TRANSCENDENTAL NUMBERS[1]

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If K is a field and L an extension field, $\alpha \in L$ is said to be algebraic over K if it satisfies an equation

$$x^m + a_1 x^{m-1} + \cdots + a_m = 0$$

with coefficients in K. If α does not satisfy any such equation, it is called transcendental over K.

This definition is formal and algebraic, and the property of being transcendental is defined in a purely negative way.

Let now K, say, be the field Γ of rational numbers, and L the field P of real numbers. We have then the problem of deciding whether a given real number, e.g.

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n,$$

is algebraic or transcendental over Γ, or as we say for shortness, algebraic or transcendental.

The problem has at this point changed and become connected with properties outside formal algebra. For the elements of P are in general defined as limits of sequences of rational numbers. Thus the basic problem we shall have to discuss is whether there is a simple connection between the property of a given real number of being more or less well approximated by rational numbers of small denominators, and its property of being algebraic or transcendental, respectively.

The first result in this direction was obtained some 120 years ago by Liouville. He proved the following theorem and used it for the actual construction of the first transcendental numbers to become known.

THEOREM. *If for every $\omega > 0$ there are infinitely many pairs of integers p, q such that*

$$\left|\alpha - \frac{p}{q}\right| \leqq q^{-\omega}, \quad q > 0, \quad (p, q) = 1.$$

then α is transcendental.

[1] Invited talk given at the Annual Meeting of the Australian Mathematical Society held at the University of Adelaide in May, 1964.

In recent years, after fundamental work by Thue, Siegel and others, this theorem has been replaced by the following one which is nearly final; it is due to Roth.

THEOREM. *If $c > 0$, and if there are infinitely many pairs of integers p, q such that*

$$(1) \qquad \left|\alpha - \frac{p}{q}\right| \leqq q^{-2-c}, \quad q > 0, \quad (p, q) = 1,$$

then α is transcendental.

That this result cannot be much improved, can be seen as follows. First, to every real irrational number α (algebraic or transcendental) there are infinitely many pairs of integers p, q such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}, \quad q > 0, \quad (p, q) = 1;$$

thus in (1) the condition $c > 0$ cannot be omitted.

Unfortunately, while (1) gives a sufficient condition for α to be transcendental, it is by no means also necessary. Thus, in the case of e,

$$\left|e - \frac{p}{q}\right| < \frac{1}{q^2 (\log q)^2}, \quad q > 0, \quad (p, q) = 1$$

has only finitely many solutions in integers p, q; and the same is true for "almost all" transcendental numbers, in the sense of Lebesgue.

Thus (1) is not a very useful tool for deciding whether a given real number is transcendental, since it can be applied in "almost no" cases! Actually there are quite interesting examples of numbers the transcendency of which may be proved by means of this theorem, e.g.

$$\sum_{0}^{\infty} 2^{-2^n}, \quad \sum_{1}^{\infty} \frac{[n\sqrt{2}]}{2^n}, \quad 0.123456789101112\cdots.$$

Since the rational approximations p/q of a real number α, or equivalent to this, the linear expressions

$$|q\alpha - p|$$

do not always give enough information for deciding whether α is algebraic or transcendental, we must look for other and stronger methods. As it turns out, we need to consider general polynomials in α.

We use the following notation. If

$$a(x) = a_0 + a_1 x + \cdots + a_m x^m$$

is any polynomial, put

$$L(a) = \sum_{j=0}^{m} |a_j|.$$

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Then the following two results hold.

I. If α is algebraic of degree m, say $a(\alpha) = 0$ where $a(x)$ has integral coefficients and is irreducible and if $b(x) = b_0+b_1x+\cdots+b_nx^n$ is any polynomial with integral coefficuents, then either $b(\alpha) = 0$, or

$$|b(\alpha)| \geqq \{L(a)^n L(b)^{m-1}\}^{-1}.$$

II. If α is real, but not algebraic of degree $\leqq n$, then for every $t > 1$ there exists a polynomial $b(x) = b_0+b_1x+\cdots+b_nx^n$ with integral coefficients such that

$$0 < |b(\alpha)| \leqq \frac{4(n+1)^n}{t^n}, \quad 0 < L(b) \leqq t$$

and thence

$$0 < |b(\alpha)| \leqq \frac{4(n+1)^n}{L(b)^n}.$$

These two lemmas show at once that

α is transcendental if and only if, given any $\omega > 0$, there exists a degree n and infinitely many distinct polynomials $b(x)$ of degree $\leqq n$ with integral coefficients satisfying

$$0 < |b(\alpha)| < \frac{1}{L(b)^\omega}.$$

More generally,

α is transcendental if, given any $\omega > 0$ there exist in-finitely many distinct polynomials $b(x)$, of arbitrary degrees satisfying

$$L(b) \geqq n^2,$$

such that

$$0 < |b(\alpha)| < \frac{1}{L(b)^\omega}.$$

It is a difficult but interesting problem which numbers satisfy infinitely many relations

$$0 < |b(\alpha)| < \frac{1}{L(b)^\omega} \quad \text{where} \quad L(b) < n^2.$$

(The exponent 2 may be replaced by any number greater than 1).

The condition for transcendency just stated has many applications and in an implicit form is the basis of many proofs of transcendency. By way of example, one can show the following results.

Let $\varepsilon > 0$ be arbitrarily small. Then for every degree m there are infinitely many polynomials $b(x)$ of degree m with integral coefficients such that

$$0 < |b(e)| < L(b)^{-(m-\varepsilon)},$$

but only finitely many with

$$0 < |b(e)| < L(b)^{-(m+\varepsilon)}.$$

By a theorem of Feldman,

$$|b(\pi)| > L(b)^{-cm(\log(m+2))^2}$$

if

$$L(b) \geqq e^{m^2(\log(m+2))^4}.$$

Further, to take linear approximations,

$$\left|\pi - \frac{p}{q}\right| > q^{-42} \quad \text{if} \quad q \geqq 2.$$

Although the transcendency of important classes of numbers has been proved, there are still constants like

C (Euler's constant), e^e, $e+\pi$, $e\pi$ for which not even the irrationality has been established.

The most important classes of numbers for which the problem of transcendency has been settled, are as follows.

(I) Solutions of linear differential equations the coefficients of which are rational functions with algebraic numerical coefficients. Here the first general result was obtained by C. Siegel who proved that the Bessel function $J_0(z)$ is transcendental for algebraic $z \neq 0$. This work has in recent years been much extended by Shidlovski who brought the theory to a certain conclusion.

(II) Periodic functions. Here the most important results are due to Gelfond and Schneider. I mention only two results.

If $\alpha \neq 0$, 1 is algebraic, α^z is algebraic if z is algebraic of degree 1 (i.e. z is rational), but is transcendental if z is algebraic of higher degree.

Let $j(z)$ be the modular function which is defined for $I(z) = y > 0$, satisfies

$$j\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) = j(z)$$

for all integers α, β, γ, δ of determinant $\alpha\delta-\beta\gamma = 1$, and has the Fourier expansion

$$j(z) = \frac{1}{q} + \sum_0^\infty a_k q^k, \quad q = e^{2\pi i z}, \quad a_k \text{ integers.}$$

Then $j(z)$ is algebraic if z, with $I(z) > 0$, is algebraic of degree 2, but is transcendental if z is algebraic of higher degree.

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