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MORE ON THE ORIGIN OF FINANCIAL ECONOMICS: EARLY CONTRIBUTIONS TO JOINT LIFE ANNUITY VALUATION

BY

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The origin of modern financial economics can be traced to early discounted expected value solutions for the price of life annuities. In contrast to the single life annuity valuations attributed to Jan de Witt and Edmond Halley, the computational complexity of joint life annuity valuation posed difficulties. Following a brief review of various joint life annuity specifications, a history of joint life annuity issuance and valuation in northern Europe from the thirteenth to the mid-eighteenth centuries is provided. With this background, the 1671 correspondence from de Witt to Jan Hudde on possible methods for valuing joint life annuities is detailed. These methods are contrasted with the geometric method described in Halley (1693), providing impetus for examination of the analytical approximations developed by Abraham de Moivre and Thomas Simpson.

I. INTRODUCTION

The traditional history of economics canon privileges the school of classical political economists led by Adam Smith, e.g., Mark Blaug (1997). As evidenced by historical narratives for subjects that have appeared as growth areas in the corpus of post-WW II economics, such as econometrics and financial economics, the traditional canon is disconnected from the central concerns of those subjects.[1] Such narratives often intersect

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[1] "Disconnected" does not mean totally absent. For example, Adam Smith (1976, pp. 916–918) discusses difficulties of life annuity issuance in the Million Act to fight a war against France (1691) and the terms for the 1695 conversion of these life annuities into a term annuity. Tontines are also discussed. However, detailing the price and amounts of public debt issuance situates the discussion more in the vein of public finance as opposed to explicitly solving for annuity price using discounted expected value. Similarly, James Steuart (1767, bk. I, ch. XIII, p. 75) has a brief examination of demographic issues that connects Halley to methods of

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with the history of subjects lying outside the academic silo reserved for 'economics.' In the case of financial economics, histories of actuarial science, demography, and mathematical statistics play an essential role. As such, narratives focusing on the origin of financial economics have been constructed from sources largely unfamiliar to the traditional canon, e.g., Geoffrey Poitras (2006), David Bellhouse (2017), and William Deringer (2018a). Though considerable progress has been made in developing historical narratives applicable to financial economics, largely unexplored avenues remain concerning the origin of financial asset pricing using discounted expected value methods.

Before the emergence of defined benefit pensions, social security plans, state pensions, superannuation, and the like, life annuities issued by public and, to a lesser extent, private borrowers served a similar role for those who could afford such securities. In the absence of accurate mortality estimates, actuarially sound life annuity valuation was not possible. Consequently, the origin of pricing methods for life annuities is closely connected to the emergence of rudimentary life tables in the late seventeenth century. Even as rudimentary tables became available, computational demands involved in determining discounted expected value led to development of analytical approximations that were easier to calculate. While this task for single life annuities was fruitful, such approximations for joint life annuity prices posed difficulties. By the time the founding work of classical political economy—the *Wealth of Nations*—appeared, analytical methods for pricing life annuities by contributors not typically found in the history of economic thought were well developed. Though narratives on single life annuity valuation relevant to the history of financial economics are available—e.g., Anders Hald (1990, ch. 9); Poitras (2000, ch. 6)—contributions detailing joint life annuity valuation are lacking.

Joint life annuities—featuring variations where the annuity payment may or may not continue until all nominees have died—have a history as long as the single life annuity.[2] Joint life annuity nominees could include a husband and wife or a parent and children or selected children. Though the secondary literature on the history of joint life annuity value approximations contains some excellent contributions covering the timeline under consideration—e.g., Francis Baily (1813, pp. iii–xli); Hald (1990, ch. 25); Bellhouse (2017)—some unresolved questions remain. In addition, recent efforts on early life annuity issuance, by financial historians such as Marc Boone, Karel Davids, and Paul Janssens (2003), have expanded knowledge about pricing conventions and contract specification prior to the seminal 1671 contribution on single life annuity valuation attributed to Jan de Witt. These efforts provide useful historical context to assess the contributions of Jan Hudde, Edmond Halley, Abraham de Moivre, and Thomas Simpson to the development of series approximations for joint life annuity valuation appearing from the late seventeenth century to the mid-eighteenth century.

pricing life annuities: "Dr Halley, and others, have calculated the value of annuities which… ought to be valued at their real worth."

[2] The single life annuity contract has three basic components: the subscriber providing the initial capital; the shareholder receiving the annuity payments; and the nominee on whose life the annuity payment is contingent. These individuals can be the same, e.g., a subscriber can purchase an annuity contingent on their life and receive annuity payments until death. Alternatively, the individuals can be different, e.g., a father can purchase a life annuity with a child as nominee and payments to be paid to the mother. In such cases, some provision is required to determine the recipient of the annuity payment if the shareholder dies before the nominee.

A variety of different cash flows from a joint life annuity contract can be specified; de Moivre (1725) provides solutions for sixteen possible variations that can be classified as joint survivor, last survivor, reversion, successor, and renewal. These different types depend on specification of the shareholder(s)—who receive the annuity payments—and the nominee(s) who determine the life contingent payout. Due to differences in the possible ages of nominees increasing the number of valuations to be determined for each contract type, computational complexity increases geometrically with the number and age of nominees involved. Against this backdrop, assessing the contribution of a specific individual to joint life annuity valuation can be obscured by lax modes of identifying specific contract types being considered, compounded by the practice of using the same notation to indicate different variables, and lack of consistency in notation used by different authors. Baily (1813, p. xxxvi) laments "a vicious and corrupt mode of expression by every author that has hitherto written on the subject of joint life annuities," resulting in difficulties in accurately distinguishing between solutions for joint survivor, last survivor, and other contract types.

Taking the simplest joint life annuity cases where only two lives (A and B) are involved, a joint survivor annuity involves A and B receiving payments until either A or B dies, and a last survivor annuity would continue until both A and B are deceased. As with a single life annuity, proof of life for the nominee(s) is required for an annuity payment to be received. For example, in a joint survivor annuity, if either A or B dies before an annuity payment date is reached, that payment is not received. Calculation of a reversion involves decomposing the solution for the joint survivor annuity from the single life annuity to determine the residual value of the annuity that B (A) would receive upon death of A (B). A succession—where A receives the annuity payment and nominates B as a successor to receive a single life annuity upon the death of A—involves determining the annuity value for two successive lives. The seemingly obscure renewal problems in de Moivre (1725) relate not to life annuities, per se, but to where a fine is paid to a landlord for renewing a lease or tenancy when a person named in the lease or tenancy dies. Solutions to renewal problems were useful in determining the value of an estate as a combination of rental income and fines, e.g., Bellhouse (2017, ch. 5, esp. p. 66).

Where the annuity involves three or more persons, contract specification for joint survivor and last survivor annuities is straightforward, though division of annuity payments between shareholders may or may not be considered. For example, a three-person (A, B, C) last survivor annuity could be specified as a tontine where A, B, and C each received one-third of the payment when all nominees are alive, one-half each when one nominee dies, and full payment when two nominees are deceased. Alternatively, division of payment between shareholders could be unspecified in the contract. In contrast, other variants for reversion contracts are possible. For example, C could receive payments when either A or B dies first, or when both A and B are deceased, or only if B (A) survives A (B). Where B and C could receive payments when A is deceased, division of the annuity payments may or may not be specified. Starting from the seminal solution for the value of a single life annuity in de Witt (1671) based on a theoretical life table, life annuity valuation evolved to consider more complicated contract specifications; to provide approximations that substantively reduced computational complexity; and, eventually, to incorporate solutions estimated using observed life tables.

II. ORIGINS OF JOINT LIFE ANNUITIES

Ancient instances of joint life contingency valuation are unknown. Likely inspired by the judicial need to enforce the Falcidian law, Ulpian's table (*Digesta* 35.2.68) for valuing single life maintenances and usufructs survives as evidence that life-contingent valuations were done. The historical context surrounding inheritances suggests the need to value contingencies for joint lives was also a concern but incomplete and "often so one-sided" sources from ancient history provide no evidence for such valuations. In conjunction with the more traditional perpetual hereditary *rente—rente héritable* and *erfrent* later known as *losrent*—the life annuity contract—*rente viagères* and *lijfrent*—became a staple of municipal and, eventually, state finance in northern Europe.[3] As surviving Roman law sources provide no evidence of *rente*-type contracts, it is generally accepted such contracts evolved from the Carolingian *census.*[4] However, evolution from ninth-century *census* arrangements to early thirteenth-century issuance of life annuity contracts by towns and cities in the *Langues d'öil* of northern Europe lacks detailed study.[5]

Conventionally, emergence of public *rentes* and *renten* is connected to the evolution of scholastic usury doctrine. Though there were some ecclesiastic dissenters—e.g., Munro (2003, pp. 520–523); Baum (1985, pp. 27–31)—accepted medieval scholastic doctrine was evidenced in 1250 by Pope Innocent IV exempting such contracts from usury sanctions, as a loan (*mutuum*) involved a return of what was borrowed, a feature absent in perpetual and life annuities. With some provisos, redemption of a perpetual *rente* was also acceptable. However, emphasis on the connection between evolution of the usury doctrine and emergence of public debt diminishes the key role played by licit private debt contracting that predates the thirteenth century, such as the annuities secured by real estate—"plots of land within the town walls" (Baum 1985, p. 25)—in Hanse towns and elsewhere that roughly correspond to redeemable perpetual *rentes.* Similarly, rudimentary single and joint life annuities are reflected in corrody transactions of twelfth-century England and elsewhere, "provided by a religious institution such as a monastery, priory, abbey or hospital," which involved an individual or couple purchasing "some agreed mixture of food, drink, heat, light, accommodation, clothing, laundry, health care, maybe a small monetary allowance and even stabling and grazing for their livestock" (Bell and Sutcliffe 2010, pp. 142–143).

[3] In Middle Dutch, *pensiones* or *pensien* was also used to refer to annuities, e.g., van Schaïk (2003, p. 112). Circa 1350, in Hamburg, Lubeck, and several other south German towns, as well as Swiss cities such as Basel, issues of life annuities were referred to as *Leibgedinge*, and, starting in 1340, Barcelona began issuing *censals vitalicios* (Tracy 2003, pp. 17, 22) at 14.24% and *censals morts* (perpetuals) at 7.5%. In 1538 Venice switched from forced loans to single life annuities at 14%.

[4] Munro (2003, p. 519) observes: "the *rente* was based on the Carolingian *census* contract that many monasteries had long utilized in order to acquire bequests of lands, on condition that the donor receive an annual usufruct income (*redditus*) from the land, in kind or money, for the rest of his life and sometimes for the lives of his heirs." Tracy (2003, pp. 14–15) details studies that identify differences in scholastic justification for the two forms of *rente* contracts. Baum (1985, pp. 29–31) discusses features that made annuities licit transactions.

[5] Tracy (2003, p. 16) attributes the first surviving document for an issue of municipal *rente viagères* to Rheims in 1218, though this issue was an isolated event. The Latin text of a *rente viagères* issued by Tournai in 1228 is provided in SGN (1898, pp. 190–192).

Impressive efforts by historians trolling municipal, notarial, and state administrative records from the thirteenth to the eighteenth centuries have provided a wealth of data about public finance in northern Europe. The bulk of these efforts are concerned with the role that issues of long-term debt—life annuities and redeemable perpetuities—and short-term bills played in the "financial revolutions" that contributed to the rise and consolidation of national governments (e.g., Tracy 1985; Munro 2003; Fritschy 2003; Gelderblom and Jonker 2011). Consequently, limited attention has been given to methods of valuation for joint life annuity contracts prior to the later seventeenth-century contributions of de Witt and Hudde (Hendricks 1853) and Halley (1693). Despite the absence of a systematic treatment, some evidence is available about issuance, investors, and contract design of such joint life annuities. This evidence reveals a decidedly more complicated interpretation for the general claim that early issues of life annuities were sold without reference to age, e.g., Poitras (2000, p. 190).[6] In addition to differences in contractual terms and investor characteristics across jurisdictions and time, in some instances life annuities were marketed to investors beyond the confines of the issuing municipality, while, in other instances, sales were restricted to local residents.

Some caution is required to interpret claims about life annuities prior to the sixteenth century appearing in the secondary literature. For example, Lorraine Daston (1988, p. 121), correctly referencing SGN (1898, p. 209), observes that, starting in 1402, Amsterdam sold municipal annuities at regular intervals, typically "charging flat rates of 9 1/11 percent for annuities on two heads and 11 13/17 percent for one, regardless of age."[7] Yet, the surviving document for this date only indicates life annuity issues were permitted to fund a loan to Duke Albert I at 10%. SGN (1898, p. 208) infers these life annuity rates from a 1464 document authorizing named Amsterdam city officials to refund outstanding loans using life annuities at "11 [years's purchase] on two heads and nine and one half on one head." The percent calculation for two heads is correct, but the 11 13/17 is an error in SGN, as 9.5 years' purchase does not convert to the stated percent. Complications associated with determining contract terms is further reflected in a 1472 last survivor annuity document from Leiden for Clais Jacobs, priest, and Lysbeth, daughter of Lambrecht, where an annuity of seven gold Rhine guilders is granted but the initial capital is not recorded. The annuity is granted "in the name of the city," with annuity payments to cumulate and be paid in full, when possible, in the event payments were suspended.

An early eighteenth-century source, Nicolas Bernoulli (1709, ch. 4, esp. pp. 26–27), documents considerable variation in the recommended relationship between the prices of joint and single life annuities. Included among these instances is a case where, on scholastic grounds, "just pricing" for single life and last survivor annuities required sale

[6] Among primary sources supporting this claim is the statement of de Witt near the end of *de Waerdye* (see de Witt 1671): "these annuities have been sold, even in the present century, first at 6 years' purchase, then at 7 and 8, and that the majority of all life annuities now current and at the country's expense were obtained at 9 years' purchase" (Hendricks 1852, p. 149).

[7] It is an oddity of historical research on late medieval and early modern annuities that the contractual method of "pricing" annuities—either payments per annum or years' purchase—is converted to "interest rates" calculated using a "current yield" percentage, i.e., the inverse of years' purchase—the price of the annuity divided by the annual annuity payment.

at the same price for those of advanced age. Referencing Pierre du Moulin (1568–1658), Bernoulli observes: "If the rent should be purchased on the lifetime of two in its entirety, a considerably more ample premium … must be constituted." Following a review of six opinions on the just price for one and two life annuities, du Moulin concludes that for individuals of comparable age and health, twelve years' purchase is recommended for an annuity on two lives and ten years' purchase for a single life. Without reference to the 1694 English government issue of £100 annuities on one life at £14/annum, two lives at £12/annum, and three lives at £10/annum (5&6 W&M), Bernoulli (1709, p. 27) recognizes the 1703 English government annuity issue (2&3 Anne c.3) that charged nine years' purchase for a single life, eleven years' purchase on two lives, twelve years' purchase on three lives, and fifteen years' purchase on a ninety-nine-year term annuity.

Closer inspection of the historical record reveals a diverse picture of contract variation and pricing across time and issuing location, providing some insight as to relative usage of joint as opposed to single life annuity contracts. The record reveals the claim that annuities were sold without reference to age is imprecise as some locales restricted sales by age, indicating usage of life annuities as a rudimentary old age "social safety net." For example, in some south German towns, Remi van Schaïk (2003, p. 112) observes, in 1457, Augsberg sold life annuities only to those forty years of age and older, and Nuremberg restricted sales to sixty and over; van Schaïk also finds similar prices for single life and last survivor annuities arising in Zutphen (1400 to 1600), although "this did not happen very often." Hans-Jörg Gilomen (2003, p. 148) provides a fascinating sample of thirty-one Swiss *rentes viagères* contracts from 1470–71 and 1479–80—the majority from Basel (Bâle)—with nineteen contracts featuring several joint life annuity variants with husband and wife or father and son as nominees. In numerous instances, both last survivor and single life annuities sold at ten years' purchase. Several last survivor contracts featured a lower payment when one of the nominees died. As some single life annuities were for widows, this provides some explanation for selection of single versus joint life contracts. Though ages are not given, with a few exceptions where last survivor annuities involved a father and child, the nominees appear to be older.

At some point, it is not clear when and where, the social safety net rationale for life annuity issues transitions from nominees and shareholders, usually older, being the same to include an investment vehicle motivation for shareholders using opportunistic selection of young, typically female, nominees with enhanced life expectancies. The rapid increase in Dutch debt following 1600—see, e.g., Oscar Gelderblom and Joost Jonker (2011, p. 11)—likely accelerated this transition, although the "Tableau of mortality" produced by Hudde from the Amsterdam register of life annuity nominees for the years from 1586 to 1590 indicate this transition was beginning earlier, (see, e.g., SGN 1898, pp. 80–81). Potential investment gains are reflected in Holland's selling single life and last survivor annuities at six and eight years' purchase in 1595 and eight and ten years' purchase in 1608 (Fritschy 2003, p. 64) without reference to age. By comparison, the initial 1672 Amsterdam single life annuity issue that took account of nominee age featured ten years' purchase for nominees between one and nineteen years, decreasing to a low of three years' purchase for nominees seventy-five years and older. According to Dirk Houtzager (1950), this pricing according to age was driven more by need to raise funds from older age groups, which were discouraged from

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purchasing life annuities sold without reference to age, rather than seeking actuarially sound pricing. The terms to annuitants were found to be so favorable to older age groups that a flood of applications ensued and a ban on sale to persons over fifty years of age became necessary.

The cost to government of the transition to young nominees in order to obtain enhanced investment returns was identified in a supplement attached to de Witt (1671): "one finds with wonderment, that in practice, when the purchaser of several life annuities comes to divide his capital which he intends to invest upon several young lives—upon ten, twenty, or more—this annuitant may be assured, without hazard or risk of the enjoyment of an equivalent, in more than sixteen times the rent which he purchases" (Hendricks 1853, p. 118). The practice of selecting young nominees was not isolated to the Dutch. Evidence of an overwhelming preponderance of young nominees appears in "A List of Names of the Several Nominees with Their Ages Subscribed" (Exchequer 1749) for the 1693 English Million Act that featured fourteen years' purchase for single life annuities—in combination with the possibility of also purchasing a tontine (e.g., Milevsky 2015, ch. 4). One of the earliest examples of an investment fund active from the early 1770s to the late 1780s—*Les trente demoiselles de Genève*—involved the sale of shares in the fund organized by Genevan banks investing in French government life annuities, usually with single life annuities but in one case at ten years' purchase with thirty nominees. These nominees were carefully selected: young females who had survived smallpox from well-to-do families in Geneva (Velde and Weir 1992; Spang 2015).

III. HUDDE, DE WITT, AND *DE WAERDYE*

One useful dictum obtainable from study of ancient history is the need to be aware of bias in interpretation arising from lack of sources. Because ancient sources are usually woefully inadequate to provide sufficient detail about historical events, historians must create a narrative for a particular event at issue based on sources "so incomplete, often so one-sided, often so naively disconnected with fundamental movements" (Frank 1910, p. 99). A tendency to develop interpretations biased toward information in meager available sources is difficult to avoid. In contrast, historiographies of the medieval to early modern periods feature increased availability of sources: books and pamphlets, government reports, journal articles, private papers, letters, and the like. However, sorting details of a specific event is, again, guided by trolling of available sources. For example, the origins of life annuity contracts in northern Europe have been identified using contract records that have survived, though such contracts may have been sold earlier and in different locations, leaving no surviving records. Is it possible that the received intellectual history of joint life annuity valuation has been adversely impacted by the availability of sources?

Intellectual history is replete with examples where seminal contributions attributed to specific individuals are later found to be inappropriate based on more detailed examination of sources, e.g., Stigler's law of eponymy. Where definitive sources are unavailable, insights, if any, must be gleaned from context and inferences. Such is the case with Hudde, where what survives of numerous contributions to mathematics, engineering, physics, astronomy, and actuarial science are largely available from other sources, especially from appendices in books by Frans van Schooten, together with letters and correspondence of Christian Huygens, de Witt, and, to lesser extent, Gottfried Liebnitz and Baruch Spinoza. As observed in Frederick Hendricks (1853, p. 97): Hudde "seems not to have taken sufficient care in the preservation of his manuscripts." There are no direct primary sources for Hudde for the period following his 1672 appointment by Stadtholder Wilhelm III as one four burgomasters of Amsterdam. Prior to that date, there is published work on mathematics, mostly provided by van Schooten, over the period 1654 to 1663, as well as letters about mathematics between Huygens and Hudde in 1663. There are also letters to Huygens on comets and correspondence with Spinoza on telescopes in 1665. Efforts related to life annuities appear in 1670–71 correspondence involving Hudde, Huygens, and de Witt.[8]

Careful consideration of historical context for the correspondence between Hudde, de Witt, and Huygens raises substantive questions about relative contributions to the seminal, if not widely distributed, publication on the valuation of single life annuities, *Waerdye van Lyf-renten naer proportie van Los-renten* (July 1671).[9] Following Hendricks's (1852) rediscovery and publication of the English translation (*Value of Life Annuities in Proportion to Redeemable Annuities)*, this contribution has invariably been attributed to the Grand Pensionary (*raadpensionaris*), Jan de Witt, author of the government report.[10] Only passing mention, if any, is given by modern scholars to the contribution of Hudde. Despite the record of correspondence between de Witt to Hudde on the valuation of single and last survivor annuities being incomplete with only selected letters from de Witt to Hudde having survived, the available evidence does indicate this attribution is, at least partially, misplaced.[11] From 1652, the year prior to his election as the Grand Pensionary, until the year of his tragic death in 1672,

[8] The primary source for the correspondence from de Witt to Hudde is Fruin (1913), in Dutch. Hendricks (1853) provides English translations for the most significant letters. Correspondence between Huygens and Hudde is available from the repository for Huygens correspondence at https://ckcc.huygens.knaw.nl/epistolarium/ in Dutch (accessed November 25, 2024).

[9] The title given for the *Waerdye* is close to the usage in Hald (1990, p. 123). Hendricks (1852) incorrectly observes the actual title is *Waardye van Lyf-renten naer proportie van Losrent.* However, though physical copies of the original source are rare, digital versions are now available and the exact title on the manuscript is that being used. Similarly, several variants of the spelling for Jan de Witt—Jan de Wit, John de Witt, Johanne de Wit, and so on—are available. The usage of Hald (1990) has been followed.

[10] Mattmüller (2014, pp. 279, 285) discusses the connection of Jacob Bernoulli with van Schooten and Huygens; the context for the unsuccessful attempt by Bernoulli to obtain a copy of de Witt (1671) from Leibniz is also identified. See also Sylla (2005).

[11] In addition to August and October 1671 correspondence from de Witt to Hudde and from de Witt to van Beuningen and his brother-in-law Deutz, available in Fruin (1913), there is also an important August 18, 1671, letter from Hudde to Huygens. The most telling source for possible attribution is the text of the *Waerdye* circulated to members of the Estates General in late July 1671 signed J. de Wit. This is followed by a brief attachment: "I, the undersigned, declare, that having attentively read and examined, at the request of My Lord the Grand Pensionary of Holland and West Friesland, the above propositions, and the conclusions thence," signed J. Hudde. Significantly, the detailed biographical source on de Witt, Pontalis (1885, p. 191), refers to Hudde as the "coadjutor." Unfortunately, available correspondence between Hudde and de Witt is for dates after the *Waerdye* was published. That Hudde was involved in production of the *Waerdye* is apparent from the August 2, 1671, letter from de Witt to Hudde, but uncovering the extent of the contributions is a task for later archival research.

de Witt was actively engaged in various reform efforts aimed at addressing Dutch government borrowing costs (see, e.g., Gelderblom and Jonker 2011, pp. 15–16). The work on life annuities with Hudde was aimed at critiquing a government proposal to issue single and last survivor annuities at fourteen and seventeen years' purchase, continuing the practice of issuing such annuities without accurately accounting for nominee age.

While the timeline for Hudde and de Witt developing actuarially sound solutions for the value of last survivor annuities roughly parallels that for single lives, details do differ. Starting with the August 2, 1671, correspondence from de Witt to Hudde, it appears a solution for the last survivor annuity was actively discussed and attempts at appropriate calculations executed.[12] Unfortunately, "the table of life annuities calculated upon two lives, in the selected class of 96 lives all aged 6 years, and also the fuller demonstration of the provisional hypothesis" provided by Hudde to de Witt in correspondence have not survived.[13] As the last survivor solution was decidedly more complex than for a single life, after a number of false starts de Witt proposed an ingenious, if not fully satisfactory, solution method involving the following theoretical (nearly) uniform death rate life table (Hendricks 1853, p. 109):

> 8 young lives (that number being given in order to avoid here too great a complication) and who are found to have lived as follows—the first to have become defunct 7 full years from the well-established date at which … has been bought … a life annuity; the second life 15 years; the third 24 years; the fourth 33 years; the fifth 41 years; the sixth 50 years; the seventh 59 years; the eighth 68 years.

Using an equally weighted average of the term annuity values for each lifespan and an interest rate of 4%, de Witt determines a single life annuity value (for one florin annual payment) of 17.22 florins:

$$\frac{A_7 + A_{15} + A_{24} + A_{33} + A_{41} + A_{50} + A_{59} + A_{68}}{8} = 17.22$$

where:

$$A_N = \frac{1}{r} - \frac{1}{\left(r(1+r)^N\right)} = \frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right)$$

and r is the annual interest rate. This approach is then adapted to produce solutions for last survivor annuities with young nominees of equal ages for two, three, four, up to eight lives to be determined by using a weighted average with weights calculated using binomial coefficients.

In a "Memoir" to an October 27, 1671, letter to Hudde, de Witt provides the following table for the binomial coefficients in the weighted average:

[12] English translation for the correspondence is reported in Hendricks (1853) and French translation in SGN (1898). The Dutch primary source is available in Fruin (1913).

[13] See letter of October 27, 1671, from de Witt to Hudde (Hendricks 1853, p. 107). The correspondence also indicates Hudde visited de Witt in the Hague in mid-October 1671, where valuation of single and joint life annuities was discussed in detail.

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Years from Purchase to Death of Nominee

7	15	24	33	41	50	59	68
	1	2	3	4	5	6	7
		1	3	6	10	15	21
			1	4	10	20	35
				1	5	15	35
			Weights		1	6	21
						1	7
							1

To apply these coefficients, de Witt gives worked examples for two, three, and four lives. For the two life last survivor annuity value calculation, de Witt gives the weighted average as: "the value of a life annuity upon 2 lives … is rightly and precisely equal to the value of a life annuity upon one life of a class of 28 lives, of which one life has lived 15 complete years; 2 each 24 years; 3 each 33 years; 4 each 41 years; 5 each 50 years; 6 each 59 years; and 7 each 68 years."

The solution of 20.76 florins (for one florin annual payment) is determined by using a weighted average with 28 total chances:

$$P_{LS} = \frac{A_{15} + (2A_{24}) + (3A_{33}) + (4A_{41}) + (5A_{50}) + (6A_{59}) + (7A_{68})}{28} = 20.766$$

From the table, the three life case would have 56 total chances and values of 1 for 24 years, 3 for 33 years, 6 for 41 years, 10 for 50 years, 15 for 59 years, and 21 for 68 years (= 21.98; 56 chances). Solutions for a last survivor annuity on four (= 22.54; 70 chances), five (= 22.85; 56 chances), six (=23.04; 28 chances), and seven (= 23.17; 8 chances) lives follow appropriately with the value for eight lives being equal to the annuity certain value for sixty-eight years (23.26 florins). While not directly stated in the correspondence, based on the maximum eighty-year lifespan (ω) used in de Witt (1671), the underlying assumption is that the lives involved are initially all twelve years old. For simplicity, de Witt uses only eight lives in calculating the solutions; de Witt refers to Hudde using calculations with eighty lives to solve the last survivor annuity, possibly also using uniformly distributed death rates. This would make for more complicated calculations but would avoid the degenerate solution given by de Witt for more than eight lives. However, details for this contribution are lost and connection to the binomial coefficient model of de Witt cannot be accurately determined.[14]

[14] The following from the letter of August 2, 1671 (Hendricks 1853, pp. 101–102), is also of interest regarding the difference between single and last survivor annuities issues for state finances: "there is a general persuasion that the life annuity upon two lives, at 17 years' purchase, is much more advantageous than that upon one life at 14 years' purchase; and that it may even be, that the life annuity upon two lives, if sold at 18 years' purchase, would be even then preferred to that upon one life at 14 years' purchase; as this might produce a notable advantage to the republic, it is, in my opinion, of the highest importance to leave people in this persuasion; therefore I have not divulged it to anyone except yourself, that according to my calculation (since I remark that yours, upon two lives, is even lower still) the purchase upon one life at 14 years' purchase

How did de Witt arrive at the binomial coefficient table for estimating the last survivor annuity? While recognizing that a significant number of potential primary sources essential to definitively addressing this question have either not survived or, one hopes, not yet surfaced from the archives, substantial inferences can be drawn from five letters from de Witt to Hudde with August and October 1671 dates. That the solution to the joint life annuity price was problematic is apparent from the first letter, dated August 2: "I have perfectly understood the estimation of the value of life annuities upon one life … you will oblige me by informing me whether the computation of the value … of life annuities upon two lives, has also been made according to certain cases of several life annuities granted upon two lives."

The statement of the binomial coefficient table appears in a "Memoir" to the last letter dated October 27. A most telling statement by de Witt appears in the October 27 letter: "I will with pleasure respond to your wish upon the subject, and in that case would establish thereon an argument *a priori*, although I have found it *a posteriori*, like in almost all inventions" (Hendricks 1853, p. 108). The beginning of this letter reveals the source of empirical evidence and importance of the correspondence from Hudde to de Witt that has been lost: "Since the departure of my last letter of the 22nd, your two letters of the 21st and 22nd instant have respectively come to hand, with the table of life annuities calculated upon two lives, in the selected class of 96 lives all aged 6 years, and also the fuller demonstration of the provisional hypothesis; namely, that out of 80 young lives, about 1 dies." As such, the binomial coefficient table provided by de Witt represents the first substantive effort to provide an analytical approximation to the price of a joint life annuity.

IV. THE METHOD OF HALLEY

The narrative for early analytical approximations to joint life annuity values is part of much broader social and economic processes taking place in the seventeenth and eighteenth centuries associated with application of numerical calculation to political reasoning (see, e.g., Deringer 2013; Brewer 1989, pt. II, ch. 4). The appearance of compound interest tables in the late sixteenth century gradually facilitated the spread of present value calculations that enabled de Witt and Hudde to estimate life annuity prices. More generally, "calculated values" were increasingly used for assessing soundness of government fiscal affairs (Deringer 2018b). Yet, the practical problem of pricing joint life annuities issued to fund government activities challenged available computational abilities and provided impetus for producing accurate life contingency data. Both themes appear in the seminal contribution to demography by Halley (1693) that accompanied emergence of the Financial Revolution in English government funding following the Glorious Revolution of 1688 that featured transplanting of Dutch fiscal methods (e.g., Dickson 1967; Milevsky 2015).

is more advantageous than that upon two lives at 17 years' purchase, and that is why I leave you to consider whether you do not judge it to be *useful for the public good that this estimate should be absolutely hidden*, and people left in their ancient persuasion, for the advantage of the State finances; because I am convinced, on this subject, that they will not be put in the track by a calculation made in round numbers, and probably no one will make a precise calculation for them" (emphasis added).

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Though the correspondence between de Witt and Hudde featured uniformly distributed death rates, it is not difficult to see the distances between yearly nodes or coefficients in the weighted average could be adjusted to reflect a theoretical life table with non-uniform death rates, as used in de Witt (1671) to solve the single life annuity value as just over sixteen florins. However, being available only in correspondence not widely recognized until Hendricks (1853) in the middle of the nineteenth century, the ingenious method of solving the last survivor annuity using binomial coefficients faded into obscurity, leaving the more cumbersome and computationally intensive approach proposed by Halley to influence development of solutions for joint life annuity values. While recommending the usefulness of logarithms for "facilitating the Computation of the Value of two, three, or more Lives," Halley does not provide completely worked solutions for any joint life annuity values. What Halley does provide is an overview of the actuarially sound compound probability method for solving joint life annuities where the ages of the nominees differ, motivated by a novel geometric analysis for individual terms in the sum that determine the joint life annuity value for two and three lives. That Halley spent considerable time and effort on the solution to the joint life annuity value is apparent. Halley (1693) dedicates less than two pages to value single life annuities with about four and one-half pages to joint life annuities, concentrated almost exclusively on the valuation for two and three lives.

While the conceptual approach to joint life annuity valuation proposed by Halley does have the desirable feature of allowing for unequal nominee lives, Halley recognizes that solving for the years' purchase of a specific combination of ages is computationally demanding. Halley (1693, p. 604) verbally describes the brute-force method for solving the joint life annuity on two lives:

> for the number of Chances of each single Life, found in the [Breslaw life] Table, being multiplied together, become the chances of Two Lives. And after any certain Term of Years, the Product of the two remaining Sums is the Chances that both Persons are living. The Product of the two Differences, being the numbers of the Dead of both Ages, are the Chances that both the Persons are dead. And the two Products of the remaining Sums of one Age multiplied by those dead of the other, shew the Chances that there are that each Party survives the other: Whence is derived the Rule to estimate the Value of the Remainder of one Life after another. Now as the Product of Two Numbers in the Table for the Two Ages proposed, is to the difference between that Product and the Product of the two numbers of Persons deceased in any space of time, so is the value of a Sum of Money to be paid after so much time, to the value thereof under the Contingency of Mortality. And as the aforesaid Product of the two Numbers answering to the Ages proposed, to the Product of the Deceased of one Age multiplied by those remaining alive of the other; So the Value of a Sum of Money to be paid after any time proposed, to the value of the Chances that the one Party has that he survives the other whose number of Deceased you made use of, in the second Term of the Proposition.

This verbal explanation is followed by a numerical illustration of the relevant calculations using the Breslaw life table for one term in the sum after eight years have elapsed and the nominees are initially eighteen and thirty-five years of age. Recognizing the difficulty involved in understanding the various calculations, Halley (1693, fig. 7) then provides a geometric motivation for these calculations (see Figure 1):

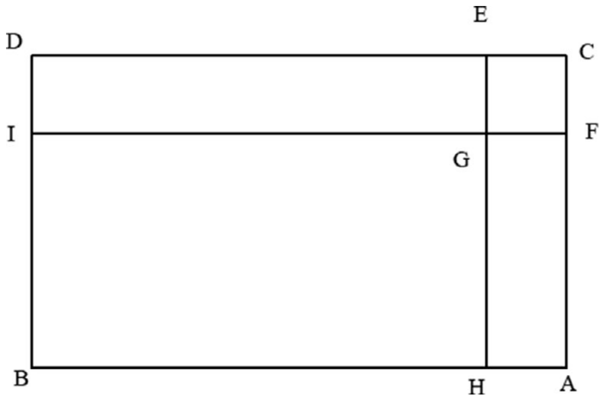

FIGURE 1. Halley's Method for a Joint Annuity on Two Lives

The whole area of *ABCD* $\left(=\mathrm{L_x L_y}\right)$ is the total number of chances for two lives at $t=0$ where $\mathrm{L_x}\left(\mathrm{L_y}\right)$ is the number of lives at starting age x (y).[15] The area of the inner rectangle *HGIB* $\left(=\mathrm{L_{x+m} L_{y+m}}\right)$ is the total number of chances both are alive at $t=m$. The two side rectangles *IDGE* $\left(=\mathrm{L_x}\left(\mathrm{L_y}-\mathrm{L_{y+m}}\right)\right)$ and *HAFG* $\left(=\mathrm{L_y}(\mathrm{L_x}-\mathrm{L_{x+m}})\right)$ are the chances one nominee is alive while the other is dead, leaving the upper rectangle *EGFC* to be the chances both nominees have died. Subtracting the ratio of *EGFC* to *ABCD* from one determines the weight—the survival rate—applicable to the discounted cash flow associated with $(1/(1+r))^m$.

Formally, because no allowance is made for the portion of joint annuity payments that would be made when there are no chances the eldest nominee will be alive, this geometric argument fully covers only valuation of the last survivor annuity where the two nominees have equal ages.[16] Significantly, Halley does provide an explicit recognition of "what value ought to be paid for the Reversion of one life after another." Stating Halley's method algebraically gives the following compound probability specification for $t = m$:

$$1-\left[\left(\mathrm{L_y}-\mathrm{L_{y+m}}\right)(\mathrm{L_x}-\mathrm{L_{x+m}})/\left(\mathrm{L_x L_y}\right)\right]) = \left(\mathrm{L_{y+m}}/\mathrm{L_y}\right)+(\mathrm{L_{x+m}}/\mathrm{L_x})-\left(\left(\mathrm{L_{y+m}L_{x+m}}\right)/\mathrm{L_x L_y}\right)$$

Halley did not observe that multiplying this result by $(1/(1+r))^m$ and summing from $t=1$ to the maximum possible age gives the value of the two nominee last survivor annuity being equal to the sum of the single life annuities for each nominee minus the value of the joint survivor annuity. As the method for solving the single life annuity value is available, the valuation of the last survivor annuity requires a solution to the joint survivor valuation problem, and vice versa.

From this, Halley (1693, fig. 8) extends the two-dimensional analysis for a joint life annuity on two lives to a three-dimensional analysis for three lives (see Figure 2).

[15] Following the discussion in Hald (1990, p. 137), L is used to denote the number living instead of the standard actuarial notation ℓ due to the vague, non-standard calculation method used by Halley that seems to average across adjacent ages.

[16] Hald (1990, p. 140) provides a helpful exposition of the geometric argument for two lives but does not recognize this point.

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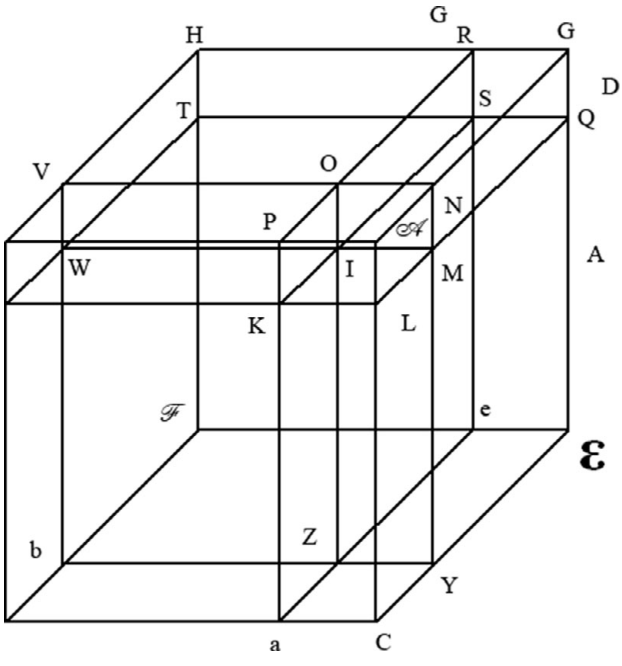

FIGURE 2. Halley's Method for a Joint Annuity on Three Lives

For Halley (1693, p. 606), the discussion for the two-dimensional case "is the Key to the Case of Three Lives." Extending the analysis to three dimensions, the fraction of total volume where all three nominees are dead is associated with the cube *KLMNOPI*. Unlike the two-dimensional case, Halley does not provide numerical calculations for a given payment. It is apparent the calculations involved are too numerous and tedious to warrant a complete resolution. Like the joint life annuity with two lives, Halley does not expand the geometrical discussion algebraically to three nominees with starting ages *x*, *y*, and *z* at *t*=*m*:

$$\left(L_{y+m}/L_y\right)+\left(L_{x+m}/L_x\right)+\left(L_{z+m}/L_z\right)-\left(\left(L_{y+m}L_{x+m}\right)/L_xL_y\right)-\left(\left(L_{y+m}L_{z+m}\right)/L_zL_y\right)-$$
$$\left(\left(L_{z+m}L_{x+m}\right)/L_xL_z\right)+\left(\left(L_{y+m}L_{x+m}L_{z+m}\right)/L_xL_zL_y\right)$$

Consequently, Halley did not discuss or observe the extension to more than three nominees. Having laid this rough foundation, Halley never returned to published efforts on this subject, leaving the next stage of development to Abraham de Moivre.

V. THE APPROXIMATIONS OF DE MOIVRE

The proximate reason given by Halley (1693, p. 654) for "not thinking of Methods for facilitating the Computation of the Value of two, three or more lives" is that the Breslaw life table was not based on "the Experience of a very great Number of Years." However, even though such a task would be "very worth while," it "seems (as I am inform'd) a Work of too much Difficulty for the ordinary Arithmetician to undertake." To this end,

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Halley "sought, if it were possible, to find a Theorem that might be more concise than the Rules there laid down, but in vain." It was this task de Moivre addressed, primarily by extending series methods used to approximate the single life annuity value assuming arithmetically declining survival rates (uniformly distributed death rates). However, while the easy-to-calculate approximation for a single life annuity works well for ages in the middle of the life table, approximations for joint life annuities were either decidedly more complicated to calculate or lacked precision, leading to considerable subsequent analysis by Simpson, de Moivre, and others seeking approximation methods and formulas for computationally demanding valuation from available life tables.

Precisely when and how Halley and de Moivre became acquainted is unclear. Drawing from the Matthew Maty (1755) biography (Bellhouse and Genest 2007), and the Ivo Schneider (1968) examination of letters and papers by de Moivre, Hald (1990, pp. 397–398) and Schneider (2004) find Halley and de Moivre becoming acquainted, possibly friends, in 1692, "likely introduced through the London Huguenot community, some of whose members were Halley's friends and neighbors in London" (Bellhouse and Genest 2007, p. 114). A credible inference is that de Moivre was a person that "inform'd" Halley of the "Difficulty" of calculating values for joint life annuities. It is interesting that de Moivre (1725, p. ii) recognizes:

> even admitting such a Table could be obtained as might be grounded on the Experience of a very great Number of Years, still the Method of applying it to the Valuation of several Lives would be extremely laborious, considering the vast Number of Operations that would be requisite to combine every Year of each Life with every Year of all the other Lives.

From this starting point, de Moivre (1725) proceeds in Problem I to provide an easy-to-calculate approximation to the value of a single life annuity exploiting the assumption of arithmetically declining survival rates demonstrating solutions for various ages from one to seventy with a maximum age at death (ω) of eighty-six years. Specific comparisons to values calculated by Halley (1693, p. 603) are provided; e.g., with $r = .06$ and $x = 30$, the approximation is 11.61 comparable with 11.72 computed by Halley (1693) from the Breslaw table. In stating the solution where $k = \omega - x$:

$$E[A_x]_L = \frac{1}{r}\left(1 - \frac{1+r}{k}A_k\right) = \frac{1}{r}\left(1 - \frac{1+r}{k}\left[\frac{1}{r} - \frac{1}{r(1+r)^k}\right]\right)$$

de Moivre refers to "a general *Theorem* in the *Doctrine of Chances*, pag. 132" (de Moivre 1718, p. 15) on a series solution "not so vulgarly known" but does not carry out a proof.

Following Halley, de Moivre typically focuses on survival rates, ignoring potential simplifications provided by death rates. Hald (1990, p. 521) provides details required to complete the complicated proof of Problem I, correctly observing that if uniform death rates are used instead of survival rates, then proving the approximation is less complicated. Observing $Prob[x,t]$, the probability of $x(y)$ surviving at time $t = n$ corresponds to the survival rate (ℓ_{x+n}/ℓ_x), the actuarially sound calculation of the single life annuity price $E[A_x]$ follows from multiplying by $(1+r)^{-t}$, and summing gives the required result, which holds for any life table. The relationship between pricing with survival rates and the death rates $((\ell_{x+n+1} - \ell_{x+n})/\ell_x)$ used by de Witt follows:

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$$E[A_x] = \frac{1}{\ell_x}\sum_{n=1}^{\omega-x-1} A_n d_{x+n} = \frac{1}{\ell_x}\sum_{n=1}^{\omega-x-1} d_{x+n}\sum_{t=1}^{n}\frac{1}{(1+r)^t}$$
$$= \frac{1}{\ell_x}\sum_{t=1}^{n}\sum_{n=1}^{\omega-x-1} d_{x+n}\frac{1}{(1+r)^t} = \frac{1}{\ell_x}\sum_{n=1}^{\omega-x-1} \ell_{x+n}\frac{1}{(1+r)^t}$$

where $\ell_{\mathrm{x+i}}$ is the number living i years past the starting age x, and $d_{\mathrm{x+i}}$ is the corresponding number dying for that year. Reconciling the arithmetically declining survival rate, the single life annuity approximation given by de Moivre with $E[A_x]_L$ calculated using uniform death rates ($k = \omega - x$) gives:

$$E[A_x]_L = \frac{1}{k}\sum_{n=1}^{\varpi-x-1}\left[\frac{1}{r} - \left(\frac{1}{r(1+r)^n}\right)\right] = \frac{1}{rk}\left(\sum_{n=1}^{\varpi-x-1}\left[1 - \left(\frac{1}{(1+r)^n}\right)\right]\right) = \frac{1}{r}\left[1 - \frac{1+r}{k}A_k\right]$$

where the last equality can be verified by induction or by evaluating the sum and manipulating.

Historical context is useful to interpret analytical development of approximations to single and joint life annuity values. Despite increasing availability of methods for actuarially sound pricing, French and British government issues of joint and single life annuities throughout the eighteenth century did not accurately account for age. While contributions such as *Essai sur les probabilités de la durée de la vie humaine* by Antoine Deparcieux (1746) and, to a lesser extent, *Calcul des Rentes viagères* by Nicolaas Struyck (1740) (see SGN 1912), substantively increased the quality of data on mortality, life annuity valuation was largely a concern for mathematicians until emergence of the life insurance industry in the later part of the eighteenth century. The numerous approximations relevant for single and joint life annuities in de Moivre (1725) were benchmarked to the rudimentary life table provided by Halley. Often motivated by de Moivre's desire to achieve mathematical solutions, some of his approximations were more successful than others, and Simpson expended considerable effort showing that direct calculation making use of actual life tables was substantially better for pricing the joint life annuity (Hald 1990, p. 532).

The key step in solving for a joint life annuity on two lives involves using the compound probability given by Halley to solve:

$$E\left[A_{xy}\right] = E[A_x] + E\left[A_y\right] - E\left[{}_xA_y\right]$$

where $E\left[A_{xy}\right]$ and $E\left[{}_xA_y\right]$ are prices of two nominee last survivor and joint survivor annuities, respectively. This exact result, applicable to any life table, follows from applying the compound probability rule given by Halley:

$$[1 - (1 - Prob[x,n])(1 - Prob[y,n])] = Prob[x,n] + Prob[y,n] - (Prob[x,n]Prob[y,n])$$

to calculate the price of the last survivor annuity $\left(E\left[A_{xy}\right]\right)$ on two lives at time $t = 0$. In contrast to approximation for single life annuity values, de Moivre struggled to achieve comparable results for joint life annuities. Taking pains to distinguish between "real lives" calculated from a life table and "fictitious lives" from the approximation, de Moivre (1725) used two approaches to solve for approximations to $E\left[{}_xA_y\right]$. One approach takes $Prob[i,j]$—"the Decrements of Life"—to be geometrically declining; the other approach uses arithmetically declining survival rates.

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More precisely, if $E[A_x]_G$ and $E\left[A_y\right]_G$ are the values of single life annuities calculated assuming geometrically declining survival rates, and $E\left[{}_xA_y\right]_G$ is the value of the "joint continuance" using geometrically declining "life probabilities," then de Moivre (1725, Problem IV) is able to show:

$$E\left[{}_xA_y\right]_G = \frac{E[A_x]_G E\left[A_y\right]_G (1+r)}{(E[A_x]_G + 1)\left(E\left[A_y\right]_G + 1\right) - \left(E[A_x]_G E\left[A_y\right]_G (1+r)\right)}$$

This result follows from the geometrically declining probability single life solution for $E[A_x]_G$ (given in Problem III). This solution is obtained by observing that for $x < 1$, then the geometric series $x + x^2 + x^3 + \ldots = (x/(1-x))$. Where $Prob[g_x]$ is the rate of geometric decline, then the solution for the $E[A_x]_G$ follows:

$$E[A_x]_G = \sum_{t=1}^{\infty} \left(\frac{Prob[g_x]}{1+r}\right)^t = \frac{\frac{Prob[g_x]}{1+r}}{1 - \frac{Prob[g_x]}{1+r}} = \frac{\frac{Prob[g_x]}{1+r}}{\frac{1+r-Prob[g_x]}{1+r}} = \frac{Prob[g_x]}{1+r-Prob[g_x]}$$

Solving:

$$Prob[g_x] = \frac{(1+r)E[A_x]_G}{E[A_x]_G + 1} \qquad Prob\left[g_y\right] = \frac{(1+r)E\left[A_y\right]_G}{E\left[A_y\right]_G + 1}$$

Using the compound probability, the approximation for a two nominee joint survivor annuity value assuming geometrically declining survival rates is:

$$E[{}_xA_y]_G = \frac{Prob[g_x]\, Prob[g_y]}{1 + r - (Prob[g_x]\, Prob[g_y])}$$

Substituting for $Prob[g_x]$ and $Prob\left[g_y\right]$ and cancelling give the desired result for $E\left[{}_xA_y\right]_G$. While this approach leads to a relatively easy-to-calculate approximation for $E\left[A_{xy}\right]_G$, casual inspection reveals that "real" survival rates do not decline geometrically. It follows that comparison with the more complicated solution using arithmetically declining survival rates $\left(E\left[A_{xy}\right]_L\right)$ is needed.

The approach in de Moivre (1725) to $E\left[A_{xy}\right]_L$—the last survivor annuity "whose Decrements are in *Arithmetic Progression*"— is unusual. The solution is not developed in the same fashion as other results, only stated without a derivation for "two lives" as a "Remark" added to Problem VII for the last survivor annuity for three lives, accompanied by some calculations to compare the arithmetic and geometric approximations. The solution requires the maximum age $\omega\,(=86)$ with starting ages x and y where $k = \omega - x$ and $j = \omega - y$. Assuming $x > y$ the solution provided is:[17]

$$E\left[A_{xy}\right]_L = \frac{1}{r} - \frac{1+r}{rk}\left(A_k + A_j\right) + \frac{1+r}{r^2 jk}\left(2j - (2+r)A_j\right)$$

A numerical example for two lives aged forty and fifty is provided and, with r = 5%, the values of $E\left[A_{xy}\right]_L$ for arithmetically and $E\left[A_{xy}\right]_G$ for geometrically declining cases are solved as 14.53 and 14.55 years' purchase, respectively. No further comparisons are

[17] The formula given in (3) of Hald (1990, p. 529) corrects an error in the Remark in de Moivre (1725 and 1731 editions, p. 47).

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provided, though Simpson (1742, pp. 36, 61) did later show the linear approximation has a better fit to values for $E[A_{xx}]$ using a "life table" determined from "ten Years Observations on the Bill of Mortality of the City of London." No indication is given in de Moivre (1725) about a solution for three lives. The $E\left[A_{xy}\right]_{\mathrm{L}}$ formula was dropped from later editions (de Moivre 1743, 1750, 1752).

Driven by the desire to provide simple-to-calculate joint life annuity formulas, de Moivre (1725) provides an extension to three lives for the geometrically declining case. The connection between joint survivor and last survivor annuities follows appropriately:

$$E\left[A_{xyz}\right] = E[A_x] + E\left[A_y\right] + E[A_z] - E\left[{}_xA_y\right] - E\left[{}_yA_z\right] - E[{}_xA_z] + E\left[{}_{xy}A_z\right]$$

where $E\left[A_{xyz}\right]$ is the value of the last survivor annuity for three nominees and $E\left[{}_{xy}A_z\right]$ is the value of the joint survivor annuity with three nominees. As for the two nominee case, this result is general and applies to any life table. Extending the solution for the two nominee case, de Moivre (1725, Problem VI) gives the geometrically declining survival rate result:

$$E\left[{}_{xy}A\right]_G = \frac{\left(E[A_x]_G + E\left[A_y\right]_G + E[A_z]_G\right)(1+r)^2}{(E[A_x]_G + 1)\left(E\left[A_y\right]_G + 1\right)(E[A_z]_G + 1) - \left(\left(E[A_x]_G + E\left[A_y\right]_G + E[A_z]_G\right)(1+r)^2\right)}$$

While no results or calculations are given for the three life joint survivor annuity using arithmetically declining life probabilities, de Moivre does indicate, with demonstration, the solution methodology for "finding the Values of as many *joint* Lives as may be assigned" (de Moivre 1725, pp. 44–45, Corollary I). The simplification provided by assuming the nominees' lives are equal is also recognized. Life tables from Halley, Willem Kersseboom, Antoine de Parcieux, and John Smart are included only in the Appendix to de Moivre (1756b).

VI. SIMPSON'S IMPROVEMENTS

Detailed examination of de Moivre's contributions to joint life annuity valuation would be incomplete without considering Simpson (1742, 2nd ed. 1775). As evidenced in the preface of de Moivre (1725; with corrections 1743), the contents of Simpson (1742) incensed de Moivre. Without identifying Simpson by name, de Moivre makes the following observations clearly directed at Simpson: "he mutilates my Propositions, obscures what is clear, makes a Shew of new Rules, and works by mine, in short confounds in his usual way, every thing with a croud of useless Symbols" (p. xii). This attack by de Moivre compelled Simpson to include in future printings an additional "Appendix containing some remarks on Mr. Demoivre's book … with answers to some personal and malignant misrepresentations, in the preface thereof." In this appendix, Simpson claims:

> It is not my design to expatiate on the unseemliness of this gentleman's usage, not to gratify a passion, which insinuations so gross must naturally excite in a mind that looks with contempt on such unfair proceedings; but only to offer a few particulars to the

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> consideration of the public, with no other view than to clear myself from a charge so highly injurious, and do justice to the foregoing work. (1742; 2nd ed. 1775, p. 129)

Simpson makes a strong case for the credibility of his contributions that is difficult to deny. In addition, it is likely that results and comments in the appendix led de Moivre to make some substantive changes in the 1743 corrected edition of 1725 and later versions.

At the time Simpson (1742) appeared, Thomas Simpson was still in the early stages of an academic career that was to produce a number of seminal advances, primarily in mathematics, earning the eponym Simpson's Rule for a method of approximating an integral using a sequence of quadratic polynomials.[18] His appointment as the head of mathematics at the Royal Military Academy at Woolwich—a position that provided Simpson with sufficient security to pursue academic interests—was not to occur until 1743. Being the son of a weaver with little formal education, Simpson had for some years made a living as an itinerant lecturer teaching in the London coffee houses. Around this time, certain coffee houses functioned as "penny universities" that provided cheap education, charging an entrance fee of a penny or two to customers who drank coffee and listened to lectures on topics specific to that coffee house (see, e.g., Poitras 2000, pp. 293–297). Popular topics were art, business, law, and mathematics. It is well known that de Moivre was a fixture at Slaughter's Coffee House in St. Martin's Lane during this period.

In support of activities at the penny universities and continuing at Woolwich, Simpson started producing a successful string of textbooks, beginning with Simpson (1737), a text on the theory of fluxions (Blanco 2014). The next book, Simpson (1740), bears a strong similarity to de Moivre (1718; see also 2nd ed. 1738, and 3rd ed. 1756a). In both Simpson (1740) and Simpson (1742), grateful references to de Moivre appear in the preface but no references in the body of the text despite obvious similarities in format and some content. The incensed reaction to Simpson in de Moivre (1725, with corrections 1743) is sometimes viewed as a response to Simpson's perceived plagiarism, though this is a generous interpretation, as the practice of borrowing without attribution was common. A precise explanation requires information that is unavailable. The seemingly damning evidence that Simpson was also accused of plagiarism by some others can, initially, be attributed to his desire to build a reputation required to sustain his livelihood and, over time, to a lack of sympathy for formal academic acknowledgment gradually gaining foothold in the eighteenth century. Even after appointment at Woolwich and election to the Royal Society (in 1745), Simpson continued to be accused of plagiarism, though not by persons with the academic stature of de Moivre.[19]

[18] It is likely Simpson's Rule was not originated by Simpson but, rather, by Newton, a point acknowledged by Simpson. As indicated by Stigler's law of eponomy, such failures of attribution are common. For example, Ypma (1995) demonstrates that the Newton-Raphson technique for solving non-linear equations is most appropriately credited to Simpson.

[19] Sources on Simpson's life are limited, the most detailed being Clarke (1929). Blanco (2014) is also useful in detailing Simpson's mathematical contributions on fluxions. Pearson (1978, p. 166) has a discussion of other sources sympathetic to Simpson that rely on Charles Hutton's 1792 introduction to the second edition of Simpson's *Select Exercises* (1752). In flourishing style typical of Karl Pearson—"to me he is a distinctly unpleasant and truculent writer of cheap textbooks, not a great mathematician like De Moivre"—Pearson (1978, pp. 176–185) takes great pains to document this view.

In contrast to the seminal theoretical work on joint (and single) life annuities accomplished by de Moivre, Simpson's contributions on joint life annuity valuation were decidedly more applied. Hald (1990, p. 511) identifies three important applied contributions: "(1) a life table based on the London bills of mortality; (2) tables of values of single- and joint-life annuities for nominees of the same age based on this life table; and (3) rules for calculating joint-life annuities for different ages from the tabulated joint-life annuities." This assessment would appear to be based largely on content of Simpson (1742) and ignores the practical usefulness of Part VI in Simpson (1752), "The Valuation of Annuities for single and joint Lives, with a Set of new Tables, far more extensive than any extant." Judging from references to Simpson, de Moivre, and Halley in the seminal Richard Price (1771), it was the practical content of Simpson's numerous worked problems and examples that had the most relative impact. While recognizing the importance of de Moivre's arithmetically declining survival rate approximations, Price (1771) dedicates Essay II to the unacceptable errors that assuming "geometrically declining life probabilities" has on the calculated values for joint life annuities on two and three lives.

Even casual reading of Simpson (1742) reveals the close connection to de Moivre (1725). The presentations are similar and various results are more general versions of the same problems. For example, corollaries III and V of Problem I give the solution for geometrically declining and arithmetically declining survival rates, respectively. Corollary III covers five or more joint lives and Corollary V applies up to three joint lives, extending results in de Moivre (1725). This said, Simpson does not mimic de Moivre's analytical approximation agenda. Results are adapted to practical ends and derivations are often either sketchy or hard to follow. To see the implications of this, consider the approximation $E\left[{}_{x}A_{y}\right]_{S}$ for the joint survivor annuity on two unequal lives given in terms of two equal lives stated by Simpson (1742, p. 50) without proof:

$$E\left[{}_{x}A_{y}\right]_{S} = E[{}_{x}A_{x}] + \frac{2E[{}_{x}A_{x}]\left(E\left[{}_{y}A_{y} - E[{}_{x}A_{x}]\right]\right)}{E\left[{}_{y}A_{y}\right]} \qquad x < y$$

This "fictitious" solution, which does not appear in de Moivre (1725), reduces the unmanageable practical problem of preparing tables for the value of joint life annuities on two unequal lives to be solved using a manageable table for joint annuity values on two equal "real" lives, tables that Simpson provides. Significantly, contributions of both de Moivre and Simpson to joint life annuity valuation had a profound impact on the seminal contributions to life insurance and pensions plans by Richard Price and, to a lesser extent, James Dodson.

VII. CONCLUSION

Against this backdrop, the joint life annuity narrative captures a fascinating level of analytical creativity and sophistication needed to achieve valuation solutions prior to the availability of sufficiently detailed demographic information used to calculate tables that appear in the early nineteenth century providing the computationally demanding discounted expected values for joint life annuities with unequal ages. The resulting narrative reveals the use of *a posteriori* reasoning by Jan de Witt to arrive at a binomial coefficient table that could be used to solve the last survivor annuity valuation problem for nominees with equal ages. Subsequent progress proceeded without the binomial coefficient approach that was seemingly known only to de Witt and Hudde, instead pursuing series approximations to the computationally intensive last survivor and joint survivor annuity problems. Despite not calculating a joint life annuity price, the geometric method used by Halley provided a compound probability rule connecting the values of joint and single life annuities that de Moivre and Simpson exploited to develop series approximations. Included in these methods is a solution provided by Simpson for the joint survivor annuity with unequal aged nominees in terms of annuity values using equal age nominees that dramatically simplifies tables needed to solve the annuity value where the nominees have arbitrary starting ages.

What relevance do early contributions to pricing joint life annuities have for the history of economic thought? Answering this question requires addressing the complicated problem of "defining economics" in the face of "expanding boundaries" (see, e.g., Backhouse and Medema 2009). In contrast to the early history of "new" economic subjects such as econometrics where narratives typically begin early in the twentieth century, substantive contributions to the early history of financial economics predate appearance of the timeline for the history of economic thought canon that privileges the school of political economists headed by Adam Smith. Substantive contributions involving individuals— Jan de Witt, Jan Hudde, Edmond Halley, Abraham de Moivre, and Thomas Simpson—using discounted expected value and series solutions to arrive at estimates for joint life annuity values are largely disconnected from the traditional political economy canon. To offset the increasing irrelevance of the traditional canon for contemporary economists populating the "expanding boundaries," perhaps a "histories of economics" strategy is needed to revive prospects for the early history of economic thought, as suggested by Geoffrey Poitras and Franck Jovanovic (2010) and Judy Klein (1997)?

COMPETING INTERESTS

The author declares no competing interests exist.

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