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An extension of the Krull–Schmidt theorem

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In the usual Krull-Schmidt-Azumaya theorem in abelian categories it is essential that each of the direct summands has a local ring of endomorphisms. A partial answer is given here to the case where this last condition is not satisfied by the indecomposable direct summands. It is found that those summands with local endomorphism rings are determined up to isomorphism and cardinality of occurrence.

The treatment here is for the category $\mathcal{M}$ of left R-modules M over the ring R. The embedding theorems for abelian categories then extend the result. However a direct treatment can be done for abelian categories following Gabriel [2]. $\mathcal{M}(M, M')$ denotes the R-module homomorphisms from M to M' and $\mathcal{M}(M) = \mathcal{M}(M, M)$. The composite $\xrightarrow{a}\xrightarrow{b}$ is written ab.

Write $M \leq M'$ if M is a submodule of M', M ds M' if M is a direct summand of M' and $M|M'$ if M ds M' and $\mathcal{M}(M)$ is a local ring. We have the following elementary facts.

(1.1) Suppose $M_1 \leq M_2 \leq M_3$. Then M_1 ds M_3 implies M_1 ds M_2. Also M_1 ds M_2 and M_2 ds M_3 imply M_1 ds M_3.

(1.2) Suppose $M_1 \leq M$ and e is an idempotent in $\mathcal{M}(M)$. Then $M = Me \oplus M(1-e) = M_1 \oplus M(1-e)$ if and only if e induces an isomorphism $M_1 \to Me$.

(1.3) Suppose the composite $M_1 \xrightarrow{a} M_2 \xrightarrow{b} M_1$ is an automorphism of M_1. Then $M_1 a$ ds M_2.

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LEMMA 2. *Let* $M = M_1 \oplus M_2$ *with* $M_1|M$ *and let* e *be the projection endomorphism of* M *onto* M_1 . *Take* a *in* $M(M)$. *Then at least one of* a *or* $1 - a$ *acts isomorphically on* M_1 *and such an isomorphic image* N *satisfies* $M = N \oplus M_2$. *The isomorphism followed by* e *induces an automorphism of* M_1 .

Proof. We have that $e = ae + (1 - a)e$. As $M(M_1)$ is local and e acts identically on M_1 , one of ae , $(1 - a)e$ induces an automorphism on M_1 . The result now follows by (1.3) and (1.2).

COROLLARY 3 . (Azumaya's Lemma) *Let* $M = M_1 \oplus \ldots \oplus M_n \oplus M'$ *with* $M_i|M$ *and let* a *belong to* $M(M)$. *Then there exist* $N_1 , \ldots , N_n$ *such that* $M = N_1 \oplus \ldots \oplus N_n \oplus M'$ *and such that each* M_i *is carried isomorphically onto* N_i *by either* a *or* $1 - a$.

Proof. Suppose by induction we have that

$$M = N_1 \oplus .. \oplus N_i \oplus M_{i+1} \oplus .. \oplus M_n \oplus M' .$$

By Lemma 2, a or $1 - a$ carries M_{i+1} isomorphically onto its image N_{i+1} and we may replace M_{i+1} by N_{i+1} . Thus the induction proceeds.

LEMMA 4. *Let* $M = \oplus_F M_f$ *with projection endomorphisms* p_f *and let* e *in* $M(M)$ *be a projection endomorphism onto* N *with* $N|M$. *If* $K = (f|p_f e$ *is an automorphism on* $N)$, *then* K *is a non-void finite set and if* f *lies in* K , *then* $Np_f|M_f$.

Proof. Take $N \neq 0$ in N . Then $n = \sum_H n_f$ with $n_f \neq 0$ in M_f . Thus H is finite. If f lies in $F - H$, then $np_f = 0$ and so f lies in $F - K$. Thus K is contained in H and so is finite. If $a = \sum_H p_f$, then $n(1 - a) = 0$ and so $1 - a$ is not an isomorphism on N . By Lemma 2, a is an isomorphism on N and $ae = \sum_H (p_f e)$ is an antomorphism on N . As H is finite and $M(N)$ is local, at least one $p_f e$ $(f$ in $H)$ is an automorphism and so K is non-void. The last statement follows from (1.3).

$M = M_F \oplus M^F$ is called a KSD (Krull-Schmidt decomposition) of M if $M_F = \oplus_F M_f$ with $M_f | M$ and $N \nmid M$ for any $N \leqq M^F$.

LEMMA 5. *Suppose* $M = M_F \oplus M^F = N_G \oplus N^G$ *are two* KSD*'s of* M *and let* $G_t = (g_1, \ldots, g_t)$ *be a finite subset of* G. *Then there exists* $F_t = (f_1, \ldots, f_t)$ *contained in* F *such that*

$$M = (\oplus_{F_t} M_f) \oplus (\oplus_{G-G_t} N_g) \oplus N^G$$

and $M_{f_i} \approx N_{g_i}$, $i = 1, \ldots, t$.

Proof. By induction, if $i < t$, we suppose that

$$M = (\oplus_{F_i} M_f) \oplus (\oplus_{G-G_i} N_g) \oplus N^G . \tag{6}$$

In the decomposition (6), let e be the projection endomorphism on $N_{g_{i+1}}$. Apply Lemma 4 to the decomposition $M = (\oplus_F M_f) \oplus M^F$. The set K of Lemma 4 cannot include the factor M^F, as M^F has no direct summands with local ring of endomorphisms. Also $p_{f_j} e = 0$ for $j = 1, \ldots, i$ and so K is contained in $F - F_i$. As K is non-void, there exists f_{i+1} in K and $N_{g_{i+1}} p_{f_{i+1}} \big| M_{f_{i+1}}$. But $M_{f_{i+1}}$ is indecomposable and so $N_{f_{i+1}} = M_{f_{i+1}}$. Also $p_{f_{i+1}} e$ is an automorphism on $N_{g_{i+1}}$ and hence e induces an isomorphism from $M_{f_{i+1}}$ to $N_{g_{i+1}}$. By (1.2), $M_{f_{i+1}}$ can now replace $N_{g_{i+1}}$ in the decomposition (6) and the induction proceeds.

THEOREM 7. *Suppose* $M = M_F \oplus M^F = N_G \oplus N^G$ *are two* KSD*'s of* M *and let* U *in* M *be such that* $M(U)$ *is a local ring. Let* m_U (n_U) *be the cardinal of summands* M_f (N_g) *with* $M_f \approx U$ $(N_g \approx U)$. *Then* $m_U = n_U$.

Proof. If n_U is finite, by Lemma 5 we have $m_U \geqq n_U$.

If n_U is infinite, by Lemma 5 so is m_U. Consider the set E of pairs (f, g) such that $q_g : M_f \stackrel{\approx}{\to} N_g$ and $N_g \approx U$. Let p_f, p' and q_g, q' be the canonical projection endomorphisms of the KSD's.

Apply Lemma 4 with $e = q_g$ and $N = N_g$. Thus there exists f in F such that $N_g p_f | M_f$ or $N_g p' | M^F$. The latter is excluded by the definition of M^F. As M_f is indecomposable, $N_g p_f = M_f$ and $q_g : M_f \stackrel{\approx}{\to} N_g$. Hence $|E| \geqq n_U$.

Apply Lemma 4 with $N = M_f$ and $e = p_f$. Thus there exists at most a finite set of g in G with $q_g : M_f \stackrel{\approx}{\to} N_g$. Hence $|E| = m_U \aleph_0$. But m_U is infinite and so $m_U \aleph_0 = m_U$ and so $m_u \geqq n_U$.

By symmetry $n_U = m_U$ and so $m_U = n_U$.

COROLLARY 8. $M_F \approx N_G$.

A module M doesn't necessarily have a KSD but a sufficient condition is that it can be written as the direct sum of indecomposable modules:

PROPOSITION 9. *Suppose* $M = \oplus_F M_f$ *is a decomposition into indecomposables with projection endomorphisms* p_f. *Put* $M_G = \oplus_G M_f$ *where* $G = (f \in F \mid M(M_f)$ *is local) and* $M^G = \oplus_{F-G} M_f$. *Then* $M = M_G \oplus M^G$ *is a* KSD.

Proof. Say $N | M^G$. By Lemma 4, there exists f in $F - G$ such that $Np_f M_f$. But M_f is indecomposable and so $Np_f = M_f$ and $M(M_f) \approx M(N)$ is local. This is a contradiction to the definition of M^G.

We can't in general be assured that the last factor M^F in a KSD is determined to within isomorphism. However this is true if F is finite.

PROPOSITION 10. *Suppose* $M = M_F \oplus M^F = N_G \oplus N^G$ *are* KSD'*s and* F *is finite. Then* $M^F \approx N^G$.

Proof. By Theorem 7, $|F| = |G|$. By Lemma 5, we have that

$M = N_G \oplus N^G = M_F \oplus N^G$. Thus $N^G \approx M/M_F \approx M^F$.

COROLLARY 11. *If a module* M *can be written as a finite direct sum of indecomposables in two fashions, then there exists an automorphism of* M *which carries each indecomposable of the first decomposition with local endomorphic ring isomorphically onto a summand of the second decomposition. and maps the sum of the remaining components of the first decomposition isomorphically onto the sum of the remaining components of the second.*

PROPOSITION 12. *Let* $M = M_F \oplus M^F = N_G \oplus N^G$ *be two* KSD'*s with projection endomorphisms* p_F , p^F , q_G , q^G . *Then* $M_F \cap N^G = N_G \cap M^F = (0)$ *and* q_G , q^G *carry* M_F , M^F *monomorphically into* N_G , N^G *respectively.*

Proof. In Lemma 2, let $a = q^G$. If q^G acts isomorphically on M_{f_0} , then

$$M = \left(\oplus_{F-(f_0)} M_f\right) \oplus M_{f_0} q^G \oplus M^F .$$

But $M_{f_0} q^G \leqq N^G$ and so by (1.1) $M_{f_0} q^G | N^G$ which is contrary to the definition of N^G . Hence q^G cannot act isomorphically on any M_f . Consider the action of q_G on a finite sum $M_{f_1} \oplus .. \oplus M_{f_n}$. As $1 - q_G = q^G$ cannot act isomorphically on any one M_{f_i} , by Corollary 3,

$$M = M_{f_1} q_G \oplus .. \oplus M_{f_n} q_G \oplus \left(\oplus_{F-(f_1,\ldots,f_n)} M_f\right) \oplus M^F ,$$

and q_G acts monomorphically on $M_{f_1} \oplus .. \oplus M_{f_n}$. Thus q_G acts monomorphically on M_F as any element of M_F is contained in $M_{f_1} \oplus .. \oplus M_{f_n}$ for some finite set $(f_1 , .. , f_n)$.

Say m lies in $M_F \cap N^G$. Then as m belongs to N^G , $mq_G = 0$.

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But q_G is monomorphic on M_F and so $m = 0$. Hence $M_F \cap N^G = (0)$. Similarly $N_G \cap M^F = (0)$. Say m in M^F is such that $mq^G = 0$. Now $\ker q^G$ is N_G and so m belongs to N_G. Thus m lies in $N_G \cap M^F = (0)$ and so q^G is monomorphic on M^F.

PROPOSITION 13. *Let* $M = M_F \oplus M^F = N_G \oplus N^G$ *be two* KSD's. *If* $M^F = (0)$, *then* $N^G = (0)$. *Thus Theorem 7 yields the usual Krull-Schmidt-Azumaya theorem* [1].

Proof. Suppose $M^F = (0)$ and so $M = M_F$. Thus from Proposition 12 we have

$$N^G = N^G \cap M = N^G \cap M_F = (0) .$$

This extension of the Krull-Schmidt-Azumaya theorem throws some light on other cases where a Krull-Schmidt statement pertains. For instance one can cite finitely generated abelian groups. The ring of endomorphisms of a cyclic group of prime power order is local. Given that a finitely generated abelian group can be written as the direct sum of cyclic subgroups, Corollary 11 now tells us that the periodic part has its invariants well determined, while the free part is determined to within isomorphism.

References

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