs $V \cap W$, $D \cap VW$ and $V \cap D$, $V \cap W$ is a compatible pair of subgroups, and hence, as before, the covering/avoidance properties of $DV \cap DW \cap VW$ and $(V \cap D)(V \cap W) = V \cap DW$ can be deduced from those of $V \cap W$, $D \cap V$ and $D \cap VW$. The proof of the theorem is complete.

We now give an example of a group G in which $\mathfrak{L}(D, W, V)$ has eighteen distinct elements and includes neither $\{1\}$ nor G.

Example. Let H be the semidirect product of a cyclic group $\langle a \rangle$ of order 25 by a cyclic group $\langle b \rangle$ of order 4, with the action of $\langle b \rangle$ on $\langle a \rangle$ being given by $a^b = a^7$. Let $K = H \text{ wr } \langle c \rangle$ (according to the regular representation), where $\langle c \rangle$ is a cyclic group of order 5, and let $G = \langle d \rangle \times K$, the direct product of K and a cyclic group $\langle d \rangle$ of order 4.

It is easy to verify that for $\mathfrak{F} = \mathfrak{N} = \mathfrak{H}$, where $\mathfrak{N}$ is the class of all finite nilpotent groups, the lattice $\mathfrak{L}(D, W, V)$ corresponding to G has eighteen distinct elements and so is a free distributive lattice on the three generators.

We conclude this paper with a remark that Theorem 1.1 is indeed capable of generalization. For instance, V in $\mathfrak{L}(D, W, V)$ could have been any subgroup of G which is p-normally embedded in G for each prime p.

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