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GRAPHS WITHOUT CYCLES OF EVEN LENGTH

THOMAS LAM

Dedicated to George Szekeres on his ninetieth birthday

In this paper we prove that a bipartite graph with parts of sizes M and N, having no cycles of even length less than or equal to $2(2k+1)$, where k is a positive integer, has at most $(NM)^{(k+1)/(2k+1)} + D_k(N+M)$ edges, where D_k only depends on k.

In particular, we show that when $k=1$, $D_1 = 1$ is possible.

1. INTRODUCTION

Paul Erdös [3] first claimed in 1965 that for every k there is a c such that any graph on n vertices with $cn^{1+(1/k)}$ edges has a cycle of length $2k$.

In 1974, Bondy and Simonovits [2] proved more generally that any graph with $100kn^{1+(1/k)}$ edges has a C_{2l} for every integer $l \in [k, kn^{1/k}]$.

Constructions of graphs with no cycles of particular even lengths have been provided by Benson in [1] and Wenger in [5]. They provide constructions of bipartite graphs with parts of size N and N: $N^{3/2} + O(N)$ edges with no C_4; $N^{4/3} + O(N)$ edges with no C_4 or C_6 and $N^{6/5} + O(N)$ edges with no C_4, C_6 or C_{10}.

Our results prove that the constructions of Benson and Wenger are exact in the following sense: a bipartite graph with parts N and M with neither a C_4 nor a C_6 can have at most $(NM)^{2/3} + N + M$ edges. When $N = M$ the leading term has the same order and constant as the construction, while the lower order term is of the same order as well. When $M = 1$ or $N = 1$, clearly we cannot have any constant less than 1 for the lower order $(N+M)$ term.

We shall prove the theorem in this form:

THEOREM 1. *Let A be a matrix with dimensions $M \times N$ whose entries are either 1's or 0's. If A does not contain any cycles of length $2l$ where $l \leqslant 2k+1$ then the number of entries with a 1 in A is no more than*

$$(NM)^{(k+1)/(2k+1)} + C_k(N+M),$$

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for some constant C_k depending only on k.

A cycle of length k in A corresponds to a possibly non-convex polygon with k vertices corresponding to entries with a 1 such that the edges are either horizontal or vertical alternatingly.

A represents the incidence matrix of a bipartite graph where the rows are one part, the columns the other part and the entries with a 1 are the edges.

2. Proof of Theorem 1

We shall be using the following defintion:

Definition 1: Suppose A is a matrix of 1s and 0s as above. A *rook move of length* n is a vector of positions $(x_0, x_1, \dots, x_n) : x_i \in A$ such that:

1. Every position x_i contains a 1.
2. x_i and x_{i+1} belong in either the same column or the same row, and this alternates, depending only on the parity of i. If x_0 and x_1 belong in the same column then the rook move is said to begin vertically, otherwise it begins horizontally.

We say that x_n can be reached from x_0 by a rook move of length n. If $x_i = x_{i+1}$ for some $0 \leqslant i \leqslant n-1$ then the rook move is called degenerate. Otherwise, it is called non-degenerate.

We shall prove the theorem by induction in N and M where N is the number of columns and M the number of rows. The result is clearly true when either N or M is 1 with $C_k = 1$. Suppose then that there is some $D \geqslant 1$ such that $X(n,m) \leqslant (nm)^{(k+1)/(2k+1)} + D(n+m)$ is true $\forall (n,m) : n < N,\ m \leqslant M$ or $n \leqslant N,\ m < M$. Let A be an $N \times M$ matrix of 1s and 0s such that the number of entries with a 1 is $X > (NM)^{(k+1)/(2k+1)} + D(N+M)$.

From now on, an *entry* is a position in A containing a 1. Throughout the proof, C will represent any positive constant which only depends on k but not N or M.

Our first step is the following lemma:

Lemma 1. *No row contains less than $CN^{(k+1)/(2k+1)}M^{(-k)/(2k+1)} + 1$ entries. No column contains less than $CM^{(k+1)/(2k+1)}N^{(-k)/(2k+1)} + 1$ entries.*

Proof: Pick a row R with r entries. Then removing this row and applying the inductive hypothesis to the resulting matrix gives:

$$X - r \leqslant \big(N(M-1)\big)^{(k+1)/(2k+1)} + D(N+M-1)$$

$$(NM)^{(k+1)/(2k+1)} + D(N+M) - r < \big(N(M-1)\big)^{(k+1)/(2k+1)} + D(N+M-1).$$

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By expanding the right hand side using the Taylor series $f(x) = x^{(k+1)/(2k+1)}$ we get the weaker inequality:

$$-r < -\frac{k+1}{2k+1} N^{(k+1)/(2k+1)} M^{(-k)/(2k+1)} - D$$

from which our first desired result immediately follows as $D \geqslant 1$. The proof for columns is identical. ∎

LEMMA 2. *No row contains more than* $CN^{(k+1)/(2k+1)}M^{(-k)/(2k+1)}$ *entries. No column contains more than* $CM^{(k+1)/(2k+1)}N^{(-k)/(2k+1)}$ *entries.*

PROOF: Pick a row R with say r entries. We want to count the number of destination entries which can be reached from any entry of R by non-degenerate rook moves of length $2k$.

Pick any entry of R, x_0. The number of entries that can be reached from x_0 by a non-degenerate rook move of length 1, starting vertically, is at least $CM^{(k+1)/(2k+1)}N^{(-k)/(2k+1)}$, using Lemma 1. Repeating this, we see that the number of non-degenerate rook moves of length $2l$ starting at x_0 is at least

$$\left(C^2 M^{(k+1)/(2k+1)} N^{(-k)/(2k+1)} N^{(k+1)/(2k+1)} M^{(-k)/(2k+1)}\right)^l = CM^{l/(2k+1)} N^{l/(2k+1)}.$$

Now set $l = k$ and note that all these non-degenerate rook moves must end on a different column (and in particular, end on different entries). For if we have two rook moves $x_0, x_1, \dots x_{2k}$ and $x_0, y_1, \dots y_{2k}$ such that x_{2k} and y_{2k} are entries of the same column, then

$$x_1, \dots x_{2k}, y_{2k}, y_{2k-1}, \dots, y_1, x_1$$

will contain a cycle of even length at most $4k$ (x_1 and y_1 are both on the same column as x_0 so are on the same column as each other).

Now consider the set of all such non-degenerate rook moves as x_0 varies over all the entries of R. Again I claim no two such rook moves say $x_0, x_1, \dots x_{2k}$ and $y_0, y_1, \dots y_{2k}$ where $x_0 \neq y_0$ end on the same column. For otherwise,

$$x_0, x_1, \dots x_{2k}, y_{2k}, y_{2k-1}, \dots, y_1, y_0, x_0$$

will contain a cycle of length at most $2(2k+1)$.

Thus we get the inequality:

$$r \times CM^{k/(2k+1)} N^{k/(2k+1)} < N.$$

So:

$$r < CN^{(k+1)/(2k+1)} M^{(-k)/(2k+1)}.$$

Similarly, no column contains more than $CM^{(k+1)/(2k+1)}N^{(-k)/(2k+1)}$ entries. ∎

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At this point we require a combinatorial lemma from [4].

Lemma 3. *Let X and $A_1, \ldots, A_n$ be finite sets for some $n \geqslant 0$, and for each $1 \leqslant i \leqslant n$ let $f_i : X \to A_i$ be a function. Then:*

$$\#\{(x_0, \ldots, x_n) \in X^{n+1} : f_i(x_{i-1}) = f_i(x_i),\ 0 \leqslant i \leqslant n\} \geqslant \frac{(\#X)^{n+1}}{\prod_{i=1}^{n} \#A_i}.$$

Let X be the set of entries which have a 1, and let A_i be the set of rows for i odd and the set of columns when i is even. Apply Lemma 3 with $n = 2k$.

The left hand side of the equation corresponds to (possibly degenerate) rook moves of length $2k$ which start horizontally. Now we observe that no two non-degenerate rook moves of length $2k$ starting horizontally can start on the same column and end on the same row, for otherwise we would easily have a cycle of length $2(2k+1)$. We now use Lemma 2 to prove:

Lemma 4. *The number of degenerate rook moves of length $2k$ is no more than $C(N+M)(NM)^{k/(2k+1)}$.*

Proof: A degenerate rook move $(x_0, x_1, \ldots, x_{2k})$ must have some i such that $x_i = x_{i+1}$. Let the row with the maximum number of entries have C_r entries, and correspondingly C_c for the columns. Fixing i, the maximum number of such rook moves is no more than $XC_r^{k-1}C_c^k$ when i is even or $XC_r^kC_c^{k-1}$ when i is odd. Since $X \leqslant NC_c$ and $X \leqslant MC_r$, we get these are less than $N(C_cC_r)^k$ and $M(C_cC_r)^k$. Using Lemma 2 to give $C_r \leqslant CN^{(k+1)/(2k+1)}M^{(-k)/(2k+1)}$ and $C_c \leqslant CM^{(k+1)/(2k+1)}N^{(-k)/(2k+1)}$ and summing over $0 \leqslant i \leqslant 2k$ (this summation just contributes to the constant as it doesn't depend on N or M) , we see that the number of degenerate rook moves is no more than:

$$C(N+M)(NM)^{k/(2k+1)}. \qquad \square$$

Combining Lemma 3 and Lemma 4 we now have:

$$\frac{X^{(2k+1)}}{N^kM^k} - C(N+M)(NM)^{k/(2k+1)} \leqslant NM$$

or

$$X^{(2k+1)} \leqslant (NM)^{k+1} + C(N+M)(NM)^{(k(2k+2))/(2k+1)}.$$

By using the first two terms of the Taylor series for $f(x) = x^{1/(2k+1)}$, this implies that:

$$(1)\quad X \leqslant (NM)^{(k+1)/(2k+1)} + \frac{C}{2k+1}(N+M)(NM)^{k(2k+2)/(2k+1)}(NM)^{-(k+1)2k/(2k+1)}$$

Thus:

$$X \leqslant (NM)^{(k+1)/(2k+1)} + \frac{C}{2k+1}(N+M).$$

Since C does not depend on N or M nor on the size of D in the inductive hypothesis, we have proved that $X \leqslant (NM)^{(k+1)/(2k+1)} + \max\big(C/(2k+1), D\big)(N+M)$. Thus, by the principle of mathematical induction, for each k there is a constant D_k such that

$$X \leqslant (NM)^{(k+1)/(2k+1)} + D_k(N+M).$$

This proves Theorem 1.

3. The case $k = 1$

Theorem 2. *When $k = 1$, $D_1 = 1$ suffices in Theorem 1.*

We begin as before, supposing that the result $X(n,m) \leqslant (nm)^{2/3} + n + m$ is true $\forall (n,m) : n < N,\ m \leqslant M$ or $n \leqslant N,\ m < M$. Let A be an $N \times M$ matrix of 1s and 0s such that the number of entries with a 1 is $X > (NM)^{2/3} + N + M$.

Lemma 1 gives us an easier way to count the number of non-degenerate rook moves which start horizontally. For $k = 1$, no row contains less than $(2/3)N^{2/3}M^{-1/3} + 1$ 1's and no column has less than $(2/3)M^{2/3}N^{-1/3} + 1$ 1's. All rook moves of length 2 are of the form (x_0, x_1, x_2). Fixing x_1 we see that the proportion of non-degenerate rook moves to total rook moves is at least:

$$\frac{(2/3)N^{2/3}M^{-1/3}}{(2/3)N^{2/3}M^{-1/3} + 1} \times \frac{(2/3)M^{2/3}N^{-1/3}}{(2/3)M^{2/3}N^{-1/3} + 1}.$$

This ratio holds for all x_1 so is true for all rook moves as a whole. So applying the same logic as before we obtain:

$$\frac{X^3}{NM} \times \frac{(2/3)N^{2/3}M^{-1/3}}{(2/3)N^{2/3}M^{-1/3} + 1} \times \frac{(2/3)M^{2/3}N^{-1/3}}{(2/3)M^{2/3}N^{-1/3} + 1} \leqslant NM.$$

Rearranging, we get:

$$X^3 \leqslant (NM)^2 \Big(1 + \frac{3}{2}N^{-2/3}M^{1/3}\Big)\Big(1 + \frac{3}{2}M^{-2/3}N^{1/3}\Big).$$

Now we use the Taylor series for $f(x) = x^{1/3}$. The third term of the Taylor series is negative, so we can preserve our inequality by using only the first two:

$$X \leqslant (NM)^{2/3}\Big(1 + \frac{1}{2}\big(N^{-2/3}M^{1/3} + M^{-2/3}N^{1/3}\big) + \frac{3}{4}N^{-1/3}M^{-1/3}\Big).$$

Multiplying out, we get:

$$X \leqslant (NM)^{2/3} + \frac{1}{2}(N + M) + \frac{3}{4}N^{1/3}M^{1/3}.$$

This is as good as our desired bound of $(NM)^{2/3} + (N + M)$ when:

$$\frac{3}{2}N^{1/3}M^{1/3} \leqslant (N + M).$$

Let us suppose that $M = aN$. Substituting this, we get:

$$\frac{3}{2} \leqslant N^{1/3}\big(a^{2/3} + a^{-1/3}\big).$$

Now a is a positive real number and an easy calculation shows that $a^{2/3} + a^{-1/3} > 1.889$. So we have proved our bound when $N > 0.5$ which is always the case. This is a contradiction to the fact that $X > (NM)^{2/3} + (N + M)$ and thus no such matrix A exists.

Hence by the inductive hypothesis, Theorem 2 is true for all N and M.

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School of Mathematics
The University of New South Wales
Sydney NSW 2052
Australia