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ARTICLE

Pollution, public debt, and growth: the question of sustainability

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Abstract
This paper examines an endogenous growth model that allows us to consider the dynamics and sustainability of debt, pollution, and growth. Debt evolves according to the financing adaptation and mitigation efforts and to the damages caused by pollution. Three types of features are important for our analysis: the technology through the negative effect of pollution on TFP; the fiscal policy; the initial level of pollution and debt with respect to capital. Indeed, if the initial level of pollution is too high, the economy is relegated to an endogenous tipping zone where pollution perpetually increases relatively to capital. If the effect of pollution on TFP is too strong, the economy cannot converge to a stable and sustainable long-run balanced growth path. If the income tax rates are high enough, we can converge to a stable balanced growth path with low pollution and high debt relative to capital. This sustainable equilibrium can even be characterized by higher growth and welfare. This last result underlines the role that tax policy can play in reconciling debt and environmental sustainability.

Keywords: Environmental damage; pollution; fiscal policy; public debt; sustainability

JEL classifications: E60; H63; Q54; Q58

1. Introduction

In the context of growing public debt and rising costs related to global pollution, it is crucial to have a clear understanding of the interplay between debt and global pollution dynamics. Today, several countries that are particularly vulnerable to the impacts of climate change also find themselves burdened with high levels of debt. The economic repercussions of the COVID-19 pandemic further exacerbated this situation (see e.g Dibley et al. 2021). This issue is widely brought up for developing countries,[1] but is also a major concern for developed countries. Indeed, since 2008, we have observed in all groups of countries an increasing trend in the share of debt in GDP (Figure 1).

Concerning expenses related to global pollution, in addition to the substantial investments linked to the transition to a less polluting economy (i.e. mitigation), the expenses associated with adaptation are expected to grow in all countries (IPCC, 2022). Meanwhile, the connections between public debt and global environmental challenges can be illustrated by the positive and substantial impacts of climate vulnerability on debt (Buhr et al. 2018), suggesting the existence of a detrimental cycle wherein vulnerability perpetuates itself through public debt management. Moreover, the rise in disaster-related losses will result in reduced tax revenues. Zenios (2024) gives an overview of the direct and indirect channels and suggests a possible doom loop between climate change and sovereign debt. We consider in this paper these different dimensions to examine the interplay between debt and pollution dynamics.

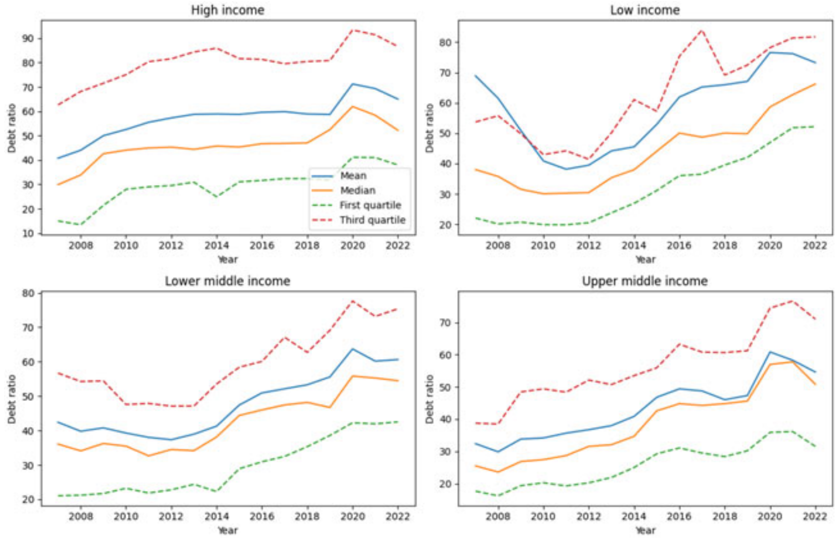

Figure 1. Evolution of the central government debt to GDP ratio by groups of countries (World bank classification in 2023). IMF global debt database.

As reported by IMF (2023), policymakers face a fundamental tradeoff. On the one hand, relying on spending-based measures to achieve emission goals and to adapt could lead to a substantial increase in public debt. On the other hand, limited environmental action exposes the world to adverse consequences from global pollution, increasing the cost of adaptation. The recent proposals from the European Commission regarding new economic governance rules highlight the interconnected nature of questions surrounding fiscal sustainability and environmental concerns. In particular, the Commission considers climate change as a structural trend representing a challenge to the financial stability of Member States' public finances.[2] The Pisani-Ferry and Mahfouz (2023) report is in line with this argument. In the case of France, it recommends using debt to finance the investments needed for the ecological transition. The use of debt should be limited to "green" investments that have a positive impact on the climate and generate long-term economic returns. This policy must be accompanied by more progressive taxation.

This paper contributes to this debate. Within an endogenous growth framework, we study the dynamic path of pollution, debt, and economic growth when public authorities finance mitigation and adaptation to tackle the damages caused by the pollution stock. From a normative perspective, we look at how fiscal and environmental policy instruments can be used to guarantee sustainability and improve welfare.

We develop an overlapping generations (OLG) model where debt, pollution, and growth are endogenous. Households live for two periods and save through two assets, capital, the source of growth, and public debt. The government issues debt securities because taxes on capital and labor incomes do not cover public expenses for pollution mitigation and adaptation, and the service of past debt. The pollution stock evolves with productive capital and mitigation measures, and is a source of damage by reducing aggregate productivity (TFP). Technology plays an important role: the final good is produced using an AK-type production function to have a simple engine of growth, but considering that the adaptation policy of the government can dampen the negative effect of pollution on TFP. In other words, the government can affect pollution using two instruments: government spending of mitigation reduces the pollution flow, whereas the government spending of adaptation offsets the negative effect of pollution on TFP. The first type of expenditure could be on waste management, the installation of filters to reduce harmful emissions or

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carbon removal solutions, while the second one includes building infrastructure to protect against rising sea levels or creating drought-resistant agriculture.

The long-run equilibria analysis shows that two balanced growth paths (BGPs) may exist, defined by constant ratios of debt and pollution over capital. One is characterized by a low pollution-to-capital ratio and a high debt-to-capital ratio while the reverse is observed for the other. When TFP vulnerability to pollution is not too high, the growth rate is higher at the BGP with low pollution and high debt, which also means higher welfare at this BGP. In that case, the higher long-term growth rate is at the expense of a high level of debt per unit of capital. This suggests a crowding-in effect of debt on growth.

The analysis of dynamics allows us to show that depending on fiscal policy, the TFP vulnerability to pollution, and initial conditions of capital, debt, and pollution stocks, the economy either converges to the BGP with low pollution over capital, collapses or experiences a perpetual increase in pollution over capital. More precisely, we show that the BGP with a high pollution-to-capital ratio is not sustainable, as the economy cannot converge to this state, while the convergence to the BGP with low pollution-to-capital may be possible for a sufficiently high labor taxation and a reasonable TFP vulnerability to pollution. We identify an extreme case in which sustainability is completely excluded. None of the two BGPs is stable and the economy either collapses or is characterized by a perpetual increase in pollution over capital. This unfavorable situation occurs when the tax rates on labor and capital income are low and TFP vulnerability to pollution is high.

If the BGP with high pollution over capital is the only unstable one, it is a saddle toward which the stock variables cannot converge. Therefore, the stable manifold of this equilibrium defines an endogenous tipping zone (ETZ). If the initial conditions on debt, pollution, and capital are such that the economy is in this zone, the dynamics of pollution relative to capital explode.[3] Interestingly, the higher the debt relative to pollution, the easier the economy can be relegated to the ETZ. It corresponds to a form of debt vulnerability: higher debt favors unsustainable dynamic paths for pollution.

Finally, in the case of a long-run sustainability, we investigate if the policy can still improve welfare. Therefore, we analyze the effect of the fiscal policy, mitigation and adaptation on the level of the stable BGP. When TFP vulnerability to pollution is not excessive, increasing taxation enhances welfare along the stable BGP, while the effects of environmental policy instruments depend greatly on their efficiency. Insufficiently efficient mitigation in reducing pollution may worsen the pollution-to-capital ratio, whereas adaptation improves welfare only if TFP responds strongly to such expenditure. In general, this analysis shows the crucial role played by adaptation.

Previous contributions studied the macroeconomic implications of the interplay between public debt and environmental factors without (Heijdra et al. 2006) and with public mitigation Fodha and Seegmuller (2012, 2014).[4] Nonetheless, debt is often considered as an exogenous instrument, and its dynamic evolution of financing adaptation and mitigation together with the dynamic path of pollution is left aside, meaning that the question of sustainability is not properly addressed. In Andersen et al. (2020), a dynamic debt scheme is designed to finance public mitigation at each period and to improve welfare. However, in contrast to our framework, they cannot discuss the implications on economic growth. In Baret and Menuet (2024), debt allows financing mitigation expenditure. However, this paper cannot address the question of sustainability by assuming a constant long-term debt-to-output ratio and a stabilizing rule ensuring convergence towards this objective. Moreover, they leave aside adaptation expenditures while they must be significantly integrated into environmental spending. We go one step further by developing a model that incorporates endogenous public debt dynamics and acknowledges the impact of environmental issues on fiscal sustainability. This dimension seems essential to address the economic consequences of global pollution and highlights how it exerts its influence on sovereign debt. Our paper is related to some recent papers that consider endogenous public debt and its dynamics. For example, Boly et al. (2022) examine the relationship between public and environmental debt in a framework with public mitigation but no adaptation. We depart from this paper focusing on fiscal sustainability and considering the economic damage entailed by pollution stock and its impact on debt. Catalano et al. (2020) examine the role of fiscal policy in climate change adaptation. They use a calibrated macroeconomic model of an open economy, that does not allow them to explicitly identify the interplay between debt dynamics, growth, and the environment. More generally, and in contrast to the literature that assumes limits on long-term public debt (Baret and Menuet, 2024; Boly et al. 2022; Seghini and Dees, 2024), we do not impose any restrictions on sovereign debt. This is crucial for studying environment-debt interactions and sustainability.

The rest of this paper is organized as follows. Section 2 presents an OLG model in which pollution is proportional to capital stock, and the government issues debt and imposes taxes on capital and labor incomes for financing adaptation and mitigation expenditures. Section 3 defines the equilibrium. Section 4 studies the balanced growth paths and examines the existence and multiplicity of BGPs. Section 5 considers the dynamics and the possibility of an endogenous tipping zone. Section 6 presents some policy implications. The final section provides the conclusion.

2. The model

We consider a dynamic model with pollution and three types of agents, firms, consumers, and a government. Time is discrete, $t = 0, 1, \dots, +\infty$, and there is no uncertainty.

2.1 Production

We consider an AK model of economic growth in which TFP decreases with pollution stock P_t.[5] Considering that pollution or climate change is detrimental to production is particularly relevant in addressing debt and environmental issues, as it allows to focus on funding adaptation efforts, extending beyond mere mitigation. The need for adaptation strategies will increase with the intensification of climate change impacts. These adaptations come with associated costs, such as building infrastructure to protect against rising sea levels, creating drought-resistant agriculture or developing air conditioning systems to cope with heat waves. Public action can provide the necessary financial resources to implement these adaptation measures, reducing the vulnerability of countries to environmental shocks. We thus assume that the capacity for adaptation reduces the incremental damage caused by pollution stock. This ability to adapt is ensured by the public authorities, who devote an amount G_{1t}, specifically for this purpose.[6]

Therefore, the production Y_t is given by:

$$Y_t = A\left(\frac{G_{1t}}{P_t}\right) K_t^{\alpha} (\bar{K}_t L_t)^{1-\alpha} \tag{1}$$

with K_t the capital, $\bar{K}_t$ the aggregate level of capital, L_t the labor, $\alpha \in (0, 1)$ and:

Assumption 1. *$A(X)$ is a strictly increasing function, with $A(0) = A_0 \geqslant 0$, $A(+\infty) = A_1 < +\infty$, and $A'(X)X/A(X) \in (0, 1)$.*

This last assumption implies an elasticity of production to the adaptation to pollution ratio lower than one. When the adaptation to pollution ratio goes up it causes a less-than-proportional increase in productivity.

Note that $A(X)$ may capture the fact that climate change destroys a part of aggregate output at each period (Golosov et al. 2014; Dietz and Stern, 2015). It can also represent the health effects of global pollution stock or the impacts of a change in temperature, which results in reduced aggregate productivity (Dasgupta et al. 2021; Burke et al. 2015).

Example: we can consider the following specifications for $A(X)$:

$$A(X) = \frac{A_1 X}{1 + X} \tag{2}$$

with $A(0) = 0$. This function is increasing and concave, with:

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$$\frac{A'(X)X}{A(X)} = \frac{1}{1+X} \in (0, 1) \tag{3}$$

Let r_t be the interest rate and w_t the wage. At equilibrium, we have $\bar{K}_t = K_t$ and assuming that the population in this economy is equal to one, labor input is $L_t = 1$. Therefore, profit maximization gives:

$$r_t = \alpha A\left(\frac{G_{1t}}{P_t}\right) \tag{4}$$

$$w_t = (1-\alpha)A\left(\frac{G_{1t}}{P_t}\right)K_t \tag{5}$$

Returns of factor being a positive function of productivity, they decrease with pollution.

2.2 Government

We consider public actions to tackle environmental issues. Public spending G_t linearly increases with GDP:

$$G_t = gY_t \tag{6}$$

and are divided into public spending which attenuates the effect of pollution on production G_{1t}, i.e. adaptation to climate effect, and mitigation G_{2t}:

$$G_{it} = g_i Y_t \tag{7}$$

for $i = 1, 2$, with $g_1 + g_2 = g$.

Environmental policy instruments consist of public spending on both mitigation and adaptation. While most of the literature has focused on their potential substitutability, these two strategies are now seen as simultaneously needed in the face of climate emergencies. This need is reflected in international ambitions to balance climate finance spending between the two strategies (Sadler et al. 2024).

Since we are in an endogenous growth framework, we assume that the government determines its public spending for adaptation and mitigation by fixing their amount per GDP unit. This means that the policy of adaptation will be determined by g_1 and the policy of mitigation by g_2.

To finance these spending, the government collects taxes on labor and capital incomes, τ_L and τ_K, and issues debt B_t. Therefore, its expenditures include repayment of debt and interest payments. The government faces the following budget constraint at each period:

$$R_t^b B_t + G_t = B_{t+1} + \tau_L w_t + \tau_K r_t K_t \tag{8}$$

with R_t^b the interest factor of debt and $B_0 > 0$ the initial stock of debt. We are in an economy with a positive initial stock of public debt. The different policy parameters as well as the interest factor of debt and the income will determine how public debt evolves through time. We will precisely study the interplay between debt accumulation, dynamics of pollution stock, and growth.

2.3 Pollution

The stock of pollution increases with the emission flow and partly leaves the atmosphere through a natural process in a share $0 < m < 1$. Emission flow is assumed to be proportional to the stock of capital. Mitigation measures G_{2t} are implemented by public authorities to further reduce pollution flows. Mitigation expenditure may include investment in carbon capture and sequestration or carbon dioxide removal solutions. The stock of pollution evolves according to:

$$P_{t+1} = (1-m)P_t - \psi G_{2t} + \mu K_t \tag{9}$$

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The parameter $\psi > 0$ captures the efficiency of public mitigation and $\mu > 0$ the pollution flow resulting from capital stock, as in Heijdra et al. (2006) or Chiroleu-Assouline and Fodha (2014). Capital is the source of pollution. Indeed, capital accumulation favors the production of pollution-intensive goods. This is consistent with the evidences (e.g. Cole et al. 2005; Andersen, 2017). For example, Cole et al. (2005) find that industrial processes that tend to be physical capital intensive generate more emissions than less capital-intensive processes. Andersen (2017) obtains that pollution emissions are higher for industries that use more intensively tangible assets, such as physical capital, than intangible assets, such as labor.

Note that since G_{2t} will increase with income, public adaptation will also have a direct negative effect on the pollution stock by increasing TFP, income and therefore mitigation. In addition, polluting capital which will be derived from a portion of savings will increase with labor income and therefore with TFP and public adaptation.

This stock of pollution only affects the real side of the economy through its negative effect on the TFP. We will not consider a direct negative effect of pollution on households welfare.

2.4 Consumers

Consumers are in overlapping generations. The population size of each generation is constant and normalized to one. Each consumer lives for two periods, consumes in both periods, and saves through two assets, public debt and capital. Capital depreciates at rate $\delta \in (0, 1)$, meaning that return on capital is given by $1 - \delta + (1 - \tau_K)r_t$.

The utility function of the generation born in t is given by:

$$U(c_t, d_{t+1}) = \ln c_t + \beta \ln d_{t+1} \tag{10}$$

with c_t the consumption when young, d_{t+1} the consumption when old, and $\beta \in (0, 1)$. The household maximizes her utility under the two budget constraints:

$$c_t + K_{t+1} + B_{t+1} = (1 - \tau_L)w_t \tag{11}$$

$$d_{t+1} = [1 - \delta + (1 - \tau_K)r_{t+1}]K_{t+1} + R^b_{t+1}B_{t+1} \tag{12}$$

The household can solve her optimal behavior in two steps. Given her saving, she determines her portfolio choice between capital and debt holding. Given this choice, she chooses her optimal saving. Maximizing the utility with respect to K_{t+1} and B_{t+1}, we obtain the following equation:

$$1 - \delta + (1 - \tau_K)r_{t+1} = R^b_{t+1} \equiv R_{t+1} \tag{13}$$

Bonds and capital assets provide the same return, which means that they are perfect substitutes.[7] Given this result, the utility maximization with respect to $K_{t+1} + B_{t+1}$ gives the saving function:[8]

$$K_{t+1} + B_{t+1} = \frac{\beta}{1+\beta}(1 - \tau_L)w_t \tag{14}$$

3. Equilibrium

We define an equilibrium as a function of capital, debt, and pollution stocks. Market clearing is obtained substituting (4), (5) and (13) in (8), (9) and (14). We get the following functions:

$$K_{t+1} + B_{t+1} = \frac{\beta}{1+\beta}(1 - \tau_L)(1 - \alpha)A\left(\frac{G_{1t}}{P_t}\right)K_t \tag{15}$$

$$B_{t+1} = R_tB_t + gA\left(\frac{G_{1t}}{P_t}\right)K_t - (\tau_L(1 - \alpha) + \tau_K\alpha)A\left(\frac{G_{1t}}{P_t}\right)K_t \tag{16}$$

$$P_{t+1} = (1 - m)P_t - \psi g_2 A\left(\frac{G_{1t}}{P_t}\right)K_t + \mu K_t \tag{17}$$

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We introduce the following variables to conduct our analysis: the growth factor $\gamma_{t+1} \equiv K_{t+1}/K_t$, debt per unit of capital $b_t \equiv B_t/K_t$, and pollution per unit of capital $\pi_t \equiv P_t/K_t$. Using the government budget (7), public adaptation per unit of capital can thus be written as a function of π_t:

$$\frac{G_{1t}}{P_t} = \frac{g_1}{\pi_t} A\left(\frac{G_{1t}}{P_t}\right) \tag{18}$$

Since $A'(x)x/A(x) \in (0, 1)$, this equation implicitly defines a decreasing function $G_{1t}/P_t = \varepsilon(\pi_t)$ if $\lim_{x\to 0} A(x)/x > \pi_t/g_1$ (see Appendix A for details).[9] Using Assumption 1 and equation (18), we further have $\varepsilon(0) = +\infty$ and $\varepsilon(+\infty) = 0$. As pollution per unit of capital increases, the amount of public spending dedicated to addressing pollution-related issues for each unit of pollution decreases. Because of the productivity cost entailed by pollution, the higher the pollution per unit of capital, the lower the resources that can be allocated to adaptation. This may be counteracted if the government decides to allocate a larger proportion of its budget to climate adaptation efforts, *i.e* if g_1 increases.

The total factor productivity can thus be expressed as a function of pollution per capital. We have $A(G_{1t}/P_t) = A[\varepsilon(\pi_t)] \equiv a(\pi_t)$, with $a'(\pi_t) < 0$, $a(0) = A[\varepsilon(0)] = A(+\infty) = A_1$, and $a(+\infty) = A[\varepsilon(+\infty)] = A(0) = A_0$. Similarly, the interest factor can be written as $R_{t+1} = 1 - \delta + (1 - \tau_K)\alpha a(\pi_{t+1}) \equiv R(\pi_{t+1})$, with $R'(\pi_{t+1}) < 0$.

Example (continued): considering our example given by equation (2), we have $\varepsilon(\pi_t) = g_1A_1/\pi_t - 1$, *which implies that:*

$$a(\pi_t) = A_1 - \frac{\pi_t}{g_1} \tag{19}$$

which requires that $\pi_t < g_1A_1$.

Now, we can rewrite the dynamic system (15)–(17) as follows:

$$\gamma_{t+1} + b_{t+1}\gamma_{t+1} = \Sigma(1-\tau_L)(1-\alpha)a(\pi_t) \tag{20}$$
$$b_{t+1}\gamma_{t+1} = R(\pi_t)b_t + ga(\pi_t) - (\tau_L(1-\alpha) + \tau_K\alpha)a(\pi_t) \tag{21}$$
$$\pi_{t+1}\gamma_{t+1} = (1-m)\pi_t - \psi g_2 a(\pi_t) + \mu \tag{22}$$

with $\Sigma \equiv \frac{\beta}{1+\beta}$ the saving rate.

Rearranging equations (20)–(22), we finally obtain:

$$\gamma_{t+1} = a(\pi_t)\left[\Sigma(1-\tau_L)(1-\alpha) + (\tau_L(1-\alpha)+\tau_K\alpha) - g - \frac{R(\pi_t)}{a(\pi_t)}b_t\right] \tag{23}$$

$$b_{t+1} = \frac{\frac{R(\pi_t)}{a(\pi_t)}b_t + g - (\tau_L(1-\alpha)+\tau_K\alpha)}{\Sigma(1-\tau_L)(1-\alpha) + (\tau_L(1-\alpha)+\tau_K\alpha) - g - \frac{R(\pi_t)}{a(\pi_t)}b_t} \tag{24}$$

$$\pi_{t+1} = \frac{(1-m)\frac{\pi_t}{a(\pi_t)} - \psi g_2 + \frac{\mu}{a(\pi_t)}}{\Sigma(1-\tau_L)(1-\alpha) + (\tau_L(1-\alpha)+\tau_K\alpha) - g - \frac{R(\pi_t)}{a(\pi_t)}b_t} \tag{25}$$

Equations (24) and (25) give the dynamics of (b_t, π_t) for $t \geq 0$, taking into account that both b_t and π_t are predetermined variables. The dynamics of debt per capital is coupled with the dynamics of pollution because the depreciation of capital is not complete. This implies that the cost of capital falls with productivity loss but not in the same proportion as growth $\left(\frac{R(\pi_t)}{a(\pi_t)}\text{ still depends on } \pi_t\right)$. The value of the growth factor γ_t is deduced from these two variables using equation (23). To conduct our analysis, we focus on relevant situations in which the growth factor, debt per unit of capital, and pollution per unit of capital are all positive. To ensure $\gamma_{t+1} > 0$, $b_{t+1} > 0$ and $\pi_{t+1} > 0$, we assume a primary deficit and the following restrictions:

$$\Sigma(1-\tau_L)(1-\alpha) > \frac{R(\pi_t)}{a(\pi_t)}b_t + g - (\tau_L(1-\alpha)+\tau_K\alpha)$$
$$\mu > a(\pi_t)\psi g_2$$

They are satisfied under the next assumption:

Assumption 2.

$$b_t < \frac{a(+\infty)}{R(+\infty)}[\Sigma(1-\tau_L)(1-\alpha) - g + \tau_L(1-\alpha)+\tau_K\alpha]$$
$$g > \tau_L(1-\alpha)+\tau_K\alpha$$
$$\mu > a(0)\psi g_2$$

The first inequality characterizes an upper bound for public debt, which increases with the amount of savings and decreases with the primary deficit, the second a primary deficit, and the third emission intensity per unit of capital higher than the efficiency of pollution abatement.

4. Balanced growth paths: multiplicity and main features

We focus here on long-run equilibria. We first show the existence and multiplicity of BGPs. Then, we investigate the main features of these equilibria. We will in particular understand how they are ranked according to the levels of debt per capital, pollution per capital, and growth.

4.1 Existence and multiplicity of BGPs

Along a balanced growth path, capital, debt, and pollution grow at a constant rate $\gamma - 1$. A balanced growth path is thus characterized by $b_t = b_{t+1} = b$ and $\pi_t = \pi_{t+1} = \pi$ solving (24) and (25). Hence, it is a stationary solution (b, π) to:

$$b = \frac{\frac{R(\pi)}{a(\pi)}b + g - (\tau_L(1-\alpha)+\tau_K\alpha)}{\Sigma(1-\tau_L)(1-\alpha) + (\tau_L(1-\alpha)+\tau_K\alpha) - g - \frac{R(\pi)}{a(\pi)}b} \tag{26}$$

$$\pi = \frac{(1-m)\frac{\pi}{a(\pi)} - \psi g_2 + \frac{\mu}{a(\pi)}}{\Sigma(1-\tau_L)(1-\alpha) + (\tau_L(1-\alpha)+\tau_K\alpha) - g - \frac{R(\pi)}{a(\pi)}b} \tag{27}$$

Given such a solution, the growth factor corresponds to:

$$\gamma = a(\pi)\left[\Sigma(1-\tau_L)(1-\alpha) + (\tau_L(1-\alpha)+\tau_K\alpha) - g - \frac{R(\pi)}{a(\pi)}b\right] \tag{28}$$

The ratio of (26) and (27) gives:

$$b = \frac{g - (\tau_L(1-\alpha)+\tau_K\alpha)}{\frac{1-m-R(\pi)}{a(\pi)} - \frac{\psi g_2}{\pi} + \frac{\mu}{a(\pi)\pi}} \tag{29}$$

Using (13), it is equivalent to:

$$b = \frac{a(\pi)\pi[g - (\tau_L(1-\alpha)+\tau_K\alpha)]}{\pi(\delta - m) - (1-\tau_K)\alpha a(\pi)\pi - \psi g_2 a(\pi) + \mu} = \frac{g - \tau_L(1-\alpha) - \tau_K\alpha}{X(\pi) - \frac{1-\delta}{a(\pi)} - (1-\tau_K)\alpha} \equiv B_1(\pi) \tag{30}$$

with $X(\pi) \equiv \frac{1-m}{a(\pi)} - \frac{\psi g_2}{\pi} + \frac{\mu}{a(\pi)\pi}$.

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Moreover, (27) can be rewritten as:

$$b = \frac{a(\pi)}{R(\pi)}\left[\Sigma(1-\tau_L)(1-\alpha)+\tau_L(1-\alpha)+\tau_K\alpha-g-X(\pi)\right]$$
$$\equiv B_2(\pi) \tag{31}$$

In the following, we assume:

Assumption 3. *$\delta \geq m$ and $\epsilon_a(\pi) < -1$.*

The first part of the assumption implies that the rate of pollution absorption is lower than the depreciation rate of capital. This is consistent in our context, as pollution stock can refer to greenhouse gases whose some will remain in the atmosphere for thousands of years. The second part of the assumption implies that total factor productivity is elastic to pollution over capital ratio, illustrating an important vulnerability to climate change (see IPCC 2022). When pollution per unit of capital increases total factor productivity falls more than proportionally. It implies that $a(\pi)\pi$ is decreasing in π. As a result, $X(\pi)$ is an increasing function of π.

Example (continued): note that with the example defined in equation (19), $\epsilon_a(\pi) < -1$ implies that $\pi > g_1A_1/2$.

This example illustrates the fact that $\epsilon_a(\pi) < -1$ could introduce a lower bound $\underline{\pi} > 0$ defined by $\epsilon_a(\underline{\pi}) = -1$ such that $\epsilon_a(\pi) < -1$ for all $\pi > \underline{\pi}$.

Under Assumptions 2 and 3, the numerator of (30) is positive and the denominator is increasing in π. We thus have $B_1'(\pi) < 0$. In addition, to ensure a positive debt along the balanced growth path, we restrict our attention to cases where:

$$X(\pi) - \frac{1-\delta}{a(\pi)} > (1-\tau_K)\alpha \tag{32}$$

New debt emissions should be higher than the cost of existing debt. Indeed, using (27) and (28), $X(\pi) = \gamma/a(\pi)$. It implies that inequality (32) is equivalent to $\gamma > R(\pi)$. Using (30), we have $b[\gamma - R(\pi)] = a(\pi)[g - \tau_L(1-\alpha) - \tau_K\alpha]$. Since we assume that the government budget is characterized by a primary deficit, a BGP should be characterized by a growth factor larger than the interest factor.

Since the left-hand side of inequality (32) is increasing in π, there exists $\pi_1 > 0$ such that $X(\pi_1) = \frac{1-\delta}{a(\pi_1)} + (1-\tau_K)\alpha$ if there is a value $\widetilde{\pi} > 0$ such that $X(\widetilde{\pi}) < \frac{1-\delta}{a(\widetilde{\pi})} + (1-\tau_K)\alpha$. Then, inequality (32) is satisfied for all $\pi > \pi_1$, with $\lim_{\pi\to\pi_1^+} B_1(\pi) = +\infty$. Note that the existence of $\pi_1 > 0$ can be compatible with Assumption 2.

Lemma 1. *Under Assumptions 1–3, assume that there exists a value $\widetilde{\pi} > 0$ such that $X(\widetilde{\pi}) < \frac{1-\delta}{a(\widetilde{\pi})} + (1-\tau_K)\alpha$. Then, there exists $\pi_1 > 0$ such that $X(\pi) > \frac{1-\delta}{a(\pi)} + (1-\tau_K)\alpha$ for $\pi > \pi_1$.*

Our example illustrates that this lemma is satisfied for a non-empty set of parameters.

Example (continued): using the example defined in equation (19), we illustrate the existence of the bound π_1. Inequality (32) writes $F(\pi) > (1-\tau_K)\alpha$, with:

$$F(\pi) \equiv \frac{\pi[g_1A_1(\delta-m)+\psi g_2]+g_1A_1(\mu-\psi g_2)}{\pi(g_1A_1-\pi)} \tag{33}$$

Since $F(\pi)$ is an increasing function and $F(g_1A_1) = +\infty$, there exists $\pi_1 \in (g_1A_1/2, g_1A_1)$ if $F(g_1A_1/2) < (1-\tau_K)\alpha$. This happens if $\delta - m < A_1(1-\tau_K)\alpha/2$ and

$$g_1 > \frac{4\mu - 2\psi g_2 A_1}{A_1[(1-\tau_K)\alpha A_1 - 2(\delta-m)]} \equiv \underline{g_1}(g_2) \tag{34}$$

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Using (31), a steady state with positive debt per unit of capital should satisfy:

$$\Sigma(1-\tau_L)(1-\alpha) > g - \tau_L(1-\alpha) - \tau_K\alpha + X(\pi) \tag{35}$$

Since the left-hand side of this inequality is constant and the right-hand side is increasing in π, there exists $\pi_2 > 0$ such that inequality (35) is satisfied for all $\pi < \pi_2$, with $B_2(\pi_2) = 0$. In this case, we also deduce that under Assumption 3, we have $B_2'(\pi) < 0$. For this, we need to have a value $\widehat{\pi} > 0$ such that $\Sigma(1-\tau_L)(1-\alpha) < g - \tau_L(1-\alpha) - \tau_K\alpha + X(\widehat{\pi})$.

Lemma 2. *Under Assumptions 1–3, assume that there exists a value $\widehat{\pi} > 0$ such that $\Sigma(1-\tau_L)(1-\alpha) < g - \tau_L(1-\alpha) - \tau_K\alpha + X(\widehat{\pi})$. The interval (π_1, π_2) is non-empty if the following inequality is satisfied:*

$$\Sigma(1-\tau_L)(1-\alpha) > g - \tau_L(1-\alpha) - \tau_K\alpha + \frac{1-\delta}{a(\pi_1)} + (1-\tau_K)\alpha \tag{36}$$

where π_2 is defined by $\Sigma(1-\tau_L)(1-\alpha) = g - \tau_L(1-\alpha) - \tau_K\alpha + X(\pi_2)$.

Example (continued) : in our example, let $\widehat{\pi} = g_1A_1$ be such that $a(\widehat{\pi}) = 0$. In this case, we have $X(\widehat{\pi}) = +\infty$, which ensures the first inequality in the lemma.

Note that in our example, we have π_1 higher but arbitrarily close to $g_1A_1/2$ if g_1 tends to $\frac{4\mu-2\psi g_2A_1}{A_1[(1-\tau_K)\alpha A_1-2(\delta-m)]}$. Therefore, inequality (36) is satisfied if $\Sigma(1-\tau_L)(1-\alpha) > g - \tau_L(1-\alpha) - \tau_K\alpha + \frac{2(1-\delta)}{A_1} + (1-\tau_K)\alpha$. This last inequality is satisfied if τ_K and A_1 are high enough and the primary deficit is not too important. It proves the existence of π_2 and of a non-empty interval (π_1, π_2).

Since $B_1(\pi_1) = +\infty > B_2(\pi_1)$ and $B_1(\pi_2) > B_2(\pi_2) = 0$, the economy may be characterized by an even number (two) of steady states.

Proposition 1. *Under Assumptions 1–3, and inequality (36), there exists $\overline{g} > \tau_L(1-\alpha) + \tau_K\alpha$ such that for $g \in (\tau_L(1-\alpha) + \tau_K\alpha, \overline{g})$, there are (at least) two BGPs, (π_I, b_I) and (π_{II}, b_{II}), with $\pi_I < \pi_{II}$ and $b_I > b_{II}$.*

Proof. *See Appendix B.* □

A primary deficit ($g > \tau_L(1-\alpha) + \tau_K\alpha$) ensures the possibility of having a positive stationary debt per unit of capital in our context where growth is higher than the interest factor. At the same time, if the environmental expenditure is too high ($g > \overline{g}$), the primary deficit is too significant to observe stationarity in debt per unit of capital. The share of GDP devoted to environmental issues has to take an intermediate value to observe stationary solutions. In addition, condition (36) has to be satisfied.

Example (continued): using the example defined in equations (2) and (19), we illustrate what the conditions for existence of BGPs imply in terms of policy.

Using Lemmas 1 and 2, we have conditions ensuring that the interval (π_1, π_2), necessary for the existence of a steady-state π_i, is non-empty. Adaptation expenditures must satisfy the constraint (34), $g_1 > \underline{g_1}(g_2)$, where adaptation g_1 must not deviate significantly from the minimum value $\underline{g_1}(g_2)$.

Thus, for any tax rates τ_L et τ_K, existence of a primary deficit and (34) determine the precise level of environmental policy expenditures g_1 and g_2. Adaptation g_1 has to be close to $\underline{g_1}(g_2)$ and mitigation g_2 adjusts so as to have a weak primary deficit. Using Assumption 3, mitigation is also constrained by the condition imposing a maximum threshold $\mu/A_1 > \psi g_2$. Considering that μ and A_1 are close or equal, this last inequality is never binding.

We also need to ensure that savings are high enough to finance the current deficit, investment, and the cost of debt, i.e. inequality (36) holds:

$$\frac{\beta}{1+\beta}(1-\tau_L)(1-\alpha) > g_1 + g_2 - (\tau_L(1-\alpha) + \tau_K\alpha) + 2\frac{(1-\delta)}{A_1} + \alpha(1-\tau_K) \tag{37}$$

To understand how considering these different conditions create constraints on the policy instruments, let us define $\kappa > 0$ small enough as being the primary deficit: $\kappa = g_1 + g_2 - [\tau_L (1-\alpha) + \tau_K \alpha]$. Substituting this expression in inequality (37), we obtain:

$$\tau_L < \frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta}\tau_K + 1 - \frac{1}{1-\alpha}\frac{1+\beta}{\beta}\left(\kappa + \alpha + 2\frac{1-\delta}{A_1}\right)$$

We show that these conditions impose interdependencies between the tax rates τ_L and τ_K, and the size of the primary deficit κ. If the tax rate on labor income is positive, it requires a sufficiently high tax rate on capital income. Moreover, the higher the primary deficit is, the higher the capital tax rate. This positive relationship allows to keep the debt burden low enough such as the debt over capital ratio can be stationnary.

The BGP characterized by the lowest pollution to capital ratio (π_I) has the highest level of debt over capital (b_I), while the one with the highest pollution-to-capital ratio (π_{II}) is also defined by the lowest debt per unit of capital (b_{II}). We can note that the decreasing relationship we observe between π and b is ensured by the sufficient TFP vulnerability ($\epsilon_a(\pi) < -1$). This property is specific to our analytical framework and is explained in detail in the following section.

4.2 Balanced growth and the role of TFP sensitivity to pollution

We want to clearly understand the links between pollution to capital, debt to capital, and growth that come from the comparison of the two BGPs.

First, we turn our attention to the growth factor γ. From (28), we see that it is a declining function of the debt per capital ratio b, through *a priori* a usual crowding-out effect of debt on investment. Moreover, as pollution generates negative external effects on production and therefore also on savings, the growth factor also depends negatively on the pollution per capital ratio π. In our context of TFP vulnerability ($\epsilon_a(\pi) < -1$), a BGP with a higher level of π is characterized by a lower debt per capital ratio b. Therefore, at the BGPs, the relationship between γ and π (or b) seems ambiguous.

To avoid this ambiguity, we exploit the fact that the growth of capital is equal to the growth of the pollution stock. Then, using (27) and (28), the growth factor can be expressed as a function of π:

$$\gamma = 1 - m + \frac{\mu - \psi g_2 a(\pi)}{\pi} \tag{38}$$

Hence, we note that the growth factor is higher than 1, i.e. growth is positive, as soon as m is low enough. The growth factor increases with the pollution flow but decreases with the current pollution stock. Indeed, since mitigation decreases with pollution through its effect on TFP, π has two opposite effects on growth, a positive one through pollution flows and a negative one through the pollution stock.

Therefore, growth is a decreasing function of pollution over capital ratio (and hence an increasing function of b) if and only if the elasticity of TFP with respect to π satisfies:

$$-\epsilon_a(\pi) < \frac{\mu - \psi g_2 a(\pi)}{\psi g_2 a(\pi)} \tag{39}$$

which may be in accordance with Assumptions 2 and 3. We thus deduce that:

Corollary 1. *Under Assumptions 1–3, inequality (36), and $\bar{g} > g > \tau_L(1-\alpha) + \tau_K\alpha$, we have:*

1. $\gamma_I > \gamma_{II}$ *iif* $-\epsilon_a(\pi) < \frac{\mu - \psi g_2 a(\pi)}{\psi g_2 a(\pi)}$ *for all* $\pi \in (\pi_1, \pi_2)$;
2. $\gamma_I < \gamma_{II}$ *iif* $-\epsilon_a(\pi) > \frac{\mu - \psi g_2 a(\pi)}{\psi g_2 a(\pi)}$ *for all* $\pi \in (\pi_1, \pi_2)$.

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In case 1, the growth rate decreases with π, so it is lower in the state with low debt and high pollution (π_{II}, b_{II}). This configuration is characterized by a not excessive TFP vulnerability to pollution. When the production is not too sensitive to pollution through the TFP, a higher level of π means a lower pollution flow over pollution stock, which implies lower growth.

In case 2, the high TFP vulnerability explains that growth is higher in the state (π_{II}, b_{II}). Indeed, a higher level of pollution over capital implies a strong increase in the pollution flow because of the decrease in public mitigation. Then, the pollution flow over the pollution stock increases, which implies higher growth.

This result is important as it reveals that as long as we consider a not excessive TFP vulnerability to pollution (case 1 of Corollary 1), all other things being equal, a BGP characterized by a lower pollution level per unit of capital is associated with higher capital growth. Recall that this BGP also has a higher level of debt over capital. A direct implication of Corollary 1 is that when TFP vulnerability is not excessive, a BGP with higher growth means a BGP with higher debt over capital. In contrast, when TFP vulnerability is very high, a BGP with lower growth means a BGP with higher debt over capital. Therefore, there is a crowding-in effect of debt on growth in the first case and a crowding-out effect in the second one.

Using (26) and (28), the intertemporal budget constraint evaluated at a BGP can be written:

$$b[\gamma - R(\pi)] = a(\pi)[g - \tau_L(1-\alpha) - \tau_K\alpha] \tag{40}$$

When pollution over capital is low, the TFP and, therefore, the interest factor are high. This means that both the primary deficit and debt services are high. Debt over capital is high, even if growth is higher than at the steady state with a higher ratio of pollution over capital (case 1 of Corollary 1). This explains that higher debt can be compatible with higher growth. This is an interesting result regarding the macroeconomic literature that mainly finds that public debt usually has a crowding-out effect on growth (see the seminal contribution by Diamond, 1965), except in the presence of some financial imperfections (see Woodford, 1990). Using (40), if the TFP is constant and, therefore, the interest factor too, we immediately deduce that public debt over capital and growth are inversely related. Public debt always has a crowding-out effect on growth.

Now, we want to understand precisely why a high TFP vulnerability to pollution is the source of a negative link between π and b when we compare both BGPs. We start by examining the extreme case in which TFP is not sensitive to pollution damage (i.e. $\epsilon_a(\pi) = 0$). It implies that $a(\pi) = a$ and $R(\pi) = R$ are constant. Equations (30) and (31) thus become:

$$B_1(\pi) = \frac{a[g - (\tau_L(1-\alpha) + \tau_K\alpha)]}{\delta - m - (1-\tau_K)\alpha a + \frac{\mu - \psi g_2 a}{\pi}} \tag{41}$$

$$B_2(\pi) = \frac{a}{R}\left[\Sigma(1-\tau_L)(1-\alpha) + \tau_L(1-\alpha) + \tau_K\alpha - g - \frac{1-m}{a} - \frac{1}{\pi}\left(\frac{\mu}{a} - \psi g_2\right)\right] \tag{42}$$

with $B_1'(\pi) > 0$ and $B_2'(\pi) > 0$. In that case, if there still exist several steady states, the one with the highest level of pollution over capital will be also characterized by the highest level of debt over capital. Comparing equations (41) and (42) with equations (30) and (31) provides insights into the differences that arise.

Equation (41) represents debt over capital as the ratio of the primary deficit over the new debt emission, which increases with growth, minus the cost of debt services measured by the interest factor. As previously mentioned, the growth of capital is equal to the growth of pollution at a BGP, which explains that it is decreasing in the pollution stock over capital and hence that there is a positive relationship between π and b. When productivity is negatively affected by pollution over capital, several adding effects may imply a reversal of this link. These effects can be perceived by using equation (30). Production being affected negatively by π, a higher π implies a lower primary deficit $a(\pi)[g - \tau_L(1-\alpha) - \tau_K\alpha]$. In addition, the cost of debt reimbursement goes down

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with productivity loss. These adding effects, due to endogenous productivity, are important when vulnerability to pollution is high enough. In that case, we have a negative relationship between debt over capital and pollution over capital. Equation (42) comes from the equilibrium on the asset market taking into account that capital growth is equal to pollution growth and considering the government budget constraint. Debt per unit of capital is equal to the difference, discounted by the interest factor, between savings and the sum of the primary deficit and the increase in capital (which is here equal to pollution growth). When productivity is constant, the only effect of π on b is positive and is due to its negative effect on pollution growth, as previously mentioned. When the TFP decreases with pollution over capital, several adding effects may overturn the link between debt over capital and pollution over capital. These effects can be perceived by using equation (31). First, a lower TFP decreases income discounted by the interest factor.[10] Second, growth per unit of TFP increases with π, which has a crowding-out effet on debt.

To summarize, TFP vulnerability to pollution implies that higher pollution to capital reduces product so that the debt to capital ratio reduces too because less deficit has to be financed and less saving is available to buy public debt. Turning to the analysis of the dynamics is now essential to determine toward which equilibria the economy will converge.

5. Dynamics, endogenous tipping zone and sustainability

The first main question we ask is whether the economy might converge to a BGP. Only in such a case, the economy will be sustainable in the long run. Otherwise, either the economy will collapse, or pollution over capital will follow an explosive dynamic path. We will especially identify a zone in terms of initial conditions, that we call the endogenous tipping zone, such that the economy will not be sustainable.

Proposition 1 states that two BGPs with positive debt, capital, and pollution may coexist. We analyze the dynamics in this interesting case. The question is to know toward which BGP the economy will converge. We aim to identify the conditions for a sustainable or explosive and unsustainable dynamic path. The objective is to highlight the respective roles of fiscal instruments, TFP vulnerability to pollution, and initial conditions on pollution debt and capital.

5.1 Stability and sustainability

The dynamics are driven by equations (24) and (25). They can alternatively be driven by a combination of (24) and (25), and equation (25).

Let $\Phi_t \equiv \frac{b_t}{\pi_t} = \frac{B_t}{P_t}$. Using (24) and (25), we obtain:

$$\Phi_{t+1} = \frac{\frac{R(\pi_t)}{a(\pi_t)}\Phi_t + \frac{g-\tau_L(1-\alpha)-\tau_K\alpha}{\pi_t}}{X(\pi_t)} \tag{43}$$

Then, $\Phi_{t+1} \geqslant \Phi_t$ is equivalent to:

$$\Phi_t \leqslant \frac{g-\tau_L(1-\alpha)-\tau_K\alpha}{\pi_t[X(\pi_t)-\frac{1-\delta}{a(\pi_t)}-(1-\tau_K)\alpha]} \equiv B_3(\pi_t) \tag{44}$$

with $B_3(\pi_t) = B_1(\pi_t)/\pi_t$, $B_3(\pi_1) = \infty$, $B_3(\pi_2) > 0$ and $B_3'(\pi_t) < 0$.

Using (25) and (31), $\pi_{t+1} \geqslant \pi_t$ rewrites $b_t \geqslant B_2(\pi_t)$. This is equivalent to $\Phi_t \geqslant B_2(\pi_t)/\pi_t \equiv B_4(\pi_t)$, given by:

$$B_4(\pi_t) = \frac{a(\pi_t)}{R(\pi_t)\pi_t}\left[\Sigma(\pi_t)(1-\tau_L)(1-\alpha)+\tau_L(1-\alpha)+\tau_K\alpha-g-X(\pi_t)\right] \tag{45}$$

with $B_4'(\pi_t) < 0$, $B_4(\pi_1) > 0$ and $B_4(\pi_2) = 0$.

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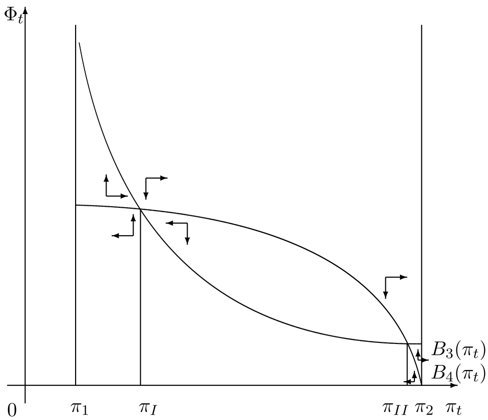

Figure 2. Dynamics with sustainability.

We can draw a phase diagram using these different ingredients and the results of the previous section. The stationary values of debt over pollution are $\Phi_I = \frac{b_I}{\pi_I}$ and $\Phi_{II} = \frac{b_{II}}{\pi_{II}}$. Since $B_3(\pi_t)$ and $B_4(\pi_t)$ are both decreasing and $\pi_I < \pi_{II}$, we deduce that $\Phi_I > \Phi_{II}$. We further note that $B_3'(\pi_I) < B_4'(\pi_I)$, while $B_3'(\pi_{II}) > B_4'(\pi_{II})$.

A qualitative picture of the dynamics is represented in Figure 2. We conjecture that the steady state (π_I, Φ_I) is stable, whereas the steady state (π_{II}, Φ_{II}) is a saddle. Since the two dynamic variables π_t and Φ_t are predetermined, a saddle is generically unstable. We will now confirm this conjecture by the analysis of local dynamics.

Proposition 2. *Under Assumptions 1–3, inequality (36), and $g \in (\tau_L(1-\alpha) + \tau_K\alpha, \bar{g})$, we have the following:*

1. *The steady state $(\pi_{II}, b_{II}, \Phi_{II})$ is a saddle;*
2. *The steady state (π_I, b_I, Φ_I) is locally stable if τ_L is high enough and $\epsilon_a(\pi)$ is not too negative.*

Proof. *See Appendix C.* □

This proposition shows that the economy never converges to the steady state with a high level of pollution over capital. As pollution, debt, and capital are all predetermined, the saddle (π_{II}, b_{II}) is never achieved and therefore delimits a pollution trap, as we will discuss later. Indeed, when initial conditions are characterized by too high levels of pollution and debt with respect to capital, the economy cannot converge to a long-run BGP. Both b_t and π_t increase across time. The too high level of pollution over capital implies a too low TFP and GDP growth to be compatible with the convergence to a stable and sustainable BGP. When pollution over capital is not too high, the economy would converge to the stable BGP characterized by the low level of pollution over capital and the highest level of debt over capital or pollution (π_I, b_I), as represented in Figure 2. Such a dynamic path could experience oscillations converging to the steady state.[11] Depending on the level of pollution over capital, the convergence to this steady state does not require a so low debt relative to capital. The stability of this BGP requires a not too strong TFP vulnerability to pollution and a high tax rate on labor income. If the first condition is not fulfilled, any increase in pollution over capital implies a strong decrease in GDP and, therefore, investment in future capital due to the high TFP vulnerability to pollution. Future pollution over capital increases even more, generating an explosive dynamic path (see for instance equation (25)). Under the last condition, savings are low enough to prevent an unsustainable buildup of debt and capital.

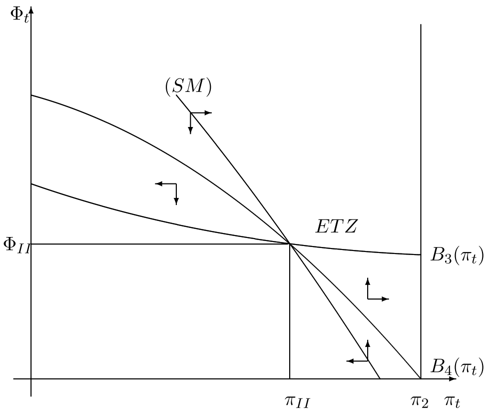

Figure 3. Endogenous tipping zone (ETZ).

5.2 Endogenous tipping zone (ETZ)

Since the steady state $(\pi_{II}, b_{II}, \Phi_{II})$ is a saddle, it has one stable and one unstable manifold. By inspection of Figure 2, we observe that the stable manifold of this steady state delineates a zone such that when the economy is on the right side of this manifold, pollution will be explosive. We call this zone an Endogenous Tipping Zone (ETZ), because it is endogenously defined by the dynamic behavior of the economy. In the following proposition, we show that:

Proposition 3. *Under Assumptions 1–3, inequality (36), and $g \in (\tau_L(1-\alpha) + \tau_K\alpha, \bar{g})$, the stable manifold of the steady state $(\pi_{II}, b_{II}, \Phi_{II})$ has a negative slope at least in the neighborhood of the steady state, while the unstable manifold has a negative slope but higher than the stable one.*

Proof. *See Appendix D.* □

The stable manifold (*SM*) of the steady state $(\pi_{II}, b_{II}, \Phi_{II})$ is clearly represented in Figure 3. Since the two dynamic variables are predetermined, the economy is generically on the left or the right side of this decreasing curve. If the initial conditions are such that the economy is on the left side of (*SM*), the dynamics could be characterized by convergence to the stable steady state with an increase in the long run of debt over pollution and capital and a decrease in pollution per unit of capital.

If the initial conditions are such that we are on the right side of (*SM*), pollution per unit of capital will increase *a priori* indefinitely. Since the unstable manifold is negatively sloped, pollution will also increase with respect to debt. The economy is in the ETZ if the stock of pollution is sufficiently high with respect to capital. Interestingly, at least around the BGP, the ETZ is delimited by a negatively slopped relationship between Φ_t and π_t. This means that the economy will experience explosive paths for lower pollution levels over capital when the debt relative to pollution is higher. When an economy is already burdened with high debt levels, it may struggle to allocate additional resources toward pollution reduction efforts and adaptation. A vicious cycle is triggered, where a high level of debt does not translate into significant spending on pollution control. Instead, it illustrates an inability to adapt and mitigate adequately, thereby increasing damages and making it more difficult to stabilize debt and pollution per unit of capital. In contrast, the lower the debt over pollution, the higher the level of pollution over capital to have an explosive path. This means that lower public debt and higher capital allow to reach a sustained growing economy more easily.

Considering the damage of pollution stock on production, we provide a theoretical mechanism explaining why, for a given pollution-to-capital ratio, the likelihood of observing explosive paths for pollution is higher in countries with higher debt. Such a detrimental situation is pointed out in the literature, but as we can remark in Zenios (2024), the papers that attempt to formalize it focus

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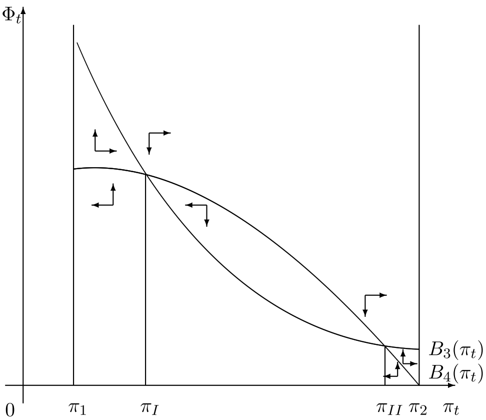

Figure 4. Dynamics with unsustainability.

mainly on the impact of climate risks on public spending; he highlights that these risks increase the cost of public debt, making public finances even more vulnerable. We obtain a similar conclusion of debt vulnerability considering a purely deterministic context.

The importance of our result is all the greater as public debt-to-GDP ratios have risen for decades and have reached record levels in a significant number of both developed and developing countries(see Figure 1 and WorldBank, 2023). This trend, combined with the heightened vulnerability to environmental issues, emphasizes the need to propose adapted policy tools to avoid a vicious circle of pollution over debt.

5.3 Unsustainability

The system can also be completely unsustainable. This happens if no steady state is stable, i.e. if the equilibrium (π_I, b_I, Φ_I) is unstable. In such a situation, pollution and debt over capital will follow an explosive dynamic paths. As we will see, it can occur if the TFP vulnerability to pollution is high and the tax rates are low. The next proposition provides sufficient conditions for such an undesirable configuration:

Proposition 4. *Under Assumptions 1–3, inequality (36), and $g \in (\tau_L(1-\alpha) + \tau_K\alpha, \bar{g})$, we have the following:*

1. *The steady state $(\pi_{II}, b_{II}, \Phi_{II})$ is a saddle;*
2. *The steady state (π_I, b_I, Φ_I) is unstable if τ_L is low enough, $\epsilon_a(\pi) < -2$, $1 + m > 2\delta$ and $1 - \tau_K > \mu/[2\alpha a(\pi_2)\pi_2]$.*[12]

Proof. *See Appendix E.* □

This proposition gives sufficient conditions to have all steady states saddle or unstable. It particularly requires sufficiently low tax rates on capital and labor incomes and a strong TFP vulnerability to pollution. Of course, the BGPs are affected by the level of the tax rates, which means that the stationary values of π_i and b_i are not similar in Proposition 4 and in Proposition 2. This explains that we do not conduct an analysis of bifurcations, but in both configurations, two BGPs coexist. When the tax rates are low, the primary deficit is important, as is the cost of debt. Both effects directly deteriorate public finance. The positive impact of low tax rates on aggregate savings is not sufficient to compensate. Low taxation creates conditions promoting unsustainable debt levels and hence hinders the government's ability to tackle pollution issues. Moreover, when

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TFP is highly vulnerable to pollution damage, the economy can never converge to a long-run BGP. As we have already seen, a small increase in pollution over capital implies a strong decrease in production and capital, which implies a larger future increase in pollution over capital. Either pollution and debt will go to infinity, characterizing a vicious circle of debt and pollution or the economy will collapse. There may exist a dynamic trajectory that diverges from the BGP with low pollution over capital to converge to the BGP with high pollution over capital. However, since the variables are predetermined, the economy will not generically experience such a dynamic path.

To summarize this section, high levels of tax rates are key ingredients to rule out an explosive accumulation of pollution and debt. However, the economy may be unsustainable for technological reasons, i.e. a high TFP vulnerability to pollution. Finally, high initial levels of debt and pollution with respect to capital promote instability of the dynamic path. Note that a high initial debt over capital is not *a priori* a source for unsustainability since it may reinforce the possibility of converging to the stable steady state with low pollution over capital. It will depend on the level of pollution over capital.

6. Policy implications

We are now interested in a configuration of possible sustainability. This means that we consider the case in which the BGP (π_I, b_I, Φ_I) is locally stable.[13] In this encouraging situation, we examine which public policy can improve welfare in the long run. Instead of focusing on first best policies,[14] we examine policies which improve welfare in the long run or after a given number of periods. Such analysis can be performed by analyzing comparative statics at the stable BGP. Starting at or close to this BGP, the economy will converge to it after the policy change. Therefore, analyzing the change of welfare at such a BGP is appropriate to evaluate the change of welfare after a certain number of periods. For this aim, we precisely study the effect of policy variables that allow the management of environmental adaptation and mitigation, g_1 and g_2, and the fiscal revenue, τ_L and τ_K, on the welfare at the BGP $i = I$.

We start by evaluating the welfare at a BGP $i = \{I, II\}$. The consumptions are given by:

$$c_{it} = (1 - \Sigma)(1 - \tau_L)w_{it} \tag{46}$$

$$d_{it+1} = R(\pi_i)\Sigma(1 - \tau_L)w_{it} \tag{47}$$

with $w_{it} = (1 - \alpha)a(\pi_i)\gamma_i^t K_0$. Substituting these two consumptions into the utility function (10), we get:

$$U(c_{it}, d_{it+1}) = \ln(1 - \Sigma) + \beta \ln \Sigma R(\pi_i) + (1 + \beta) \ln(1 - \tau_L)(1 - \alpha)a(\pi_i)\gamma_i^t K_0$$

Therefore, the main factor determining the utility evaluated at a BGP is the growth factor γ_i. A higher growth factor has a dominant positive effect on the welfare. Since $R(\pi)$ and $a(\pi)$ are decreasing in π, a lower pollution over capital also positively affects welfare. Based on (38), we see that policy variables affect γ_i through their impact on π_i. In addition, we note that the policy variable g_2 has also a direct negative effect on the growth rate.

One necessary step to determine how policy variables modify welfare is to examine their effects on (π_i, b_i). For this aim, we redefine all relevant equations as functions of (π_i, b_i) and policy instruments $(\tau_L, \tau_K, g_1, g_2)$. Productivity can thus be defined as a function of g_1 and π_i:

$$A(G_1/P) = A[\theta(\pi_i, g_1)] \equiv \Theta(\pi_i, g_1)$$

where $\theta(\pi_i, g_1)$ is defined by (18), and is increasing in g_1 and decreasing in π_i. We deduce the sign of the two following derivatives: $\Theta_{g_1}(\pi_i, g_1) > 0$, $\Theta_{\pi_i}(\pi_i, g_1) < 0$. The function $\Theta_{g_1}(\pi_i, g_1)$ captures how productivity responds to a variation in mitigation.

We present the influence of fiscal policy, via τ_L and τ_K, and environmental policy, through g_1 and g_2, focusing on the potentially stable and sustainable steady state (b_I, π_I) (see Proposition 2).[15]

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Lemma 3. *Under Assumptions 1–3 and inequality (36), we have the following properties:*

- *An increase in τ_L and/or τ_K increases b_I and reduces π_I;*
- *An increase in g_1 reduces π_I if $\Theta_{g_1}(\pi, g_1)$ is high enough, τ_L is high and $g - \tau_L(1-\alpha) - \tau_K\alpha$ is not too small;*
- *There exists $\bar{\psi} > 0$, such that if $\psi < \bar{\psi}$, an increase in g_2 reduces b_I and increases π_I.*

Proof. *See Appendix F.* □

Lemma 3 emphasizes that at the low pollution BGP (b_I, π_I), more stringent taxation reduces pollution over capital. This is because more fiscal revenue reduces the primary deficit and hence promotes capital investment. However, even though the government earns more revenue from taxation, the reduction in pollution to capital ratio goes with an increase in the debt ratio. Higher fiscal pressure leading to a higher debt over capital can appear counter-intuitive. It is driven by two key assumptions: productivity reacts to the fall in pollution per unit of capital and there is a primary deficit ($g > \tau_L(1-\alpha) + \tau_K\alpha$). In this context, the increase in the debt ratio is a positive side effect of productivity gains, which is facilitated by the reduction in pollution damage. Note that even if the qualitative effect of tax on capital and labor is similar, it masks different mechanisms. Labor tax reduces labor income while capital taxation reduces the cost of capital and debt.

Concerning environmental policy variables, an increase in the share of the public budget allocated to adaptation leads to the following consequences: the rise in g_1 generates competing effects on productivity and hence on pollution to capital ratio. First, it puts pressure on the primary deficit, leading to a reduction in capital accumulation due to a crowding-out effect. This increases pollution per unit of capital (π) and reduces productivity. Second, the increase in g_1 has a direct positive effect on productivity, as a larger share of public spending is allocated to adaptation. When productivity is sufficiently sensitive to g_1 ($\Theta_{g_1}(\pi, g_1)$ high enough), the negative effect driven by the crowding-out is surpassed by the direct positive effect of g_1. Pollution per unit of capital (π) goes down while productivity goes up.

Concerning the effect of mitigation, the efficiency of public spending to reduce pollution flow (captured by ψ) is crucial, because it determines how pollution and debt respond to an increase in g_2 at the BGP. More precisely, we can observe a backfire effect of mitigation policy in the sense that it can increase pollution per unit of capital and reduce debt. Indeed, as for g_1, the rise in g_2 puts pressure on the primary deficit, leading to a reduction in capital accumulation due to a crowding-out effect. This, in turn, leads to an increase in pollution per unit of capital (π), resulting in decreased productivity $a(\pi)$. At the same time, the increase in g_2 has a direct negative effect on pollution, whose importance depends on ψ, as a larger share of public spending is allocated to mitigation. However, when mitigation is not sufficiently efficient to reduce pollution flow, this direct effect of an increase in g_2 on pollution is low. The negative feedback effects of an increase in mitigation on capital accumulation, and hence on productivity, surpasses the initial effect of the policy.

Now, we can identify welfare-improving policy scenarios along the BGP $i = I$, i.e. policy scenarios that increase the growth factor γ_I and decrease pollution over capital. In addition, using Corollary 1, we recall that as $-\epsilon_a(\pi)$ is lower than $\frac{\mu - \psi g_2 a(\pi)}{\psi g_2 a(\pi)}$, the growth rate evaluated at a BGP decreases with pollution over capital. Using this relationship, the one between π_I and policy instruments presented in Lemma 3, and equation (38), we have the following result:

Proposition 5. *Under Assumptions 1–3, inequality (36) and $-\epsilon_a(\pi) < \frac{\mu - \psi g_2 a(\pi)}{\psi g_2 a(\pi)}$, the welfare along the BGP $i = I$*

- *increases with taxation τ_L and τ_K*

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- *increases with the share of budget allocated to adaptation g_1 if $\Theta_{g_1}(\pi, g_1)$ is high enough, τ_L is high and $g - \tau_L(1-\alpha) - \tau_K\alpha$ is not too small*
- *decreases with the share of budget allocated to mitigation g_2 if ψ is sufficiently low*

Proof. *See Appendix G.* □

When the TFP vulnerability to pollution is still not too high, a more stringent fiscal policy is a means to increase welfare as it allows for a reduction in the pollution-to-capital ratio. This is not so evident for environmental instruments because their financing costs can heavily burden the primary budget. This creates a crowding-out effect that outweighs the benefits when the effectiveness of these policies in achieving their main objectives (reducing pollution or increasing adaptation) is limited. However, when TFP reacts sufficiently to adaptation expenditures, this spending can improve growth and welfare despite the fact the primary deficit enlarges.

To illustrate in particular that the condition $\Theta_{g_1}(\pi, g_1)$ high can be in accordance with an elasticity of productivity with respect to pollution ratio satisfying $-\epsilon_a(\pi) < \frac{\mu - \psi g_2 a(\pi)}{\psi g_2 a(\pi)}$, let us come back on our example.

Example (continued): in our example, $-\epsilon_a(\pi) = \frac{\pi/g_1}{A_1 - \pi/g_1} = \frac{A_1}{a(\pi)} - 1$. Thus, condition (39) is equivalent to $\mu > A_1\psi g_2$, which holds under Assumption 2. This means that $-\epsilon_a(\pi) < \frac{\mu - \psi g_2 a(\pi)}{\psi g_2 a(\pi)}$. Moreover, in our example, we have $\Theta_{g_1}(\pi, g_1) = \pi/g_1^2$ which is high if g_1 is low. In such a case, an increase in g_1 can improve welfare, as shown in Proposition 5 .

Finally, according to Assumption 2, mitigation is not highly efficient. Consequently, relying solely on this instrument is not a viable option for increasing welfare, as it does not directly dampen the fall in productivity caused by global pollution.

Showing that an increase in a tax (on labor or capital) improves welfare on a stable BGP means that, starting near this BGP before the policy change, the economy converges to the modified BGP. After a certain number of periods, the generations will benefit from the welfare increase because the economy is not too far from the new BGP. Of course, when the increase in tax (on labor or capital) is implemented, this is not beneficial to the current and first generations after the policy change takes place. This is a well-known result, shown in a different context by Fodha et al. (2018). Indeed, by inspection of equations (46) and (47), we easily see that an increase in the tax rates has a direct negative effect on at least one consumption, while the increase in growth and the reduction of pollution over capital will take some times to have a significant effect. However, after a certain number of periods, the economy will converge and become closer to the BGP with a higher level of growth and a lower pollution over capital. Hence, after a certain time, the welfare of all the successive generations will be higher. This does not imply that intergenerational equity issues are absent during the convergence process because of potential oscillations in the dynamics. Nonetheless, the cost of these fluctuations can be viewed as relatively small compared with the positive effect on welfare associated with the new BGP.

Two last remarks are at stake. First, we have in mind to consider policies that do not change the stability conditions of the BGP with the lowest level of pollution over capital. This means that we do not consider too drastic changes, although we can also note that increasing the tax rates promotes stability. Second, when the policy reduces pollution over capital at the lowest BGP (b_I, π_I), it increases pollution over capital at the saddle BGP (b_{II}, π_{II}). This suggests that the ETZ will occur for higher levels of pollution over capital. Therefore, this zone is reducing and the convergence to the stable BGP (b_I, π_I) could take place for a larger range of initial conditions.

7. Conclusion

This paper examines the complex interplay between public debt, environmental quality, and economic growth, particularly pertinent in the context of growing public debt and global pollution

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concerns. We address these issues within an endogenous growth framework that incorporates public debt dynamics, adaptation and mitigation spending, and feedback effects of pollution on productivity. We identify two balanced growth paths characterized by varying levels of pollution and debt relative to capital. Depending on fiscal policy, initial conditions, and the responsiveness of productivity to pollution and adaptation, the economy either converges to a sustainable BGP, collapses, or experiences perpetual increases in debt and pollution. Unsustainable debt and pollution arise particularly when pollution-induced damage severely impacts productivity, underscoring the importance of policy interventions for environmental and fiscal stability.

Our results suggest that technological efficiency is fundamental: improving technology to adapt to the impacts of climate change is important for welfare at a sustainable path. Government should start tackling global emissions (mitigation and adaptation) before a pollution threshold is reached; crossing this threshold, on the other hand, pushes the economy into a tipping zone of unsustainability. Finally, since environmental policies can be financed through both public debt and income tax revenues, we find conditions under which fiscal policy is a key element in reconciling debt sustainability and environmental sustainability.

Our findings suggest that increasing taxation, particularly in contexts where productivity is highly sensitive to pollution damage, can enhance welfare along stable growth paths. This emphasizes the crucial role of efficient environmental policy instruments in promoting adaptation.

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Notes

1 In December 2023, among 67 low-income countries, the Debt Sustainability Analysis provided by the World Bank Group and the IMF identified 28 countries with a high risk of overall debt distress and 11 countries already in distress. This worrisome observation is coupled with the increase in the frequency and intensity of extreme weather events in these vulnerable countries.

2 See https://ec.europa.eu/commission/presscorner/detail/en/ip_23_2393.

3 Note that Bacchiocchi et al. (2024) also exploit the saddle property of a steady state to exhibit a form of unsustainability of public debt. However, they are interested on the interplay between debt and inflation, whereas we are concerned with pollution, debt, and growth.

4 There is also a literature analyzing the link between environmental quality and production in the absence of public debt. For a recent reference, see Menuet et al. (2024) where public pollution abatement is introduced into the production function and positively affects the efficiency of pollution as an input.

5 Burke et al. (2015) finds a nonlinear decline in macroeconomic productivity following a change in temperature, across sectors, in 166 rich and poor countries since the 1960s. This finding is confirmed by Kalkuhl and Wenz (2020) which looks at the effects of rising global mean surface temperature on production.

6 Adaptation is formalized as a flow, that reduces the damage of pollution stock. This implies that adaptation expenses are recurrent during each period. In this time perspective, we suppose that changing climatic conditions necessitate ongoing adjustments, updates, or improvements.

7 As it is usually the case in models with pollution, production and public debt, we consider that there is no market imperfection on the asset market. Then, our results will not depend on a specific financial market imperfection.

8 In the working paper version of the paper Davin et al. (2024), the analysis is extended to the case of a CES utility function where consumption in both periods are substitutable. In such a case, the saving function is increasing in the real interest factor R_{t+1}.

9 Note that it is always satisfied if $A(0) > 0$. If $A(0) = 0$, the condition is equivalent to $\lim_{x\to 0} A'(x) > \pi_t/g_1$.

10 We can notice that the negative relationship between b and π may be observed because the depreciation of capital is not complete ($\delta \neq 1$). Otherwise, when $\delta = 1$, $R(\pi)/a(\pi)$ is constant. From equation (26), the debt ratio b does no longer evolve with π. The reduction in the primary deficit and the cost of debt induced by the increase in π are proportional to the decrease in savings so that the debt-to-capital ratio does not depend on productivity and thus not on π.

11 Oscillatory convergence is one possible scenario we may observe, and discussion concerning such a convergence is not the central focus of our study.
12 Note that the assumption $1 + m > 2\delta$ fit with $\delta \geq m$. Indeed, since $m < 1$, the interval $[m, (1 + m)/2)$ is non-empty and we can choose δ in this interval.
13 Based on Proposition 2, this implies a TFP vulnerability to pollution not too important. We also consider in this section the case in which growth is decreasing with pollution ratio π, i.e. $-\epsilon_a(\pi) < \frac{\mu - \psi g_2 a(\pi)}{\psi g_2 a(\pi)}$.
14 The analysis of the optimal allocation and its decentralization using fiscal instruments is out of the scope of this paper. It could create several problems linked to the existence of a solution because of the Ak production function already characterized by increasing returns and where the TFP is a function of both the adaptation policy and the stock of pollution.
15 The effects of comparative statics on the other steady state (b_{II}, π_{II}) are the opposite.

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Appendix.

A. The relationship between G_{1t}/P_t and π_t

Let $x_t = G_{1t}/P_t$. Then, equation (18) writes:

$$x_t = \frac{g_1}{\pi_t} A(x_t)$$

Differentiating this equation with respect to x_t and π_t, we obtain:

$$\frac{dx_t}{d\pi_t} = \frac{g_1 A(x_t)/\pi_t^2}{\frac{g_1}{\pi_t} A'(x_t) - 1}$$

Since $\frac{g_1}{\pi_t} = \frac{x_t}{A(x_t)}$, we finally have:

$$\frac{dx_t}{d\pi_t} = \frac{x_t/\pi_t}{\frac{x_t A'(x_t)}{A(x_t)} - 1} < 0$$

because $x_t A'(x_t) < A(x_t)$ under Assumption 1. This also means that $A(x_t)/x_t$ is decreasing in x_t. Since $A(+\infty) < +\infty$ under Assumption 1, $\lim_{x_t \to +\infty} \frac{A(x_t)}{x_t} < \frac{\pi_t}{g_1}$. There is a unique solution $x_t = \varepsilon(\pi_t)$ to the equation $\frac{\pi_t}{g_1} = \frac{A(x_t)}{x_t}$ if $\lim_{x_t \to 0} \frac{A(x_t)}{x_t} > \frac{\pi_t}{g_1}$.

B. Proof of Proposition 1

There exist (at least) two BGPs if there are two solutions $\pi \in (\pi_1, \pi_2)$ to the equation $B_1(\pi) = B_2(\pi)$. Since $B_1(\pi_1) > B_2(\pi_1)$ and $B_1(\pi_2) > B_2(\pi_2)$, it requires the existence of at least one value of $\pi \in (\pi_1, \pi_2)$ such that $B_1(\pi) < B_2(\pi)$.

Using equations (30) and (31), the inequality $B_1(\pi) < B_2(\pi)$ is equivalent to:

$$\Omega(\pi) > g - \tau_L(1-\alpha) - \tau_K \alpha \tag{B.1}$$

with

$$\Omega(\pi) \equiv \frac{a(\pi)}{R(\pi)} \left[X(\pi) - \frac{1-\delta}{a(\pi)} - (1-\tau_K)\alpha \right] \left[\Sigma(1-\tau_L)(1-\alpha) + \tau_L(1-\alpha) + \tau_K \alpha - g - X(\pi) \right] \tag{B.2}$$

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By construction, we have $\Omega(\pi_1)=\Omega(\pi_2)=0$ and $\Omega(\pi)>0$ for all $\pi\in(\pi_1,\pi_2)$. Let us consider $\widetilde{\pi}=\lambda\pi_1$, with $\lambda\in(1,\pi_2/\pi_1)$ a constant independent of g. Then, $\widetilde{\pi}$ does not depend on g and $\Omega(\widetilde{\pi})>0$. When g tends to $\tau_L(1-\alpha)+\tau_K\alpha$, inequality (B.1) evaluated at $\pi=\widetilde{\pi}$ is satisfied. By continuity, there exists $\bar{g}>\tau_L(1-\alpha)+\tau_K\alpha$ such that $\Omega(\widetilde{\pi})>g-\tau_L(1-\alpha)-\tau_K\alpha$ for all $g\in(\tau_L(1-\alpha)+\tau_K\alpha,\bar{g})$. This proves the existence of two solutions π_I and π_{II}, with $\pi_1<\pi_I<\pi_{II}<\pi_2$. Since $B_1(\pi)$ and $B_2(\pi)$ are decreasing functions, the associated stationary values b_I and b_{II} are ranked in the following way: $b_I>b_{II}$.

C. Proof of Proposition 2

The dynamic system we consider is given by equation (43) and equation (25) which rewrites:

$$\pi_{t+1}=\frac{X(\pi_t)\pi_t}{\Sigma(1-\tau_L)(1-\alpha)+(\tau_L(1-\alpha)+\tau_K\alpha)-g-\frac{R(\pi_t)}{a(\pi_t)}\pi_t\Phi_t}\tag{C.1}$$

Let us note $R(\pi)/a(\pi)=(1-\delta)/a(\pi)+(1-\tau_K)\alpha\equiv\widetilde{R}(\pi)$. We have $X'(\pi)>\widetilde{R}'(\pi)$ and, using (32), $X(\pi)>\widetilde{R}(\pi)$.

Differentiating equation (43), we get:

$$\frac{d\Phi_{t+1}}{\Phi}=\frac{\widetilde{R}(\pi)}{X(\pi)}\frac{d\Phi_t}{\Phi}+\left[\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)}\frac{\widetilde{R}(\pi)}{X(\pi)}-\frac{g-\tau_L(1-\alpha)-\tau_K\alpha}{\pi\Phi X(\pi)}-\frac{X'(\pi)\pi}{X(\pi)}\right]\frac{d\pi_t}{\pi}\tag{C.2}$$

Differentiating (C.1), we get:

$$\frac{d\pi_{t+1}}{\pi}=\frac{R(\pi)\pi\Phi}{\gamma}\frac{d\Phi_t}{\Phi}+\left[1+\frac{X'(\pi)\pi}{X(\pi)}+\frac{R(\pi)\pi\Phi}{\gamma}\left(\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)}+1\right)\right]\frac{d\pi_t}{\pi}\tag{C.3}$$

The trace T and the determinant D of the associated Jacobian matrix are given by:

$$T=\frac{\widetilde{R}(\pi)}{X(\pi)}+1+\frac{X'(\pi)\pi}{X(\pi)}+\frac{R(\pi)\pi\Phi}{\gamma}\left(\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)}+1\right)>1\tag{C.4}$$

$$D=\frac{\widetilde{R}(\pi)}{X(\pi)}\left(1+\frac{X'(\pi)\pi}{X(\pi)}+\frac{R(\pi)\pi\Phi}{\gamma}\right)+\frac{R(\pi)\pi\Phi}{\gamma}\left(\frac{g-\tau_L(1-\alpha)-\tau_K\alpha}{\pi\Phi X(\pi)}+\frac{X'(\pi)\pi}{X(\pi)}\right)\tag{C.5}$$

and define the characteristic polynomial $P(\lambda)\equiv\lambda^2-T\lambda+D=0$. We have $P(0)=D>0$. Since $P(-\infty)=+\infty$, $P(+\infty)=+\infty$, and $T>1$, the two roots are positive or complex conjugates.

Using $\widetilde{R}(\pi)=R(\pi)/a(\pi)$, $\gamma=a(\pi)X(\pi)$ and equation (43) at a steady state, T and D rewrite:

$$T=\frac{\widetilde{R}(\pi)}{X(\pi)}+1+\frac{X'(\pi)\pi}{X(\pi)}+\frac{\widetilde{R}(\pi)\pi\Phi}{X(\pi)}\left(\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)}+1\right)>1\tag{C.6}$$

$$\begin{aligned}D&=\frac{\widetilde{R}(\pi)}{X(\pi)}\left(1+\frac{X'(\pi)\pi}{X(\pi)}+\frac{\widetilde{R}(\pi)\pi\Phi}{X(\pi)}\right)+\frac{\widetilde{R}(\pi)\pi\Phi}{X(\pi)}\left(1-\frac{\widetilde{R}(\pi)}{X(\pi)}+\frac{X'(\pi)\pi}{X(\pi)}\right)\\&=\frac{\widetilde{R}(\pi)}{X(\pi)}\left(1+\frac{X'(\pi)\pi}{X(\pi)}\right)(1+\pi\Phi)\end{aligned}\tag{C.7}$$

We deduce that:

$$\begin{aligned}P(1)&=1-T+D\\&=-\left(1-\frac{\widetilde{R}(\pi)}{X(\pi)}\right)\frac{X'(\pi)\pi}{X(\pi)}+\frac{\widetilde{R}(\pi)\pi\Phi}{X(\pi)}\left(\frac{X'(\pi)\pi}{X(\pi)}-\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)}\right)\end{aligned}\tag{C.8}$$

Using (44) and (45), we rewrite $B_3(\pi)$ and $B_4(\pi)$ as follows:

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$$B_3(\pi) = \frac{g - \tau_L(1-\alpha) - \tau_K\alpha}{\pi\,[X(\pi) - \widetilde{R}(\pi)]} \tag{C.9}$$

$$B_4(\pi) = \frac{1}{\widetilde{R}(\pi)\pi}\left[\Sigma(1-\tau_L)(1-\alpha) + \tau_L(1-\alpha) + \tau_K\alpha - g - X(\pi)\right] \tag{C.10}$$

We deduce that:

$$\frac{B_3'(\pi)\pi}{B_3(\pi)} = -1 - \frac{X'(\pi) - \widetilde{R}'(\pi)}{X(\pi) - \widetilde{R}(\pi)}\pi \tag{C.11}$$

$$\frac{B_4'(\pi)\pi}{B_4(\pi)} = -1 - \frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)} - \frac{X(\pi)}{\Phi\widetilde{R}(\pi)\pi}\frac{X'(\pi)\pi}{X(\pi)} \tag{C.12}$$

After some computations, we can show that $\frac{B_3'(\pi)\pi}{B_3(\pi)} < \frac{B_4'(\pi)\pi}{B_4(\pi)}$ is equivalent to:

$$\left(1 - \frac{\widetilde{R}(\pi)}{X(\pi)}\right)\left(-\frac{X'(\pi)\pi}{X(\pi)}\right) + \frac{\widetilde{R}(\pi)\pi\,\Phi}{X(\pi)}\left(\frac{X'(\pi)\pi}{X(\pi)} - \frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)}\right) > 0 \tag{C.13}$$

We recall that $B_3'(\pi_I) < B_4'(\pi_I)$ and $B_3'(\pi_{II}) > B_4'(\pi_{II})$. By inspection of equations (C.8) and (C.13), we deduce that $P(1) > 0$ at the steady state (π_I, Φ_I) and $P(1) < 0$ at the steady state (π_{II}, Φ_{II}). Therefore, the steady-state (π_{II}, Φ_{II}) is a saddle, with one eigenvalue between 0 and 1 and one higher than 1. The steady-state (π_I, Φ_I) is stable if $D < 1$ and unstable if $D > 1$.

Using (C.7) and $\pi\Phi = b$, $D < 1$ is equivalent to:

$$\widetilde{R}(\pi)(1+b) < \frac{X(\pi)}{1 + \frac{X'(\pi)\pi}{X(\pi)}} \tag{C.14}$$

Using the expression of $\widetilde{R}(\pi)$ and equation (31), the left-hand side of inequality (C.14) is given by:

$$\begin{aligned}\widetilde{R}(\pi)(1+b) &= \Sigma(\pi)(1-\tau_L)(1-\alpha) - [g - \tau_L(1-\alpha) - \tau_K\alpha] - [X(\pi) - \widetilde{R}(\pi)] \\ &< \Sigma(\pi)(1-\tau_L)(1-\alpha)\end{aligned} \tag{C.15}$$

which is arbitrarily low if the tax rate τ_L is high. Using $X(\pi) = \frac{1-m}{a(\pi)} - \frac{\psi g_2}{\pi} + \frac{\mu}{a(\pi)\pi}$, we also have:

$$\frac{X'(\pi)\pi}{X(\pi)} = \frac{-\frac{1-m}{a(\pi)}\epsilon_a(\pi) + \frac{\psi g_2}{\pi} - \frac{\mu}{a(\pi)\pi}(1+\epsilon_a(\pi))}{\frac{1-m}{a(\pi)} - \frac{\psi g_2}{\pi} + \frac{\mu}{a(\pi)\pi}} \tag{C.16}$$

Therefore, $X'(\pi)\pi/X(\pi)$ is not too high if $\epsilon_a(\pi)$ is not too negative.

We deduce that we have $D < 1$, which ensures the stability of the steady state (π_I, Φ_I), if τ_L is sufficiently high and close to 1 and $\epsilon_a(\pi)$ is not too negative. Note that at least when D is close to 1, the eigenvalues are complex conjugates, meaning that the dynamic path converges with oscillations around the steady state.

D. Proof of Proposition 3

The Jacobian matrix of the linearized dynamic system is given by:

$$J = \begin{pmatrix} \frac{\widetilde{R}(\pi)}{X(\pi)} & \frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)}\frac{\widetilde{R}(\pi)}{X(\pi)} - \frac{g-\tau_L(1-\alpha)-\tau_K\alpha}{\pi\Phi X(\pi)} - \frac{X'(\pi)\pi}{X(\pi)} \\ \frac{R(\pi)\pi\Phi}{\gamma} & 1 + \frac{X'(\pi)\pi}{X(\pi)} + \frac{R(\pi)\pi\Phi}{\gamma}\left(\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)} + 1\right) \end{pmatrix}$$

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Let $E_s = (1, e_s)$ be the eigenvector associated to the stable eigenvalue $\lambda_s \in (0, 1)$. We have $JE_s = \lambda_s E_s$. Using the second equation of this system, we deduce that:

$$\frac{R(\pi)\pi\Phi}{\gamma} = e_s\left[\lambda_s - \left[1 + \frac{X'(\pi)\pi}{X(\pi)} + \frac{R(\pi)\pi\Phi}{\gamma}\left(\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)} + 1\right)\right]\right]$$

Since we have:

$$\lambda_s < 1 < 1 + \frac{X'(\pi)\pi}{X(\pi)} + \frac{R(\pi)\pi\Phi}{\gamma}\left(\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)} + 1\right)$$

we deduce that $e_s < 0$. This means that the eigenvector associated with the stable eigenvalue has a negative slope. On the stable manifold, we have $\frac{d\Phi_t}{\Phi} / \frac{d\pi_t}{\pi} = 1/e_s < 0$ in the neighborhood of the steady state.

Let $E_u = (1, e_u)$ be the eigenvector associated to the unstable eigenvalue $\lambda_u > 1$. $det(J - \lambda_u I) = 0$ is equivalent to:

$$\left(\frac{\widetilde{R}(\pi)}{X(\pi)} - \lambda_u\right)\left[1 + \frac{X'(\pi)\pi}{X(\pi)} + \frac{R(\pi)\pi\Phi}{\gamma}\left(\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)} + 1\right) - \lambda_u\right]$$
$$= -\frac{R(\pi)\pi\Phi}{\gamma}\left[\frac{g - \tau_L(1-\alpha) - \tau_K\alpha}{\pi\Phi X(\pi)} + \frac{X'(\pi)\pi}{X(\pi)} - \frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)}\frac{\widetilde{R}(\pi)}{X(\pi)}\right]$$

Since $X'(\pi) > \widetilde{R}'(\pi)$, $X(\pi) > \widetilde{R}(\pi)$ and $\lambda_u > 1$, the right-hand side of this equation is strictly negative, which implies that:

$$\lambda_u < 1 + \frac{X'(\pi)\pi}{X(\pi)} + \frac{R(\pi)\pi\Phi}{\gamma}\left(\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)} + 1\right)$$

Using

$$\frac{R(\pi)\pi\Phi}{\gamma} = e_u\left[\lambda_u - \left[1 + \frac{X'(\pi)\pi}{X(\pi)} + \frac{R(\pi)\pi\Phi}{\gamma}\left(\frac{\widetilde{R}'(\pi)\pi}{\widetilde{R}(\pi)} + 1\right)\right]\right]$$

we deduce that $e_u < 0$, which means that the eigenvector associated to the unstable eigenvalue has a negative slope. On the unstable manifold, we have $\frac{d\Phi_t}{\Phi} / \frac{d\pi_t}{\pi} = 1/e_u < 0$ in the neighborhood of the steady state. Since $\lambda_u > \lambda_s$, we even have $1/e_u > 1/e_s$, which means that around the steady state, the negative slope of the unstable manifold is greater than the one of the stable manifold.

E. Proof of Proposition 4

Using the proof of Proposition 2, the steady state (π_{II}, Φ_{II}) is saddle. We know that the steady state (π_I, Φ_I) is unstable if $D > 1$, i.e.

$$\widetilde{R}(\pi)(1+b) > \frac{X(\pi)}{1 + \frac{X'(\pi)\pi}{X(\pi)}} \tag{E.1}$$

which requires τ_L sufficiently low and $\epsilon_a(\pi)$ sufficiently negative.

Using $X(\pi) = \frac{1-m}{a(\pi)} - \frac{\psi g_2}{\pi} + \frac{\mu}{a(\pi)\pi}$ and (C.16), the inequality $X'(\pi)\pi/X(\pi) > 1$ can be written:

$$-\frac{1-m}{a(\pi)}\epsilon_a(\pi) + \frac{\psi g_2}{\pi} - \frac{\mu}{a(\pi)\pi}(1 + \epsilon_a(\pi)) > \frac{1-m}{a(\pi)} - \frac{\psi g_2}{\pi} + \frac{\mu}{a(\pi)\pi}$$

which is equivalent to:

$$-\frac{1-m}{a(\pi)}(1 + \epsilon_a(\pi)) + 2\frac{\psi g_2}{\pi} - \frac{\mu}{a(\pi)\pi}(2 + \epsilon_a(\pi)) > 0$$

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This inequality is satisfied for $\epsilon_a(\pi) < -2$. Using the fact that $X'(\pi)\pi/X(\pi) > 1$, inequality (E.1) is satisfied if:

$$\widetilde{R}(\pi)b > \frac{X(\pi)}{2} - \widetilde{R}(\pi) \tag{E.2}$$

The left-hand side of this inequality is positive, whereas the right-hand side is strictly negative if $X(\pi) < 2\widetilde{R}(\pi)$, i.e.

$$-\frac{\psi g_2}{\pi} < \frac{1 - 2\delta + m}{a(\pi)} + 2(1 - \tau_K)\alpha - \frac{\mu}{a(\pi)\pi} \tag{E.3}$$

Since $a(\pi)\pi$ is decreasing, this last inequality is satisfied if $1 + m > 2\delta$ and $1 - \tau_K > \mu/[2\alpha a(\pi_2)\pi_2]$. It gives a sufficient condition to have $D > 1$.

F. Proof of Lemma 3

Using (26)–(28), the system that defines the stationary solutions can be rewritten as:

$$b_i = \frac{g_1 + g_2 - (\tau_L(1-\alpha) + \tau_K\alpha)}{z(\pi_i, g_1, g_2) - y(\pi_i, g_1, \tau_K)} \equiv \mathcal{B}_1(\pi_i, g_1, g_2, \tau_K, \tau_L) \tag{F.1}$$

$$\begin{aligned} b_i &= \frac{\left[\Sigma(1-\tau_L)(1-\alpha) + \tau_L(1-\alpha) + \tau_K\alpha - (g_1+g_2) - z(\pi_i, g_1, g_2)\right]}{y(\pi_i, g_1, \tau_K)} \\ &\equiv \mathcal{B}_2(\pi_i, g_1, g_2, \tau_K, \tau_L) \end{aligned} \tag{F.2}$$

with

$$y(\pi_i, g_1, \tau_K) = (1-\delta)/\Theta(\pi_i, g_1) + \alpha(1-\tau_K).$$

The sign of partial derivatives are: $y_{\pi_i} > 0$; $y_{g_1} < 0$; $y_{\tau_K} < 0$; and

$$z(\pi_i, g_1, g_2) = \frac{1-m}{\Theta(\pi_i, g_1)} - \frac{\psi g_2}{\pi_i} + \frac{\mu}{\Theta(\pi_i, g_1)\pi_i}.$$

The sign of partial derivatives are: $z_{\pi_i} > 0$; $z_{g_1} < 0$; $z_{g_2} < 0$.

Total differentiation of Equations (F.1) and (F.2) gives:

$$\mathcal{C} \times \begin{bmatrix} \mathrm{d}b \\ \mathrm{d}\pi \end{bmatrix} = \mathcal{D} \times \begin{bmatrix} \mathrm{d}g_1 \\ \mathrm{d}g_2 \\ \mathrm{d}\tau_L \\ \mathrm{d}\tau_K \end{bmatrix}$$

where

$$\mathcal{C} = \begin{bmatrix} 1 & -B_1'(\pi) \\ 1 & -B_2'(\pi) \end{bmatrix}$$

with $B_1'(\pi) = \frac{\partial \mathcal{B}_1(\pi, g_1, g_2, \tau_K, \tau_L)}{\partial \pi}$ and $B_2'(\pi) = \frac{\partial \mathcal{B}_2(\pi, g_1, g_2, \tau_K, \tau_L)}{\partial \pi}$ and where

$$\mathcal{D} = \begin{bmatrix} \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 & \mathcal{D}_4 \\ \mathcal{D}_5 & \mathcal{D}_6 & \mathcal{D}_7 & \mathcal{D}_8 \end{bmatrix}$$

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$$= \begin{bmatrix} \frac{1+\frac{b\Theta_{g_1}}{(\Theta)^2}(\delta-m+\mu/\pi)}{z-y} & \frac{\pi+\psi b}{\pi(z-y)} & \frac{\alpha-1}{z-y} & \frac{-\alpha(1+b)}{z-y} \\ \frac{-1+\frac{\Theta_{g_1}}{(\Theta)^2}(b(1-\delta)+\mu/\pi+1-m)}{y} & \frac{\psi-\pi}{\pi y} & \frac{(1-\alpha)(1-\Sigma)}{y} & \alpha\left(\frac{1+b}{y}\right) \end{bmatrix} \tag{F.3}$$

The determinant of matrix $\mathcal{C}$ is $\det \mathcal{C}(\pi) = B_1'(\pi) - B_2'(\pi)$. We recall that $B_1'(\pi_I) < B_2'(\pi_I)$ and $B_1'(\pi_{II}) > B_2'(\pi_{II})$. We thus have $\det \mathcal{C}(\pi_I) < 0$ and $\det \mathcal{C}(\pi_{II}) > 0$. Moreover, under Assumptions 1–3, we have $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_7, \mathcal{D}_8 > 0$, $\mathcal{D}_3, \mathcal{D}_4 < 0$. The sign of $\mathcal{D}_5$ and $\mathcal{D}_6$, referring to the impact of environmental policy variables, g_1 and g_2, depends on policy and model parameters and is discussed later.

We obtain the effects of policy variables on stationary variables examining:

$$\begin{bmatrix} \mathrm{d}b \\ \mathrm{d}\pi \end{bmatrix} = \frac{1}{\det \mathcal{C}(\pi)} \times \begin{bmatrix} -B_2'(\pi) & B_1'(\pi) \\ -1 & 1 \end{bmatrix} \times \mathcal{D} \times \begin{bmatrix} \mathrm{d}g_1 \\ \mathrm{d}g_2 \\ \mathrm{d}\tau_L \\ \mathrm{d}\tau_K \end{bmatrix}$$

- Effect of g_1

$$\frac{\mathrm{d}b}{\mathrm{d}g_1} = \frac{-B_2'(\pi)\mathcal{D}_1 + B_1'(\pi)\mathcal{D}_5}{\det \mathcal{C}(\pi)}$$

$$\frac{\mathrm{d}\pi}{\mathrm{d}g_1} = \frac{-\mathcal{D}_1 + \mathcal{D}_5}{\det \mathcal{C}(\pi)}$$

Using the expression for $\mathcal{D}_1$ and $\mathcal{D}_5$ given in (F.3), we have:

$$\frac{\mathrm{d}\pi}{\mathrm{d}g_1} = \left(\frac{-1+\frac{\Theta_{g_1}}{\Theta^2}\left(b(1-\delta)+\frac{\mu}{\pi}+1-m\right)}{y} - \frac{1+\frac{b\Theta_{g_1}}{\Theta^2}\left(\delta-m+\frac{\mu}{\pi}\right)}{z-y}\right) \times \frac{1}{\det \mathcal{C}(\pi)}$$

Rewriting the previous equation, we obtain:

$$\frac{\mathrm{d}\pi}{\mathrm{d}g_1} = \left(\frac{-z}{y(z-y)} + \frac{\Theta_{g_1}}{y\Theta^2}\left[b(1-\delta)+\frac{\mu}{\pi}+1-m - b\frac{y}{z-y}\left(\frac{\mu}{\pi}+\delta-m\right)\right]\right) \times \frac{1}{\det \mathcal{C}(\pi)} \tag{F.4}$$

Using (F.1) and (F.2), we have:

$$\frac{y}{z-y} < \frac{x(1-\tau_L)(1-\alpha)}{g-\tau_L(1-\alpha)-\tau_K\alpha}$$

The right-hand side of this inequality is small if τ_L is high and $g-\tau_L(1-\alpha)-\tau_K\alpha$ is not too small. In this case, the term into brackets in equation (F.4) is positive such that for Θ_{g_1} high enough, we have $\operatorname{sgn}\left(\frac{\mathrm{d}\pi_I}{\mathrm{d}g_1}\right) < 0$ and $\operatorname{sgn}\left(\frac{\mathrm{d}\pi_{II}}{\mathrm{d}g_1}\right) > 0$. Given the expression for $\frac{\mathrm{d}b}{\mathrm{d}g_1}$, we do not conclude concerning the impact of g_1 on b in this configuration. We can note that

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as long as Θ_{g_1} is low enough, $\mathcal{D}_5 < 0$. This implies $\frac{db}{dg_1} < 0$ and $\frac{d\pi}{dg_1} > 0$ for the BGP $i = I$ and the reverse for the BGP $i = II$.

- Effect of g_2

$$\frac{db}{dg_2} = \frac{-B_2'(\pi)\mathcal{D}_2 + B_1'(\pi)\mathcal{D}_6}{\det \mathcal{C}(\pi)}$$

$$\frac{d\pi}{dg_2} = \frac{-\mathcal{D}_2 + \mathcal{D}_6}{\det \mathcal{C}(\pi)} = \frac{1}{\pi}\left(\frac{\psi - \pi}{y} - \frac{\pi + \psi b}{z - y}\right)$$

Note that the two steady states exist with strictly positive values π_I and π_{II} even if ψ tends to 0. This means that there exists $\bar{\psi} > 0$ such that $\psi < \pi_I < \pi_{II}$ for all $\psi < \bar{\psi}$. In this case, we have $\mathcal{D}_6 < 0$. Then, an increase in g_2 means that π_I and b_{II} increases while π_{II} and b_I decrease.

- Effect of τ_L

$$\frac{db}{d\tau_L} = \frac{-B_2'(\pi)\mathcal{D}_3 + B_1'(\pi)\mathcal{D}_7}{\det \mathcal{C}(\pi)}$$

$$\frac{d\pi}{d\tau_L} = \frac{-\mathcal{D}_3 + \mathcal{D}_7}{\det \mathcal{C}(\pi)}$$

We have $\frac{d\pi}{d\tau_L} < 0$ and $\frac{db}{d\tau_L} > 0$ for the BGP $i = I$ and the reverse, $\frac{d\pi}{d\tau_L} > 0$ and $\frac{db}{d\tau_L} < 0$, for the BGP $i = II$.

- Effect of τ_K

$$\frac{db}{d\tau_K} = \frac{-B_2'(\pi)\mathcal{D}_4 + B_1'(\pi)\mathcal{D}_8}{\det \mathcal{C}(\pi)}$$

$$\frac{d\pi}{d\tau_K} = \frac{-\mathcal{D}_4 + \mathcal{D}_8}{\det \mathcal{C}(\pi)}$$

We have $\frac{d\pi}{d\tau_K} < 0$ and $\frac{db}{d\tau_K} > 0$ for the BGP $i = I$ and the reverse, $\frac{d\pi}{d\tau_K} > 0$ and $\frac{db}{d\tau_K} < 0$, for the BGP $i = II$.

G. Proof of Proposition 5

We examine how policy variables affect γ_I to deduce the welfare effect along the stable BGP $i = I$. Using (38), we have:

$$\gamma_I = 1 - m + \frac{\mu - \psi g_2 a(\pi_I)}{\pi_I} \tag{G.1}$$

The proof of Lemma 3 reveals that $\frac{d\pi_I}{d\tau_L} < 0$ and $\frac{d\pi_I}{d\tau_K} < 0$ and gives the conditions to have $\frac{d\pi_I}{dg_1} < 0$ or > 0. In the configuration where the fall in π_I is good for growth $\left(-\epsilon_a(\pi) < \frac{\mu - \psi g_2 a(\pi)}{\psi g_2 a(\pi)}\right)$, we directly have the policy scenarios for fiscal and adaptation policies that increase welfare.

As regards mitigation g_2, it exerts a direct negative effect on growth, in addition to its effect through π. Lemma 3 gives a condition such that $\frac{d\pi_I}{dg_2} > 0$, meaning that in the configuration where the fall in π_I is good for growth, the increase in g_2 has a double negative effect on welfare.

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