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A CHARACTERIZATION OF THE GAMMA DISTRIBUTION

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The aim of this note is to prove the following characterization.

Theorem. The positive number α is fixed, X and Y are two positive independent random variables and the distribution of Y is defined by

$$\mathbb{E}(Y^s) = \left(1+\frac{s}{\alpha}\right)^{\alpha+s} \quad \text{for} \quad s>0.$$

Then $X \exp(-X/\alpha)\ Y$ and X have the same distribution if and only if the distribution of X is

$$\gamma_\alpha(dx) = \exp(-x)x^{\alpha-1}\mathbb{1}_{(0,\infty)}(x)\frac{dx}{\Gamma(\alpha)}.$$

Klamkin [3] and Donald J. Newman (quoted in [3]) have made the following observation: if U and V are independent random variables with uniform distribution in $[0,1]$, then $(UV)^{UV}$ and U^U have the same distribution. Writing $X=-\log U$ and $Y=-\log V$, a translation of this result is: if X and Y are independent random variables with distribution γ_1, then $(X+Y)\exp(-(X+Y))$ and $X\exp(-X)$ have the same distribution. Observing that $X+Y=X_1$ has distribution γ_2, this result admits an easy extension, as follows.

Proposition 1. The positive number α is fixed, X and X_1 have respective distributions γ_α and $\gamma_{\alpha+1}$. Then:

(1) $X\exp(-X/\alpha)$ and $X_1\exp(-X_1/\alpha)$ have the same distribution

(2) $\mathbb{E}((X\exp(-X/\alpha))^s) = \dfrac{\Gamma(\alpha+s)}{\Gamma(\alpha)}\left(1+\dfrac{s}{\alpha}\right)^{-(\alpha+s)}$ if $s>-\alpha$.

Proof. Taking $s>-\alpha$, one gets (2) by

$$\mathbb{E}((X\exp(-X/\alpha))^s) = \int_0^\infty x^{s+\alpha-1}\exp\left(-x\left(1+\frac{s}{\alpha}\right)\right)\frac{dx}{\Gamma(\alpha)}.$$

In the same way

$$\mathbb{E}((X_1\exp(-X_1/\alpha))^2) = \frac{\Gamma(\alpha+1+s)}{\Gamma(\alpha+1)}\left(1+\frac{s}{\alpha}\right)^{-(\alpha+1+s)}$$

which is $\mathbb{E}((X\exp(-X/\alpha))^s)$ by using $\Gamma(z+1)=z\Gamma(z)$; this proves (1).

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The interesting point in (2) is that $\mathbb{E}(X^s)=\Gamma(\alpha+s)/\Gamma(\alpha)$. Therefore, if we are able to prove that there exists a positive random variable Y such that

$$\mathbb{E}(Y^s)=\left(1+\frac{s}{\alpha}\right)^{\alpha+s} \quad \text{for} \quad s>-\alpha,$$

(2) implies that $X\exp(-X/\alpha)Y$ and X have the same distribution (in terms of 'arithmetic of laws', if X has distribution γ_α, the distribution of $-(X/\alpha)+\log X$ divides the distribution of $\log X$). Such a Y does exist, as we now demonstrate.

Proposition 2

(1) There exists a probability density f on $\mathbb{R}$ which is real-analytic such that

$$s^s=\int_{-\infty}^{+\infty}\exp(-sx)f(x)\,dx \quad \text{for} \quad s>0.$$

(2) The positive random variable Y such that $\log Y$ has density $\exp(-\alpha x)f(x-\log\alpha)$ satisfies

$$\mathbb{E}(Y^s)=\left(1+\frac{s}{\alpha}\right)^{\alpha+s} \quad \text{for} \quad s>-\alpha.$$

Proof.

(1) It is well known that f is the density of a stable law with index 1 (see Berg et al. [1], p. 218): hence f is real-analytic (see Lukacs [4], Theorem 5.7.5.).

(2) Consider $Z=-\log Y$. Then

$$\mathbb{E}(Y^s)=\mathbb{E}(\exp(-sZ))=\int_{-\infty}^{+\infty}\exp(-sz)\exp(-\alpha z)f(z-\log\alpha)\,dz$$

$$=\alpha^{-(\alpha+s)}\int_{-\infty}^{+\infty}\exp(-sx)\exp(-\alpha x)f(x)\,dx=\alpha^{-(\alpha+s)}(\alpha+s)^{\alpha+s}, \quad \text{from (1).}$$

Proof of the theorem. The 'if' part is obvious from Proposition 1(2). To prove the 'only if' part, we consider a sequence $X_0, Y_1, Y_2, \cdots$ of independent positive random variables such that the Y_j have the distribution of Y. We introduce now the Markov chain $(X_n)_{n=0}^{\infty}$ on $(0,+\infty)$ defined by

$$X_{n+1}=X_n\exp(-X_n/\alpha)Y_{n+1} \quad \text{for} \quad n\geqq 0.$$

Clearly $X\exp(-X/\alpha)Y$ and X have the same distribution if and only if the distribution of X is stationary for the above chain. We already know that γ_α is a stationary distribution; the remaining point is to see that it is the only one.

If A is measurable and contained in $(0,+\infty)$, it will be called a closed set if $P(x\exp(-x/\alpha)\;Y\in A)=1$ for any x in A. Since the density f described in Proposition 2(1) is real-analytic, the density of Y is strictly positive on $(0,+\infty)$ except on a countable number of points. Therefore if A is a closed set, $(0,+\infty)\backslash A$ has Lebesgue measure 0 and two closed sets cannot be disjoint: this means that the chain is indecomposable and has at most one stationary distribution (see Breiman [2], Theorem 7.16).

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