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EXISTENCE OF NON-TRIVIAL DEFORMATIONS OF INSEPARABLE EXTENSION FIELDS III*

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Let K be an extension of a field k and p denotes the characteristic. In [4] and [5], we proved that if K is inseparable algebraic over k, then considered as an algebra over k, K is not rigid. In this note we shall prove the following

THEOREM. *Let K be a finitely generated inseparable extension of a field k of characteristic $p \neq 0$. Then considered as an algebra over k, K is not rigid.*

Throughout this note, we assume $p \neq 0$.

Let φ be a derivation of K over k and f_t the one-parameter family of deformations of K constructed from φ in [1]. Then f_t is expressible in the form

$$f_t(a, b) = ab + tF_1(a, b) + t^2F_2(a, b) + \cdots,$$

for $a, b \in K \otimes_k k((t))$, where F_i is a bilinear mapping defined over k and $F_1 = Sq_p\varphi$. Now we assume f_t is trivial, i.e., there exists a non-singular linear mapping Φ_t of $K \otimes_k k((t))$ onto itself of the form

$$\Phi_t = 1 + t\varphi_1 + t^2\varphi_2 + \cdots,$$

where φ_i is a linear mapping defined over k, such that $f_t(a, b) = \Phi_t^{-1}(\Phi_t a \cdot \Phi_t b)$.

LEMMA 1. *Let φ, f_t be as above and a be an element of ker φ. If f_t is trivial, then φ_r satisfies the following conditions;*

1) *If r is not divisible by p, then $\varphi_r(a^p) = 0$.*
2) *If r is not divisible by $p^m (m > 0)$, then $\varphi_r(a^{p^{m+1}}) = 0$.*
3) $\varphi_{p^m}(a^{p^{m+1}}) = 0$.

Proof. By [5, Lem. 2], $F_i(a, b) = 0$ for every $b \in K$ and $i \geq 1$.

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1) Since $\delta\varphi_1(a, a^n) = F_1(a, a^n) = 0$, $\varphi_1(a^n) = na^{n-1}\varphi_1(a)$. Hence $\varphi_1(a^p) = 0$. By [5, Lem. 3],

$$(\delta\varphi_r + \sum_{i=1}^{r-1} \varphi_i \cup \varphi_{r-i})(a, a^{n-1}) = 0,$$

i.e.,

$$\varphi_r(a^n) = a\varphi_r(a^{n-1}) + a^{n-1}\varphi_r(a) + \sum_{i=1}^{r-1} \varphi_i(a)\varphi_{r-i}(a^{n-1}).$$

Hence if we set $x_i = \varphi_i(a)$, $x'_i(n) = \varphi_i(a^n)$ and $y = a$, then, by [5, Cor. 1], $\varphi_r(a^p) = 0$, where r is not divisible by p.

2) and 3) We shall prove by induction on m.

i) The case $m = 1$. By [5, Lem. 3],

$$(\delta\varphi_r + \sum_{i=1}^{r-1} \varphi_i \cup \varphi_{r-i})(a^p, a^{(n-1)p}) = 0.$$

Set $x_1 = \varphi_i(a^p)$, $x'_i(n) = \varphi_i(a^{np})$ and $y = a^p$. By [5, Cor. 1], if r is not divisible by p, $x'_r(p) = \varphi_r(a^{p^2}) = 0$ and $\varphi_p(a^{p^2}) = x'_p(p) = x_1^p = \{\varphi_1(a^p)\}^p = 0$.

ii) The case $m > 1$. By [5, Lem. 3], $(\delta\varphi_r + \sum_{i=1}^{r-1} \varphi_i \cup \varphi_{r-i})(a^{p^m}, a^{(n-1)p^m}) = 0$. Set $x_i = \varphi_i(a^{p^m})$, $x'_i(n) = \varphi_i(a^{np^m})$ and $y = a^{p^m}$. By [5, Cor. 1], if r is not divisible by p, then $x'_r(p) = \varphi_r(a^{p^{m+1}}) = 0$, if $r = up^v\,(1 < u < p,\ 1 \leqq v < m)$, then $x'_r(p) = (x_{up^{v-1}})^p = \{\varphi_{up^{v-1}}(a^{p^m})\}^p = 0$ and if $r = p^m$, then $x_{p^m}(p) = (x_{p^{m-1}})^p = \{\varphi_{p^{m-1}}(a^{p^m})\}^p = 0$. This ends the proof.

LEMMA 2. *Let φ, f_t be as in Lemma 1. If f_t is trivial, then $\varphi_{p^m}(a^{p^{m+1}}b) = a^{p^{m+1}}\varphi_{p^m}(b)$ for $a \in \ker\varphi$ and $b \in K$.*

Proof. By Lemma 1, $\varphi_r(a^{p^{m+1}}) = 0$ for $r \leqq p^m$. Therefore, by [5, Lem. 4], $\delta\varphi_{p^m}(a^{p^{m+1}}, b) = 0$ and $\varphi_{p^m}(a^{p^{m+1}}b) = a^{p^{m+1}}\varphi_{p^m}(b)$. This ends the proof.

Let $K = k(x_1, \cdots, x_g)$ be a finitely generated inseparable extension of a field k. Then we may assume that there exist non-negative integers $e < f \leqq g$ such that K is separable algebraic over $L = k(x_1, \cdots, x_f)$, the set $\{x_1, \cdots, x_e\}$ is a transcendency base, $x_i(e+1 \leqq i \leqq f)$ is inseparable algebraic over $M = k(x_1, \cdots, x_e)$ and the set $\{x_1, \cdots, x_f\}$ is a p-base of K over k (see [3, Ch. IV, 7]).

If there exists $i(e+1 \leqq i \leqq f)$ such that x_i is algebraic over k, then K is not rigid ([5, Th. 1]).

So it is assumed that every $x_i(e+1 \leqq i \leqq f)$ is not algebraic over k and x_{e+1} is an element of exponent α.

Let $\lambda(X) = \sum_{i=0}^{\beta} a_i X^{ip^\alpha}$ be the minimum polynomial of $x_{e+1} = \theta$ over M. Let P_i, Q_i be the polynomials in $k[x_1, \cdots, x_e]$ such that $a_i = \frac{P_i}{Q_i}$ and they are relatively prime, and let b be the least common miltiple of Q_i. Then $b\lambda(X) = \sum_{i=0}^{\beta} b_i X^{ip^\alpha} \in k[x_1, \cdots, x_e, X]$.

First we assume every $b_i \in k[x_1^{p^\alpha}, \cdots, x_e^{p^\alpha}]$. Let φ be a derivation of K such that $\varphi(\theta) = 1$ and $\varphi(x_i) = 0$ for $1 \leqq i \leqq e$. Then $b_i = \sum_j c_{ij} b_i^{p^\alpha}$, where $b_{ij} \in ker\ \varphi$ and $c_{ij} \in k$. By Lemma 1, 2 and [5, Prop. 1],

$$\begin{aligned}\varphi_{p^\alpha-1}(b\lambda(\theta)) &= \sum_{i=0}^{\beta} \varphi_{p^\alpha-1}(b_i\theta^{ip^\alpha}) \\ &= \sum_{i=1}^{\beta} ib_i\, \theta^{(i-1)p^\alpha} = 0.\end{aligned}$$

Therefore θ is an inseparable element of exponent $> \alpha$ over k. Hence, in this case, K is not rigid.

Next we assume that there exists i such that $b_i \notin k[x_1^{p^\alpha}, \cdots, x_e^{p^\alpha}]$. Let $H(x_1, \cdots, x_e, X) = \sum_{i=0}^{\beta} c_i X^{ip^\alpha}$ be a poylnomial of X with coefficients c_i in $k[x_1, \cdots, x_e]$ satisfies the following conditions;

1) $H(x_1, \cdots, x_e, \theta) = 0$.

2) If $H'(x_1, \cdots, x_e, X) = \sum_{i=0}^{\beta} c_i' X^{ip}$ is a polynomial of X with coefficients c_i' in $k[x_1, \cdots, x_e]$ satisfying the condition 1), then $\sum_{i=0}^{\beta} deg\, c_i' \geqq \sum_{i=0}^{\beta} deg\, c_i$.

Set $c_i = \sum_h c_{hij} x_j^{\gamma(h,i,j)} + c_{ij}$, were $\gamma(h, i, j)$ is a positive integer and $c_{hij}, c_{ij} \in k[x_1, \cdots, \hat{x}_j, \cdots, x_e, X]$ (the symbol $\wedge$ over x_j means that x_j is omitted). Let v be the minimal integer of p-degree of $\gamma(h, i, j)$ such that $c_{hij} \neq 0$ and j_0 an integer such that $v = p$-degree of $\gamma(h, i, j_0)$ for some i and h. Then we may write $H^p = \sum_{i=0}^{n} d_i^p x_{j_0}^{ip^{v+1}}$, where $d_i \in k[x_1^{p^v}, \cdots, \hat{x}_{j_0}^{p^v}, \cdots, x_e^{p^v}, X^{p^v}]$, and there exists $d_i \neq 0$ such that i is not divisible by p. Let φ be a derivation of K such that $\varphi(x_{j_0}) = 1$ and $\varphi(x_i) = 0$ for $i \neq j_0$ and $1 \leqq i \leqq e+1$. By Lemma 1, 2 and [5, Prop. 1],

$$\varphi_{p^v}(H(x_1, \cdots, x_e, \theta)^p)$$
$$= \sum_{i=1}^{n} i\, d_i(x_1^{p^v}, \cdots, \hat{x}_{j_0}^{p^v}, \cdots, x_e^{p^v}, \theta^{p^v})^p x_{j_0}^{(i-1)p^{v+1}}$$
$$= 0.$$

Therefore

$$\sum_{i=1}^{n} i\, d_i(x_1^{p^v}, \cdots, \hat{x}_{j_0}^{p^v}, \cdots, x_e^{p^v}, \theta^{p^v}) x_{j_0}^{(i-1)p^v} = 0.$$

Hence the polynomial $\sum_{i=0}^{n} i\, d_i\, x_{j_0}^{(i-1)p^v} = \sum_{i=0}^{\beta} c_i'' X^{ip^\alpha}$ satisfies the condition 1). On the other hand, it is trivial that $\sum_{i=0}^{\beta} deg\, c_i'' < \sum_{i=0}^{\beta} deg\, c_i$. This is contradiction. Hence Theorem has been proved.

By [1, p 29, Cor. 2] and Theorem, we have the following

COROLLARY. *Let K be a finitely generated extension field of a field k. Then K is separable over k if and only if, considered as an algebra over k, K is rigid.*

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