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A VANISHING THEOREM FOR THE η-INVARIANT AND HURWITZ GROUPS

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Abstract. In this paper we discuss a relationship between the spectral asymmetry and the surface symmetry. More precisely, we show that for every automorphism of a Hurwitz surface with the automorphism group $\mathrm{PSL}(2, \mathbb{F}_q)$, the η-invariant of the corresponding mapping torus vanishes if q is sufficiently large.

§1. Introduction

The *η-invariant* of the signature operator for an oriented closed Riemannian 3-manifold was introduced by Atiyah, Patodi and Singer in [2]. It can be regarded as the correction term of the Hirzebruch signature theorem when one applies it to a 4-manifold with boundary. Namely the η-invariant of a closed 3-manifold is equal to the integral of the first Pontryagin form minus the signature of a bounding 4-manifold, which allows us to compute it without using analytic tools. For a homeomorphism φ of an oriented closed surface Σ, we can construct the mapping torus $M_\varphi = \Sigma \times [0, 1]/(p, 1) \sim (\varphi(p), 0)$. In this paper we consider the case where φ is of finite order and endow M_φ with the metric which is induced from the standard metric of S^1 and φ-invariant metric of Σ. An explicit formula of the η-invariant of M_φ was first given by Meyerhoff–Ruberman in [13] using the Dedekind sum (see Section 2.1 for details). Another explicit formula using Meyer's signature cocycle was given by the author in [14].

Let X be a compact Riemann surface of genus $g \geqslant 2$ and $G = \mathrm{Aut}(X)$ the group of conformal automorphisms of X. A theorem of Hurwitz states that $|G|$ is bounded above by $84(g - 1)$. The surface X for which this bound is attained is called a *Hurwitz surface* and G is known as a *Hurwitz group*. It is

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also known that G is a Hurwitz group if and only if there is a homomorphism from the $(2,3,7)$ triangle group onto G (see Conder [5] for details). There are infinitely many values of g for which one can find a Hurwitz group of order $84(g-1)$ and the first values for such g are $3, 7$ and 14. The smallest Hurwitz group is $\mathrm{PSL}(2,\mathbb{F}_7)$ of order 168 which is the automorphism group of the *Klein surface* of genus 3. The next one is $\mathrm{PSL}(2,\mathbb{F}_8)$ of order 504 which corresponds to the *Macbeath surface* of genus 7 (see Macbeath [9]). In the case of genus 14, it is known that there are three nonisomorphic Hurwitz surfaces and G is isomorphic to $\mathrm{PSL}(2,\mathbb{F}_{13})$ of order 1092 (see Macbeath [10]).

In our previous papers [16, 19], we showed that the *reducibility* (see Section 2.4 for the definition) of automorphisms of the above Hurwitz surfaces with genera $g=3,7$ and 14 is characterized by vanishing of the η-invariant of the corresponding mapping torus (see Proposition 2.5). The purpose of the present paper is to prove the following vanishing theorem for the η-invariant and Hurwitz groups, which is a natural generalization of these results.

THEOREM 1.1. *Let X be a Hurwitz surface with $\mathrm{Aut}(X)=\mathrm{PSL}(2,\mathbb{F}_q)$. If q is sufficiently large, then every $\varphi\in\mathrm{PSL}(2,\mathbb{F}_q)$ is reducible and $\eta(M_\varphi)=0$.*

This paper is organized as follows. In the next section, we quickly review several basic facts about the fixed point data, the η-invariant of mapping tori with finite monodromies, conjugacy classes of $\mathrm{PSL}(2,\mathbb{F}_q)$ and Hurwitz groups. The proof of Theorem 1.1 will be given in Section 3.

§2. Preliminaries

In this section, we recall some basic notions which appeared in Theorem 1.1.

2.1 Fixed point data and the η-invariant

Let Σ_g be an oriented closed surface of genus g and $\mathcal{M}_g$ the mapping class group of Σ_g, the group of all isotopy classes of orientation preserving homeomorphisms of Σ_g. Let $\varphi:\Sigma_g\to\Sigma_g$ be a homeomorphism of order m. We denote the set of points of Σ_g at which $\langle\varphi\rangle\cong\mathbb{Z}/m$ does not act freely by $\mathrm{Fix}\langle\varphi\rangle$. Let $\{x_i\}$ be a set of representatives of the orbits of $\mathrm{Fix}\langle\varphi\rangle$ under $\langle\varphi\rangle$ and $\alpha_i=|\mathrm{stab}_{\langle\varphi\rangle}(x_i)|$, the order of the stabilizer at x_i. Then φ^{m/α_i} generates $\mathrm{stab}_{\langle\varphi\rangle}(x_i)$ so it acts faithfully by rotation on the tangent space at x_i. Let β_i be an integer such that $\varphi^{\beta_i m/\alpha_i}$ acts by rotation through $2\pi\sqrt{-1}/\alpha_i$.

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The integer β_i is well-defined modulo α_i and $(\alpha_i, \beta_i) = 1$, so that β_i/α_i is uniquely determined as an element of $\mathbb{Q}/\mathbb{Z}$. By the *fixed point data* of φ, we mean the collection $\sigma(\varphi) = \langle g, m \mid \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r\rangle$, where $\beta_i/\alpha_i \in \mathbb{Q}/\mathbb{Z}$ are not ordered. Moreover, the fixed point data satisfies the relation

$$\sum_{l=1}^{r} \frac{\beta_l}{\alpha_l} \equiv 0 \mod \mathbb{Z}.$$

Now, by using the fixed point data, the η-invariant of the mapping torus M_φ equipped with the metric as in Introduction is given by

$$\eta(M_\varphi) = -4 \sum_{l=1}^{r} s(\beta_l, \alpha_l), \tag{2.1}$$

where $s(\beta_l, \alpha_l)$ denotes the *Dedekind sum* and is defined by the following formula:

$$s(\beta, \alpha) = \sum_{l=1}^{\alpha-1} \left(\left(\frac{l}{\alpha}\right)\right) \left(\left(\frac{l\beta}{\alpha}\right)\right).$$

Here $((r)) \in \mathbb{R}$ is defined to be $r - [r] - 1/2$ if $r \notin \mathbb{Z}$ and 0 if $r \in \mathbb{Z}$, where $[r]$ is the greatest integer less than or equal to r.

As a basic property of the η-invariant, it is known that if two elements $\varphi, \psi \in \mathcal{M}_g$ are conjugate, then $\eta(M_\varphi) = \eta(M_\psi)$ holds. Moreover, for the inverse φ^{-1}, we have $\eta(M_{\varphi^{-1}}) = -\eta(M_\varphi)$.

Remark 2.1. The formula (2.1) is due to Meyerhoff and Ruberman [13, Theorem 1.3]. On the other hand, another explicit formula of $\eta(M_\varphi)$ using Meyer's signature cocycle [12] was proved in [14, Main Theorem]. See also [18, Theorem 3.1] for a formula of the η-invariant in terms of Meyer's function.

2.2 Conjugacy classes of $\mathrm{PSL}(2, \mathbb{F}_q)$

In this paper we use the same notations for conjugacy classes of $\mathrm{PSL}(2, \mathbb{F}_q)$ as in [1]. Let $\mathbb{F}_q$ be the finite field with $q = p^n$ elements where p is a prime number and $\mathrm{GL}(2, \mathbb{F}_q)$ the group of the 2×2 matrices over $\mathbb{F}_q$ with nonzero determinant. Let Z be the center of $\mathrm{GL}(2, \mathbb{F}_q)$, namely the subgroup consisting of all the scalar matrices in $\mathrm{GL}(2, \mathbb{F}_q)$, and $\mathrm{SL}(2, \mathbb{F}_q) = \{A \in \mathrm{GL}(2, \mathbb{F}_q) \mid \det A = 1\}$. Then $\mathrm{PSL}(2, \mathbb{F}_q)$ is defined to be

$$\mathrm{PSL}(2, \mathbb{F}_q) = \mathrm{SL}(2, \mathbb{F}_q)/Z \cap \mathrm{SL}(2, \mathbb{F}_q).$$

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Table 1.
Conjugacy classes of $\mathrm{PSL}(2,\mathbb{F}_q)$ for $q\equiv 1 \mod 4$.

Class	I	$c_2(1)$	$c_2(\Delta)$	$c_3(x)$	$c_3(\sqrt{-1})$	$c_4(z)$
Number	1	1	1	$\frac{q-5}{4}$	1	$\frac{q-1}{4}$
|Class|	1	$\frac{q^2-1}{2}$	$\frac{q^2-1}{2}$	$q(q+1)$	$\frac{q(q+1)}{2}$	$q(q-1)$
Order d	1	p	p	$d\ (\neq 2) \mid \frac{q-1}{2}$	2	$d \mid \frac{q+1}{2}$

Table 2.
Conjugacy classes of $\mathrm{PSL}(2,\mathbb{F}_q)$ for $q\equiv -1 \mod 4$.

Class	I	$c_2(1)$	$c_2(\Delta)$	$c_3(x)$	$c_4(\delta)$	$c_4(z)$
Number	1	1	1	$\frac{q-3}{4}$	1	$\frac{q-3}{4}$
|Class|	1	$\frac{q^2-1}{2}$	$\frac{q^2-1}{2}$	$q(q+1)$	$\frac{q(q-1)}{2}$	$q(q-1)$
Order d	1	p	p	$d \mid \frac{q-1}{2}$	2	$d\ (\neq 2) \mid \frac{q+1}{2}$

It is known that the order of $\mathrm{GL}(2,\mathbb{F}_q)$ is $(q+1)q(q-1)^2$ and $|\mathrm{SL}(2,\mathbb{F}_q)| = (q+1)q(q-1)$. Moreover, the order of $\mathrm{PSL}(2,\mathbb{F}_q)$ is $(q+1)q(q-1)/2$ if q is odd and $(q+1)q(q-1)$ if q is even.

Let $\mathbb{E}$ be the unique quadratic extension of $\mathbb{F}_q$. If q is odd, we choose $\Delta\in\mathbb{F}_q^*-(\mathbb{F}_q^*)^2$ and write $\mathbb{E}=\mathbb{F}_q(\delta)$ where $\delta=\sqrt{\Delta}$. For $z\in\mathbb{E}^*$, let $\overline{z}=z^q$. This is the action of the nontrivial element of the Galois group of $\mathbb{E}$ over $\mathbb{F}_q$. The *norm map* $N:\mathbb{E}^*\to\mathbb{F}_q^*$ is defined to be $N(z)=z\overline{z}=z^{q+1}\in\mathbb{F}_q$. Let $\mathbb{E}^1$ be the kernel of the norm map, which has order $q+1$.

Now we assume that p is an odd prime number (namely q is odd). Then the conjugacy classes of $\mathrm{PSL}(2,\mathbb{F}_q)$ are described by using the following four types of matrices (see [1, Section 5] for details):

(i) $I=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$;
(ii) $c_2(\gamma)=\left(\begin{smallmatrix}1&\gamma\\0&1\end{smallmatrix}\right)$ $(\gamma\in\{1,\ \Delta\})$;
(iii) $c_3(x)=\left(\begin{smallmatrix}x&0\\0&x^{-1}\end{smallmatrix}\right)$ $(x\neq\pm1)$, $c_3(x)=c_3(-x)=c_3(x^{-1})=c_3(-x^{-1})$;
(iv) $c_4(z)=\begin{pmatrix}x&\Delta y\\y&x\end{pmatrix}$ $(z=x+\delta y\in\mathbb{E}^1,\ z\neq\pm1)$, $c_4(z)=c_4(\overline{z})=c_4(-z)=c_4(-\overline{z})$.

Using these matrices, we have all the conjugacy classes of $\mathrm{PSL}(2,\mathbb{F}_q)$ as in Tables 1 and 2 above. In Table 2, 'Number' of the class $c_4(z)$ is different from

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the one in the table in [1, Section 6.4] (probably $(q-7)/4$ appeared there would be a typographic error). Here 'Order d' in Tables 1 and 2 means the order of each representative in $\mathrm{PSL}(2,\mathbb{F}_q)$. For example in Table 1, $(c_2(1))^p = I$ and $(c_3(x))^d = I$ hold, where d $(\neq 2)$ is a divisor of $(q-1)/2$.

We easily see from Tables 1 and 2 that the maximum order d_0 of elements in $\mathrm{PSL}(2,\mathbb{F}_q)$ is $d_0 = (q+1)/2$ if $q = p^n$ $(n > 1)$ and $d_0 = p$ if $q = p$.

2.3 Hurwitz groups

In [10], Macbeath proved that the Hurwitz groups of type $\mathrm{PSL}(2,\mathbb{F}_q)$ satisfy the following conditions, where $q = p^n$ (p is a prime number).

PROPOSITION 2.2. (Macbeath [10]) *The finite group* $\mathrm{PSL}(2,\mathbb{F}_q)$ *is a Hurwitz group if and only if either*

(i) $q = 7$ *or*
(ii) $q = p \equiv \pm 1 \mod 7$ *or*
(iii) $q = p^3$ *where* $p \equiv \pm 2$ *or* $\pm 3 \mod 7$.

In cases (i) and (iii) there is only one Riemann surface on which $\mathrm{PSL}(2,\mathbb{F}_q)$ *acts as a Hurwitz group. In case (ii) there are three Riemann surfaces for each* q.

By Proposition 2.2, we see that there are infinitely many Riemann surfaces that admit Hurwitz groups as their automorphism groups. As mentioned in Introduction, first three Hurwitz groups are of the form $\mathrm{PSL}(2,\mathbb{F}_q)$, namely $q = 7, 8$ and 13 which correspond to Riemann surfaces of genera $g = 3, 7$ and 14. The next one is of order 1344 which acts on a surface of genus 17, but in this case, it is known that G is isomorphic to an extension of the abelian group $(\mathbb{Z}/2)^3$ by $\mathrm{PSL}(2,\mathbb{F}_7)$. See Conder [4] for details.

Next we consider the number of fixed points $\nu(\varphi) = \#\mathrm{Fix}\langle\varphi\rangle$ for elements of Hurwitz groups $\mathrm{PSL}(2,\mathbb{F}_q)$. The following proposition is a special case of [11, Theorem 2].

PROPOSITION 2.3. (Macbeath [11]) *Let* $G = \mathrm{PSL}(2,\mathbb{F}_q)$ *be one of the Hurwitz groups as in Proposition 2.2. If* $\varphi \in G$ *has order* $d > 1$ *and* $q = p^n$

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is odd, then

$$\nu(\varphi)=\begin{cases}(q-1)\displaystyle\sum_{d\,|\,m_i}\frac{1}{m_i} & \text{if}\quad d\,\Big|\,\dfrac{q-1}{2};\\ (q+1)\displaystyle\sum_{d\,|\,m_i}\frac{1}{m_i} & \text{if}\quad d\,\Big|\,\dfrac{q+1}{2};\\ \dfrac{(n,2)}{2}p^{n-1}(p-1)\displaystyle\sum_{m_i=p}1 & \text{if}\quad d=p,\end{cases}$$

where $(n,2)$ *denotes the greatest common divisor of* n *and* 2*, and* $\{m_1,m_2,m_3\}=\{2,3,7\}$.

Using Proposition 2.3, we can evaluate the number of fixed points explicitly. Moreover, we can easily see that an automorphism of order d in $\mathrm{PSL}(2,\mathbb{F}_q)$ has no fixed point when $d\neq 2,3,7$. This is a key fact for our purpose.

2.4 Some known results

An *essential* 1-*submanifold* of Σ_g is a disjoint union of simple closed curves in Σ_g each component of which does not bound a 2-disk in Σ_g, and no two components of which are homotopic. A homeomorphism $\varphi:\Sigma_g\to\Sigma_g$ is *reducible* if it leaves some essential 1-submanifold of Σ_g invariant. An *irreducible* homeomorphism is one which is *not* reducible.

REMARK 2.4. By the classification of surface homeomorphisms due to Nielsen and Thurston (see [3]), there are three types of mapping classes: (1) finite order, (2) reducible and (3) pseudo-Anosov. We easily see that (1) and (2) have some overlap, although (3) does not have any intersection with (1) nor (2). It is known that there are some characterizations of the reducibility of $\varphi\in\mathcal{M}_g$ of finite order.

(i) Gilman shows in [7, Theorem 3.1] that $\varphi\in\mathcal{M}_g$ of finite order is irreducible if and only if the quotient orbifold $\Sigma_g/\langle\varphi\rangle$ is homeomorphic to the 2-sphere with three cone points.
(ii) Kasahara shows in [8, Theorem 4.1] that for $\varphi\in\mathcal{M}_g$ of order m, if φ is irreducible, then $m\geqslant 2g+1$; and if φ is reducible, then $m\leqslant 2g+2$, moreover if the genus g is odd, then $m\leqslant 2g$.

A characterization of reducible automorphisms of a Hurwitz surface by means of the η-invariant first appeared in [16]. This result was generalized to the Hurwitz surfaces with genera 7 and 14 in [19]. Namely we have:

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Proposition 2.5. ([16, Theorem 1], [19, Theorem 1.1]) *Let X be the Klein surface of genus 3 or the Macbeath surface of genus 7, or one of the three Hurwitz surfaces of genus 14. An automorphism φ of X is reducible if and only if the η-invariant of the corresponding mapping torus M_φ vanishes.*

Remark 2.6. For a surface with small genus, the same statement as in Proposition 2.5 holds (see [15, Theorem 4.1, Remark 4.2]).

However, this kind of theorem does not hold in general (see [17, Theorem 3.1] for instance). The next example was first pointed out by Toshiyuki Akita.

Example 2.7. For any odd prime number p, there exists an automorphism φ of order p acting on a Riemann surface Y with the genus $g = p\overline{g} + (p-1)/2$ and $\nu(\varphi) = 3$, where $\overline{g}$ is the genus of $Y/\langle\varphi\rangle$ (see [6, Chapter V]). Then we obtain $\eta(M_\varphi) \neq 0$. Actually φ has the fixed point data $\sigma(\varphi) = \langle g, p \mid 1/p, 1/p, (p-2)/p\rangle$ and by the formula (2.1) in Section 2.1, the η-invariant of M_φ is given by

$$\eta(M_\varphi) = -4\left\{s(1,p) + s(1,p) + s(p-2,p)\right\} = -8s(1,p) + 4s(2,p).$$

Here we have used the following property: if $\beta' \equiv \pm\beta \mod \alpha$, then $s(\beta', \alpha) = \pm s(\beta, \alpha)$. Moreover, using the *reciprocity law* for the Dedekind sum:

$$s(\beta, \alpha) + s(\alpha, \beta) = \frac{\alpha^2 + \beta^2 + 1 - 3\alpha\beta}{12\alpha\beta},$$

we have

$$s(1,p) = \frac{(p-1)(p-2)}{12p} \qquad \text{and} \qquad s(2,p) = \frac{(p-1)(p-5)}{24p},$$

because $s(p,1) = s(p,2) = 0$. Therefore, it follows that

$$\eta(M_\varphi) = -\frac{(p-1)^2}{2p} \neq 0.$$

By Remark 2.4(i), we can see that φ is irreducible if $\overline{g} = 0$ and reducible if $\overline{g} \neq 0$. Namely there exists a reducible automorphism $\varphi \in \mathrm{Aut}(Y)$ such that $\eta(M_\varphi) \neq 0$.

§3. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. Let us assume that q is odd (namely we assume that $p \neq 2$). See [19, Theorem 1.1] for the case of $q = 2^3 = 8$.

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3.1 Reducibility

As pointed out at the end of Section 2.2, the maximum order d_0 of elements in $\mathrm{PSL}(2,\mathbb{F}_q)$ is $(q+1)/2$ if $q=p^n$ $(n>1)$ and p if $q=p$.

For a Hurwitz surface X with genus g and $\mathrm{Aut}(X)=\mathrm{PSL}(2,\mathbb{F}_q)$, we have $84(g-1)=(q+1)q(q-1)/2$. If d_0 is less than $2g+1$, then we see from Remark 2.4(ii) that every element in $\mathrm{PSL}(2,\mathbb{F}_q)$ is reducible. In fact, for $q=p^3$ as in Proposition 2.2(iii), we have

$$\begin{aligned} 2g+1-d_0 &= \tfrac{1}{84}(q+1)q(q-1)+3-\tfrac{1}{2}(q+1) \\ &= \tfrac{1}{84}(q+1)\left\{q(q-1)-42\right\}+3>0, \end{aligned}$$

because $q\geqslant 27$. Similarly, for $q=p$ as in Proposition 2.2(ii), we have

$$\begin{aligned} 2g+1-d_0 &= \frac{1}{84}(q+1)q(q-1)+3-q \\ &= \frac{q}{84}\left\{(q+1)(q-1)-84\right\}+3>0, \end{aligned}$$

because $q\geqslant 13$. When $q=7$, a similar inequality does not hold. In fact, it is known that the Klein surface admits an irreducible automorphism of order 7 (see [16] for example).

Therefore, for a sufficiently large q, we can conclude that every automorphism of a Hurwitz surface X with $\mathrm{Aut}(X)=\mathrm{PSL}(2,\mathbb{F}_q)$ is reducible. This completes the proof of the first claim in Theorem 1.1.

3.2 Vanishing of the η-invariant

It is enough to show that the η-invariant of a mapping torus vanishes for each conjugacy class of $\mathrm{PSL}(2,\mathbb{F}_q)$ because of the conjugacy invariance of $\eta(M_\varphi)$.

Let $\varphi\in\mathrm{PSL}(2,\mathbb{F}_q)$ be of order $d>1$. As mentioned at the end of Section 2.3, if $d\neq 2,3,7$, then by Proposition 2.3 we have $\nu(\varphi)=0$. Since we are now assuming that q is sufficiently large (hence p is also large), for elements of order p we obtain $\nu(c_2(1))=\nu(c_2(\Delta))=0$. Hence we can conclude that $\eta(M_\varphi)=0$ by the formula (2.1).

Next for $\varphi=c_3(\sqrt{-1})$ or $c_4(\delta)$, it is easy to see that $\eta(M_\varphi)=0$ holds because they are involutions (see [14, Example 3.2] for instance).

Finally let us consider the case where $\varphi\in\mathrm{PSL}(2,\mathbb{F}_q)$ of order d is appeared as a subgroup of a cyclic group of order $(q-1)/2$ or $(q+1)/2$. Namely we assume that $\mathbb{Z}/d=\langle\varphi\rangle$ is $\langle c_3(x)\rangle$ or $\langle c_4(z)\rangle$. However, in these cases, we can check that φ is conjugate to φ^{-1} (see Section 2.2(iii) and (iv)).

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Therefore, we have $\eta(M_\varphi) = 0$ because $\eta(M_\varphi) = \eta(M_{\varphi^{-1}}) = -\eta(M_\varphi)$ holds. This completes the proof of Theorem 1.1.

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References

[1] J. Adams, *Character tables for* GL(2), SL(2), PGL(2) *and* PSL(2) *over a finite field*, preprint, 2002.
[2] M. F. Atiyah, V. K. Patodi and I. M. Singer, *Spectral asymmetry and Riemannian geometry I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.
[3] A. J. Casson and S. A. Bleiler, *Automorphisms of surfaces after Nielsen and Thurston*, Cambridge University Press, Cambridge, 1988.
[4] M. Conder, *The genus of compact Riemann surfaces with maximal automorphism group*, J. Algebra **108** (1987), 204–247.
[5] M. Conder, *Hurwitz groups: a brief survey*, Bull. Amer. Math. Soc. New Ser. **23** (1990), 359–370.
[6] H. M. Farkas and I. Kra, *Riemann surfaces*, 2nd ed., Graduate Texts in Mathematics **71**, Springer, New York, 1992.
[7] J. Gilman, *Structures of elliptic irreducible subgroups of the modular group*, Proc. Lond. Math. Soc. (3) **47** (1983), 27–42.
[8] Y. Kasahara, *Reducibility and orders of periodic automorphisms of surfaces*, Osaka J. Math. **28** (1991), 985–997.
[9] A. M. Macbeath, *On a curve of genus 7*, Proc. Lond. Math. Soc. (3) **15** (1965), 527–542.
[10] A. M. Macbeath, "*Generators of the linear fractional groups*", in *Number Theory, Proc. Sympos. Pure Math., Vol. XII (Houston, Tex., 1969)*, American Mathematical Society, Providence, RI, 1969, 14–32.
[11] A. M. Macbeath, *Action of automorphisms of a compact Riemann surface on the first homology group*, Bull. Lond. Math. Soc. **5** (1973), 103–108.
[12] W. Meyer, *Die Signatur von Flächenbündeln*, Math. Ann. **201** (1973), 239–264.
[13] R. Meyerhoff and D. Ruberman, *Cutting and pasting and the* η*-invariant*, Duke Math. J. **61** (1990), 747–761.
[14] T. Morifuji, *The* η*-invariant of mapping tori with finite monodromies*, Topology Appl. **75** (1997), 41–49.
[15] T. Morifuji, *On the reducibility and the* η*-invariant of periodic automorphisms of genus 2 surface*, J. Knot Theory Ramifications **6** (1997), 827–831.
[16] T. Morifuji, *A note on the reducibility of automorphisms of the Klein curve and the* η*-invariant of mapping tori*, Proc. Amer. Math. Soc. **126** (1998), 1945–1947.
[17] T. Morifuji, *A remark on the* η*-invariant for automorphisms of hyperelliptic Riemann surfaces*, Bull. Austral. Math. Soc. **62** (2000), 177–182.
[18] T. Morifuji, *On Meyer's function of hyperelliptic mapping class groups*, J. Math. Soc. Japan **55** (2003), 117–129.
[19] T. Morifuji, *Reducibility of automorphisms of Hurwitz surfaces and the* η*-invariant*, Internat. J. Math. **25** (2014), 1450119 (8 pages).

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