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INEQUALITIES CONCERNING MAXIMUM MODULUS AND ZEROS OF RANDOM ENTIRE FUNCTIONS

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Abstract Let $f(z)=\sum_{j=0}^{\infty} a_j z^j$ be a transcendental entire function and let $f_\omega(z)=\sum_{j=0}^{\infty}\chi_j(\omega)a_j z^j$ be a random entire function, where $\chi_j(\omega)$ are independent and identically distributed random variables defined on a probability space $(\Omega,\mathcal{F},\mu)$. In this paper, we first define a family of random entire functions, which includes Gaussian, Rademacher and Steinhaus entire functions. We prove that, for almost all functions in the family and for any constant $C>1$, there exist a constant $r_0=r_0(\omega)$ and a set $E\subset[e,\infty)$ of finite logarithmic measure such that, for $r>r_0$ and $r\notin E$,

$$|\log M(r,f)-N(r,0,f_\omega)|\le (C/A)^{\frac{1}{B}}\ \log^{\frac{1}{B}}\ \log M(r,f)+\log\log M(r,f),\qquad a.s.$$

where A,B are constants, $M(r,f)$ is the maximum modulus and $N(r,0,f)$ is the integrated zero-counting function of f. As a by-product of our main results, we prove Nevanlinna's second main theorem for random entire functions. Thus, the characteristic function of almost all functions in the family is bounded above by an integrated counting function, rather than by two integrated counting functions as in the classical Nevanlinna theory. For instance, we show that, for almost all Gaussian entire functions f_ω and for any $\epsilon>0$, there is r_0 such that, for $r>r_0$,

$$T(r,f)\le N(r,0,f_\omega)+\left(\tfrac{1}{2}+\epsilon\right)\log T(r,f).$$

Keywords: random variables; random Taylor series; Gaussian; Rademacher and Steinhaus entire functions; value-distribution theory; Nevanlinna's second main theorem

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1. Introduction

Let f be a transcendental entire function of the form

$$f(z)=\sum_{j=0}^{\infty}a_jz^j, \tag{1.1}$$

where $z, a_j \in \mathbb{C}$.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, where $\mathcal{F}$ is a σ-algebra of subset of Ω and μ is a probability measure on $(\Omega, \mathcal{F})$. Along with the function (1.1), we consider the random functions on the probability space $(\Omega, \mathcal{F}, \mu)$ as follows:

$$f_\omega(z)=\sum_{j=0}^{\infty}\chi_j(\omega)a_jz^j, \tag{1.2}$$

where $z, a_j \in \mathbb{C}$, $\omega \in \Omega$, $\chi_j(\omega)$ $(j = 0, 1, 2, \ldots)$ are independent and identically distributed complex-valued random variables. Further, we assume that the expectation and variance of χ_j are zero and one, respectively. It is clear that $f_\omega(z)$ is an entire function for almost all $\omega \in \Omega$ (see [6]).

In general, we consider three cases regarding $\chi_j(\omega)$. Gaussian entire functions: χ_j $(j = 0, 1, \ldots)$ are complex-valued Gaussian random variables with standard Gaussian distribution; Rademacher entire functions: χ_j $(j = 0, 1, \ldots)$ are Rademacher random variables, which take the values ±1 with probability $1/2$ each; Steinhaus entire functions: $\chi_j = \mathrm{e}^{2\pi i\theta_j}$ $(j = 0, 1, \ldots)$ are Steinhaus random variables, where θ_j $(j = 0, 1, \ldots)$ are independent real-valued random variables with uniform distribution in the interval [0,1].

The study of random polynomials was initiated by Bloch and Pólya in 1932. Since then, there are a lot of publications on random polynomials. Moreover, the research on random transcendental entire functions, especially, on Gaussian, Rademacher and Steinhaus entire functions, has drawn a lot of attention, too (e.g. [1, 4, 5, 9, 10, 13, 14, 16, 17]). Recently, Nazarov *et al.* [13, 14] made a breakthrough on the logarithmic integrability of Rademacher Fourier series and obtained several important results on the distribution of zeros of Rademacher entire functions. Their results extended earlier work of Littlewood and Offord [7, 8]. Also, in 1982, Murai [12] proved the Nevanlinna defect identity for Rademacher entire functions. In 2000, Sun and Liu [18] obtained the Nevanlinna defect identity for $f(z) + X(\omega)g(z)$ (where f, g are entire, g is a small function of f and $X(\omega)$ is a non-degenerated complex-valued random variable). Later, Mohola and Filevych [9, 10] obtained Nevanlinna's second main theorem for Steinhaus entire functions.

In this paper, we first define a family $\mathcal{Y}$ of random entire functions, which includes Gaussian, Rademacher and Steinhaus entire functions. Thus, we can deal with these three classes of famous random entire functions all together. Then, we prove several inequalities concerning the maximum modulus $M(r, f)$, $\sigma(r, f)$ and the integrated counting function $N(r, a, f_\omega)$ for the random entire functions in the family $\mathcal{Y}$. These inequalities show that the zero-counting functions of almost all randomly perturbed functions f_ω are close to the maximum modulus of *f*, up to an error term. We also carefully treat the error terms in these inequalities. Our Lemma 4.3 verifies that the family $\mathcal{Y}$ includes Gaussian, Rademacher and Steinhaus entire functions. The ingredients in our proofs involve the techniques used by Nazarov–Nishry–Sodin, Mohola–Filevych and Offord. As a by-product of our results, we also establish Nevanlinna's second main theorems for random entire functions with a careful treatment of its error term. Thus, we obtain that the characteristic function of almost all functions in the family is bounded above by an integrated counting function, rather than by two integrated counting functions as in the classical Nevanlinna theory.

The paper is organized as follows. We devote § 2 to some preliminaries and previous results. In § 3, we state our main results and Nevanlinna's second main theorems for random entire functions. In § 4, we give some lemmas, which are needed in the proofs of our results, where Lemma 4.3 is one of the key lemmas in the section. In § 5, we first prove Theorem 3.1, with which, then, we prove a lemma that has its own interests and is needed in the proof of Theorem 3.2. All corollaries are proved in this section, too.

2. Preliminaries

Let X be a complex-valued random variable. We denote the expectation and the variance of X by $\mathbb{E}(X)$ and $\mathbb{V}(X)$, respectively. In particular, if X is either a standard complex-valued Gaussian random variable (its probability density function is $\mathrm{e}^{-|z|^2}/\pi$ with respect to Lebesgue measure m in the complex plane), or a Rademacher random variable or a Steinhaus random variable, then $\mathbb{E}(X) = 0$ and $\mathbb{V}(X) = \mathbb{E}(|X|^2) = 1$. We also denote the probability of an event A by $\mathbb{P}(A)$. For a set $E \subset [1, +\infty)$, we say E has a finite logarithmic measure if $\int_E 1/t\,\mathrm{d}t < +\infty$.

For the reader's convenience, we recall some standard notation in function theory and state some important theorems in Nevanlinna theory for meromorphic functions g in the complex plane $\mathbb{C}$. These notation and theorems will be used to prove new theorems in Nevanlinna theory as corollaries of our main results for random entire functions. In the sequel, the values of constants, such as C, C_1, r_0 and r_1, may be different in each appearance of these constants.

We define the proximity function of g by

$$m(r, g) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |g(re^{it})|\,\mathrm{d}t,$$

and for any $a \in \mathbb{C}$, we define

$$m(r, a, g) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|g(re^{it}) - a|}\,\mathrm{d}t,$$

and the (integrated) counting function of a-value of g by

$$N(r, a, g) = \int_0^r \frac{n(t, a, g) - n(0, a, g)}{t}\mathrm{d}t + n(0, a, g)\log r,$$

where $n(t,a,g)$ is the number of zeros of $g-a$ in the disk $D(0,t)$. For $a=\infty$, $N(r,\infty,g)$, sometimes expressed as $N(r,g)$, is called the counting function of poles of g. We denote the Nevanlinna characteristic function of g by

$$T(r,g) = m(r,g) + N(r,\infty,g),$$

and the maximum modulus of g by

$$M(r,g) = \max_{|z|=r} |g(z)|.$$

Theorem 2.1. (Jensen Formula, e.g. [2, 3]). *If g is a meromorphic function, then*

$$\log|c_g(0)| + N(r,0,g) = \frac{1}{2\pi}\int_0^{2\pi} \log|g(re^{it})|\,\mathrm{d}t,$$

where $c_g(0)$ is the first non-zero coefficient of Laurent series of g(z) in the neighbourhood of the point z = 0.

The Jensen formula also implies the so-called Nevanlinna's first main theorem.

Theorem 2.2. (First Main Theorem, e.g. [2, 3]). *Let g be a meromorphic function in the complex plane and $a \in \mathbb{C}$. Then*

$$T(r,a,g) = T(r,g) - \log|c_g(0)| + \epsilon(a,r),$$

where $|\epsilon(a,r)| \leq \log^+|a| + \log 2.$

There are many versions of the second main theorem in Nevanlinna theory. Here, when g is an entire function, we use the one with a better error term.

Theorem 2.3. (Second Main Theorem, e.g. [2, 3]). *Let g be an entire function in the complex plane and let d_j $(j=1,2)$ be two distinct complex numbers. Then*

$$T(r,g) \leq N(r,d_1,g) + N(r,d_2,g) + S(r,g)$$

for all large r outside a set E of finite Lebesgue measure, where the error term is

$$S(r,g) \leq \log T(r,g) + 2\,\log\log T(r,g) + O(1).$$

It is known (e.g. [2, 19]) that the coefficient 1 in the front of $\log T(r,g)$ in the inequality is the best possible, and, clearly, the term $O(1)$ depends on $c_g(0)$ and d_j.

We say that functions defined in Equation (1.2) have a certain property almost surely (a.s.) if there is a set $F \subset \Omega$ such that $\mu(F)=0$ and the functions with $\omega \in \Omega \setminus F$ possessing the said property.

For $\omega \in \Omega$, we define

$$\sigma^2(r, f_\omega) = \sum_{j=0}^{\infty} |a_j \chi_j(\omega)|^2 r^{2j} = \int_0^{2\pi} |f_\omega(r\,\mathrm{e}^{i\theta})|^2 \,\frac{\mathrm{d}\theta}{2\pi},$$

and $\sigma^2(r, f) = \sum_{j=0}^{\infty} |a_j|^2 r^{2j}$. Further, if $\mathbb{E}(\chi_j) = 0$ and $\mathbb{V}(\chi_j) = 1$, then

$$\sigma^2(r, f) = \mathbb{E}(|f_\omega(r\,\mathrm{e}^{i\theta})|^2) = \sum_{j=0}^{\infty} |a_j|^2 r^{2j}.$$

Set

$$\hat{f}_\omega(r\,\mathrm{e}^{i\theta}) \overset{def}{=} \frac{f_\omega(r\,\mathrm{e}^{i\theta})}{\sigma(r, f)} = \sum_{j=0}^{\infty} \chi_j(\omega) \frac{a_j r^j}{\sigma(r, f)} \mathrm{e}^{ij\theta} \overset{def}{=} \sum_{j=0}^{\infty} \chi_j(\omega) \widehat{a_j}(r) \mathrm{e}^{ij\theta}, \tag{2.1}$$

where $\sum_{j=0}^{\infty} |\widehat{a_j}(r)|^2 = 1$ for all r. Let

$$X_r = \frac{1}{2\pi} \int_0^{2\pi} |\log|\hat{f}_\omega(r\,\mathrm{e}^{i\theta})||\mathrm{d}\theta, \quad \text{for } r \in \mathbb{R}^+.$$

Definition 2.1. *Let f and f_ω be defined as in Equations (1.1) and (1.2), respectively. Then, the random entire function f_ω belongs to the family $\mathcal{Y}$ if and only if f_ω satisfies Condition Y, i.e., there are three positive constants A, B and C such that for all $r > 0$,*

$$\text{Condition } Y: \qquad \mathbb{E}(\exp(AX_r^B)) < C.$$

In § 4, we will prove that all Gaussian, Rademacher and Steinhaus entire functions are in family $\mathcal{Y}$. Indeed, if f_ω is Gaussian, Rademacher or Steinhaus, then f_ω satisfies Condition Y when we choose $A \in (0, 2)$ and $B = 1$; A is close to zero and $B = 1/6$; $A \in (0, 1/3)$ and $B = 1$, respectively.

Observe that if χ_j $(j = 0, 1, 2, \ldots)$ are standard complex-valued Gaussian random variables, then $\mathbb{E}(X_r)$ is a positive constant. Therefore, for any Gaussian entire function f_ω,

$$\sup_{r>0} \mathbb{E}(|N(r, 0, f_\omega) - \log \sigma(r, f)|) \le C,$$

where C is a constant.

In 2010 and 2012, Mahola and Filevych proved the following result, which can be regarded as a version of Nevanlinna's second main theorem.

Theorem 2.4. ([9, 10], Theorem 1). *Let f be an entire function as defined in Equation (1.1) and let $f_\omega(z)$ be a Steinhaus or a Gaussian entire function on* $(\Omega, \mathcal{F}, \mu)$

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of the form (1.2). Then, there is a set E of finite logarithmic measure on $(0,\infty)$ such that for every $a \in \mathbb{C}$, the inequality

$$\log\sigma(r,f) \le N(r,a,f_\omega) + C_1 \log\log\sigma(r,f) + O(1) \quad a.s.\ (r \ge r_1(\omega,a), r \notin E),$$

holds, where $C_1 > 0$ is an absolute constant.

Remark 1. Mahola and Filevych [9] proved a similar inequality to that in Theorem 2.4 for the Steinhaus entire functions. In 2012, they proved Theorem 2.4 and other interesting results in [10] for the Steinhaus entire functions. Further, in 2012, Filevych stated that the inequality in Theorem 2.4 is also true for the Gaussian entire functions. Recently, Filevych told one of the authors that although the proof of the statement has not been published, it is essentially a repetition of the considerations from Mahola's Ph.D. dissertation [11].

Nazarov, Nishry and Sodin proved

Theorem 2.5. ([14], Theorem 1.1). *Let f_ω be a Rademacher entire function. There exists a set $E \subset [1,\infty)$ (depending on $|a_k|$ only) of finite logarithmic length such that*

(i) *for almost every $\omega \in \Omega$, there exists $r_0(\omega) \in [1,\infty)$ such that for every $r \in [r_0(\omega),\infty) \setminus E$ and every $\gamma > 1/2$,*

$$|n(r,0,f_\omega) - r\,\frac{\mathrm{d}}{\mathrm{d}r}\log\sigma(r,f)| \le C(\gamma)\left(r\,\frac{\mathrm{d}}{\mathrm{d}r}\,\log\sigma(r,f)\right)^{\gamma};$$

(ii) *for every $r \in [1,\infty) \setminus E$ and every $\gamma > 1/2$,*

$$\mathbb{E}|n(r,0,f_\omega) - r\,\frac{\mathrm{d}}{\mathrm{d}r}\log\sigma(r,f)| \le C(\gamma)\left(r\,\frac{\mathrm{d}}{\mathrm{d}r}\,\log\sigma(r,f)\right)^{\gamma}.$$

3. Our results

In this section, we state several inequalities concerning the maximum modulus $M(r,f)$, $\sigma(r,f)$ and the integrated counting function $N(r,0,f_\omega)$ for the random entire functions in the family $\mathcal{Y}$ with careful treatment of their error terms. A relationship between $\log\sigma(r,f_\omega)$ and $\log\sigma(r,f)$ is stated and proved in § 5.

Theorem 3.1. *If $f_\omega \in \mathcal{Y}$, then, for any constant $C > 1$, there exists a constant $r_0 = r_0(\omega)$ such that, for $r > r_0$,*

$$|\log\sigma(r,f) - N(r,0,f_\omega)| \le (C/A)^{1/B}\,\log^{1/B}\,\log\sigma(r,f), \qquad a.s.$$

where the constants A and B are from Condition Y.

Remark 2. Theorem 3.1 tells us that the number of zeros of almost all f_ω can be controlled from above and below by $\log\sigma(r,f)$ and an error term, which are independent of ω.

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Sometimes, it is easier for one to calculate $M(r,f)$ rather than $\sigma(r,f)$. By Lemma 4.6, we obtain:

Corollary 3.1. *If $f_\omega \in \mathcal{Y}$, then, for any constant $C>1$, there are a constant $r_0 = r_0(\omega)$ and a set $E \subset [e,\infty)$ of finite logarithmic measure such that, for $r > r_0$ and $r \notin E$,*

$$|\log M(r,f) - N(r,0,f_\omega)| \le (C/A)^{1/B} \log^{1/B} \log M(r,f) + \log\log M(r,f), \qquad a.s.,$$

where the constants A and B are from Condition Y.

Example. Let $f(z) = e^z$ and its random perturbation function f_ω in the family $\mathcal{Y}$. Then, the corollary tells us that, for almost all f_ω, its integrated zero-counting function in the disk $D(0,r)$ is close to r although e^z does not take the value zero at all.

Now, we state Nevanlinna's second main theorem (involving the integrated zero-counting function only) for random entire functions as corollaries of above results.

Theorem 3.1 and Lemma 4.4 easily imply the next corollary.

Corollary 3.2. *If $f_\omega \in \mathcal{Y}$, then, for any constant $C>1$, there exists a constant $r_0 = r_0(\omega)$ such that, for $r > r_0$,*

$$T(r,f) \le N(r,0,f_\omega) + (C/A)^{1/B} \log^{1/B} T(r,f), \qquad a.s.$$

and

$$T(r,f_\omega) \le N(r,0,f_\omega) + (C/A)^{1/B} \log^{1/B} T(r,f_\omega), \qquad a.s.$$

where the constants A and B are from Condition Y.

When f_ω is a Gaussian, or Rademacher or Steinhaus entire function, we have the following corollary.

Corollary 3.3. *Let f and f_ω be defined as in Equations (1.1) and (1.2), respectively. Then, for any $\epsilon>0$, there exists $r_0 = r_0(\omega,\epsilon)$ such that, for $r > r_0$,*

(i) *if f_ω is a Gaussian entire function, then*

$$T(r,f) \le N(r,0,f_\omega) + \frac{1+\epsilon}{2} \log T(r,f) \qquad a.s.$$

(ii) *if f_ω is a Rademacher entire function, then*

$$T(r,f) \le N(r,0,f_\omega) + \left(\left(\frac{eC_0}{6}\right)^6 + \epsilon\right) \log^6 T(r,f) \qquad a.s.,$$

where the constant C_0 is from Lemma 4.1.

(iii) *if f_ω is a Steinhaus entire function, then*

$$T(r,f) \le N(r,0,f_\omega) + (3+\epsilon)\log T(r,f) \qquad a.s.$$

Now we consider the case when f_ω takes any value $a \in \mathbb{C}$.

Theorem 3.2. *Let $f_\omega \in \mathcal{Y}$ and define*

$$f_\omega^*(z) = zf_\omega'(z) = \sum_{j=1}^{\infty} ja_j\chi_j z^j.$$

If f_ω^* satisfies Condition Y (maybe with different constants A and B), then, for any constant $C > 1$, there exists a set E of finite logarithmic measure such that, for every $a \in \mathbb{C}$, there is $r_1 = r_1(\omega, a)$ such that, for $r > r_1$ and $r \notin E$,

$$|\log\sigma(r,f) - N(r,a,f_\omega)| \le (C/A)^{1/B} \log^{1/B}\log\sigma(r,f) + (1+o(1))\log\log\sigma(r,f), \qquad a.s.$$

where the constants A and B are from Condition Y.

Remark 3. The first error term of the above inequality only appears in the lower bound for $N(r,a,f_\omega)$. In addition, if f is a Gaussian, or Rademarcher or Steinhaus entire function, then it follows from Lemma 4.3 that both f_ω and f_ω^* satisfy Condition Y.

The following corollary is a straightforward consequence of the above theorem and Lemma 4.6.

Corollary 3.4. *Under the assumptions of Theorem 3.2, we have that, for any constant $C > 1$, there exists a set E of finite logarithmic measure such that, for every $a \in \mathbb{C}$, there is $r_1 = r_1(\omega, a)$ such that, for $r > r_1$ and $r \notin E$,*

$$|\log M(r,f) - N(r,a,f_\omega)| \le (C/A)^{1/B} \log^{1/B}\log M(r,f) + (2+o(1))\log\log M(r,f), \ a.s.,$$

where the constants A and B are from Condition Y.

When f_ω is Gaussian, Rademacher or Steinhuas, Theorem 3.2 and Lemma 4.3 give:

Corollary 3.5. *Let f and f_ω be defined as in Equations (1.1) and (1.2), respectively. Then, for any $\epsilon > 0$, there exists a set E of finite logarithmic measure such that, for every $a \in \mathbb{C}$, there exists $r_0 = r_0(\omega, \epsilon, a)$ such that, for $r > r_0$ and $r \notin E$, we have:*

(i) *if f_ω is a Gaussian entire function, then*

$$\log\sigma(r,f) \le N(r,a,f_\omega) + (3/2+\epsilon)\log\log\sigma(r,f) \qquad a.s.$$

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(ii) *if f_ω is a Rademacher entire function, then*

$$\log\sigma(r,f) \le N(r,a,f_\omega) + \left(\left(\frac{eC_0}{6}\right)^6 + \epsilon\right)\log^6\log\sigma(r,f) \qquad a.s.,$$

where the constant C_0 is from Lemma 4.1.

(iii) *if f_ω is a Steinhaus entire function, then*

$$\log\sigma(r,f) \le N(r,a,f_\omega) + (4+\epsilon)\log\log\sigma(r,f) \qquad a.s.$$

Remark 4. Corollary 3.5 shows that the constant in the error term is $3/2+\epsilon$ and $2+\epsilon$ in Gaussian and Steinhaus cases, rather than a constant $C_1>0$ in Theorem 2.4. It is interesting to know whether these coefficients are the best possible coefficients in these error terms.

The following is Nevanlinna's second main theorem for random entire functions. It verifies that the characteristic function for almost all random entire functions can be bounded above by one integrated counting function, rather than two integrated counting functions as in the classical case (e.g. Theorem 2.3). The proof of the following corollary is a straightforward consequence of Theorem 3.2 and Lemma 4.4 as we have seen the proof of Corollary 3.2.

Corollary 3.6. *If f_ω and f^*_ω satisfy Condition Y, then, for any constant $C>1$, there exists a set E of finite logarithmic measure such that, for every $a\in\mathbb{C}$, there is $r_1 = r_1(\omega,a)$ such that, for $r>r_1$ and $r\notin E$,*

$$T(r,f) \le N(r,a,f_\omega) + (C/A)^{1/B}\log^{1/B} T(r,f) + (1+o(1))\log T(r,f), \qquad a.s.,$$

and

$$T(r,f_\omega) \le N(r,a,f_\omega) + (C/A)^{1/B}\log^{1/B} T(r,f_\omega) + (1+o(1))\log T(r,f_\omega), \qquad a.s.,$$

where the constants A and B are from Condition Y.

4. Some lemmas

In this section, in order to prove our main results, we give several lemmas.

Lemma 4.1. (Log-integrability, [13]). *Let f_ω be a Rademarcher entire function. Then, for any $p\ge 1$,*

$$\mathbb{E}\left(\int_0^{2\pi} |\log|\hat{f}_\omega||^p \frac{\mathrm{d}\theta}{2\pi}\right) \le (C_0 p)^{6p},$$

where C_0 is an absolute constant.

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Lemma 4.2. (Offord [15]). *Let $f_\omega(z)$ be a Steinhaus entire function on $(\Omega, \mathcal{F}, \mu)$ of the form (1.2), and let $\hat{f}_\omega$ be of the form (2.1). For all $t \geq 0$, $\phi \in [0, 2\pi)$, set*

$$A^* = \{\omega \in \Omega : |\mathrm{Re}(\hat{f}_\omega)\cos\phi + \mathrm{Im}(\hat{f}_\omega)\sin\phi| < t\}.$$

Then,

$$\mathbb{P}(A^*) \leq C \max\left\{t, t^{1/3}\right\},$$

where C is an absolute constant.

Lemma 4.3. *Let $f_\omega(z) \in \mathcal{Y}$ and let $\hat{f}_\omega(re^{i\theta})$ be defined by Equation (2.1). Then for any positive constant C and all $x > 1$, there is a positive constant C_1 such that*

$$\mathbb{P}\left(X_r \geq \left(\frac{C}{A}\log x\right)^{1/B}\right) \leq \frac{C_1}{x^C}. \tag{4.1}$$

In particular, we have the following:

(i) *If f_ω is a Gaussian entire function, then for any $\tau > 0$, there is a constant $C_1 = C_1(\tau)$ such that*

$$\mathbb{P}\left(X_r \geq \frac{1+2\tau}{2}\log x\right) \leq \frac{C_1}{x^{((1+2\tau)/(1+\tau))}}.$$

(ii) *If f_ω is a Rademacher entire function, then for any $\tau > 0$, $\epsilon \in (0, (6/(eC_0))^6)$ (C_0 is from Lemma 4.1), there is a constant $C_1 = C_1(\epsilon)$ such that*

$$\mathbb{P}\left(X_r \geq \left(\frac{1+\tau}{\epsilon}\right)^6 \log^6 x\right) \leq \frac{C_1}{x^{1+\tau}}.$$

(iii) *If f_ω is a Steinhaus entire function, then for any $\tau > 0$, there is a constant $C_1 = C_1(\tau)$ such that*

$$\mathbb{P}\big(X_r \geq 3(1+\tau)^2 \log x\big) \leq \frac{C_1}{x^{1+\tau}}.$$

Proof. By Markov's inequality, we obtain

$$\mathbb{P}\left(X_r \geq \left(\frac{C}{A}\log x\right)^{1/B}\right) \leq \frac{\mathrm{E}(\exp(AX_r^B))}{\exp(C\,\log x))} \overset{def}{=} \frac{C_1}{x^C}.$$

Now. we give the proof of (i). Since χ_j are independent standard complex-valued Gaussian random variables, then

$$\mathbb{E}(\hat{f}_\omega) = 0 \quad \text{and} \quad \mathbb{V}(\hat{f}_\omega) = \mathbb{E}(|\hat{f}_\omega|^2) = 1.$$

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It follows that $\hat{f}_\omega$ is a standard complex-valued Gaussian random variable. For any $x > 0$,

$$\mathbb{P}(|\log|\hat{f}_\omega|| < x) = \mathbb{P}(-x < \log|\hat{f}_\omega| < x) = \mathrm{e}^{-\mathrm{e}^{-2x}} - \mathrm{e}^{-\mathrm{e}^{2x}}.$$

Consequently, the probability density function of $|\log|\hat{f}_\omega||$ is $2\,\mathrm{e}^{-\mathrm{e}^{-2x}}\mathrm{e}^{-2x} + 2\mathrm{e}^{-\mathrm{e}^{2x}}\mathrm{e}^{2x}$ for $x > 0$ and is 0 for $x \leq 0$. It follows that the expected value $\mathbb{E}|\log|\hat{f}_\omega||$ is independent of θ. Thus, we have

$$\mathbb{E}(\mathrm{e}^{\frac{2}{1+\tau}X_r}) = \sum_{n=0}^{\infty} \frac{2^n}{n!(1+\tau)^n}\mathbb{E}X_r^n$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{n!(1+\tau)^n}\mathbb{E}\left(\frac{1}{2\pi}\int_0^{2\pi}|\log|\hat{f}_\omega||\mathrm{d}\theta\right)^n$$

$$\leq \sum_{n=0}^{\infty} \frac{2^n}{n!(1+\tau)^n}\left(\frac{1}{2\pi}\int_0^{2\pi}\mathbb{E}|\log|\hat{f}_\omega||^n\mathrm{d}\theta\right)$$

$$= \mathbb{E}\left(\mathrm{e}^{\frac{2}{1+\tau}|\log|\hat{f}_\omega||}\right)$$

$$= \int_0^{\infty} \mathrm{e}^{\frac{2x}{1+\tau}}\left(2\mathrm{e}^{-\mathrm{e}^{-2x}}\mathrm{e}^{-2x} + 2\mathrm{e}^{-\mathrm{e}^{2x}}\mathrm{e}^{2x}\right)\mathrm{d}x \overset{def}{=} C_1 < \infty, \tag{4.2}$$

where C_1 is a positive constant. It follows that Gaussian entire functions are in the family $\mathcal{Y}$ by taking $A = (2/(1+\tau))$ and $B = 1$. Set $C = ((1+2\tau)/(1+\tau))$. Then, by Equation (4.1) and for $x \geq 1$, we get

$$\mathbb{P}\left(X_r \geq \frac{1+2\tau}{2}\log x\right) \leq \frac{C_1}{x^{((1+2\tau)/(1+\tau))}}.$$

This completes the proof of (i).

Next, we prove (ii). By Lemma 4.1, we have, for any positive integer $n \geq 6$,

$$\mathbb{E}X_r^{n/6} = \mathbb{E}\left(\int_0^{2\pi}|\log|\hat{f}_\omega||\frac{\mathrm{d}\theta}{2\pi}\right)^{n/6} \leq \mathbb{E}\left(\int_0^{2\pi}|\log|\hat{f}_\omega||^{n/6}\frac{\mathrm{d}\theta}{2\pi}\right) \leq \left(\frac{C_0 n}{6}\right)^n,$$

where C_0 is the constant from Lemma 4.1. Thus, when $\epsilon \in (0, (6/eC_0)^6)$,

$$C_1 \overset{def}{=} \mathbb{E}\left(\exp\left(\epsilon X_r^{1/6}\right)\right) = \sum_{n=0}^{\infty}\frac{\mathbb{E}|\epsilon X_r|^{n/6}}{n!} \leq \sum_{n=6}^{\infty}\left(\frac{C_0\epsilon^{1/6}n}{6}\right)^n\frac{1}{n!} + O(1) < +\infty. \tag{4.3}$$

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Therefore, Rademacher entire functions satisfy the condition Y by choosing $A=\epsilon$ and $B=1/6$. Using the inequality (4.1) for $C=1+\tau$, we get

$$\mathbb{P}\left(X_r \geq \left(\frac{1+\tau}{\epsilon}\right)^6 \log^6 x\right) = \mathbb{P}\left(\exp\left(\epsilon X_r^{1/6}\right) \geq x^{1+\tau}\right) \leq \frac{\mathbb{E}\left(\exp\left(\epsilon X_r^{1/6}\right)\right)}{x^{1+\tau}} = \frac{C_1}{x^{1+\tau}}.$$

Now, we prove (iii).

For any non-negative integer j and any $\varphi \in [0,2\pi)$, set

$$b_j = \widehat{a_j}(r)\chi_j\,\mathrm{e}^{ij\theta} = \widehat{a_j}(r)\,\mathrm{e}^{ij\theta}\,\mathrm{e}^{i2\pi\theta_j} \quad \text{and} \quad B_j = \mathrm{Re}(b_j)\cos\varphi + \mathrm{Im}(b_j)\sin\varphi.$$

Thus, $\hat{f}_\omega = \sum_{j=0}^{\infty} b_j$, and B_j is a real random variable. Further, we deduce that $B_j = u\,\cos(2\pi\theta_j) + v\,\sin(2\pi\theta_j)$, where

$$\begin{aligned} u &= \mathrm{Re}(\widehat{a_j}(r))\cos(j\theta)\cos\varphi + \mathrm{Re}(\widehat{a_j}(r))\sin(j\theta)\sin\varphi \\ &\qquad - \mathrm{Im}(\widehat{a_j}(r))\sin(j\theta)\cos\varphi + \mathrm{Im}(\widehat{a_j}(r))\cos(j\theta)\sin\varphi, \\ v &= \mathrm{Re}(\widehat{a_j}(r))\cos(j\theta)\sin\varphi - \mathrm{Re}(\widehat{a_j}(r))\sin(j\theta)\cos\varphi \\ &\qquad - \mathrm{Im}(\widehat{a_j}(r))\cos(j\theta)\cos\varphi - \mathrm{Im}(\widehat{a_j}(r))\sin(j\theta)\sin\varphi, \end{aligned}$$

and $u^2+v^2=|b_j|^2$. The characteristic function of B_j is

$$\int_0^1 \exp[it(u\,\cos(2\pi x) + v\,\sin(2\pi x))]\,\mathrm{d}x,$$

which depends only on $|b_j|$. Similarly, we obtain that, for $\varphi \in [0,2\pi)$,

$$\mathrm{E}(|\log|\mathrm{Re}(\hat{f}_\omega)\cos\varphi + \mathrm{Im}(\hat{f}_\omega)\sin\varphi||^n)$$

is independent of φ for any non-negative integer n.

Since

$$\mathrm{Re}(\hat{f}_\omega)\cos\varphi + \mathrm{Im}(\hat{f}_\omega)\sin\varphi = \sqrt{\mathrm{Re}^2(\hat{f}_\omega) + \mathrm{Im}^2(\hat{f}_\omega)}\sin(\varphi+\varphi_0) = |\hat{f}_\omega|\sin(\varphi+\varphi_0),$$

(where $\sin\varphi_0 = \mathrm{Re}(\hat{f}_\omega)/|\hat{f}_\omega|$ and $\cos\varphi_0 = \mathrm{Im}(\hat{f}_\omega)/|\hat{f}_\omega|$), it follows that

$$\begin{aligned} \log|\hat{f}_\omega| &= \int_0^{2\pi} \log|\mathrm{Re}(\hat{f}_\omega)\cos\varphi + \mathrm{Im}(\hat{f}_\omega)\sin\varphi|\,\frac{\mathrm{d}\varphi}{2\pi} - \int_0^{2\pi} \log|\sin(\varphi+\varphi_0)|\,\frac{\mathrm{d}\varphi}{2\pi} \\ &= \int_0^{2\pi} \log|\mathrm{Re}(\hat{f}_\omega)\cos\varphi + \mathrm{Im}(\hat{f}_\omega)\sin\varphi|\,\frac{\mathrm{d}\varphi}{2\pi} + \log 2. \end{aligned}$$

This together with Jensen inequality gives, for any $A>0$,

$$\mathbb{E}(e^{AX_r}) = \sum_{n=0}^{\infty} \frac{A^n}{n!}\mathbb{E}X_r^n$$

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$$
\begin{aligned}
&\leq \mathbb{E}(\mathrm{e}^{A|\log|\hat{f_\omega}||}) \\
&\leq 2^A \mathbb{E}\left(\mathrm{e}^{A|(1/2\pi)\int_0^{2\pi}\log|\mathrm{Re}(\hat{f_\omega})\cos\varphi+\mathrm{Im}(\hat{f_\omega})\sin\varphi|\,\mathrm{d}\varphi|}\right) \\
&\leq 2^A \sum_{n=0}^{\infty}\frac{A^n}{n!}\frac{1}{2\pi}\int_0^{2\pi}\mathbb{E}(|\log|\mathrm{Re}(\hat{f}_\omega)\cos\varphi+\mathrm{Im}(\hat{f}_\omega)\sin\varphi||^n\,\mathrm{d}\varphi) \\
&\leq C_0\mathbb{E}\left(\mathrm{e}^{A|\log|\mathrm{Re}(\hat{f_\omega})\cos\varphi+\mathrm{Im}(\hat{f_\omega})\sin\varphi||}\right),
\end{aligned}
$$

where C_0 is a positive constant. For fixed $\theta\in(0,2\pi]$, let

$$V_\omega(\varphi,r)=|\mathrm{Re}(\hat{f}_\omega)\cos\varphi+\mathrm{Im}(\hat{f}_\omega)\sin\varphi|,$$

and

$$Y_1=\{\omega\in\Omega: V_\omega(\varphi,r)>1\},\qquad Y_2=\{\omega\in\Omega: V_\omega(\varphi,r)\leq 1\}.$$

Since $\mathbb{E}(|\hat{f}_\omega|^2)=1$, it follows that, for $0<A<2$,

$$
\begin{aligned}
\int_{Y_1}\mathrm{e}^{A\ \log V_\omega(\varphi,r)}\,\mathrm{d}\mathbb{P}(\omega) &= \int_{Y_1}|\mathrm{Re}(\hat{f}_\omega)\cos\varphi+\mathrm{Im}(\hat{f}_\omega)\sin\varphi|^A\,\mathrm{d}\mathbb{P}(\omega) \\
&\leq \int_{Y_1}|\hat{f}_\omega|^A\,\mathrm{d}\mathbb{P}(\omega)\leq\left(\int_\Omega|\hat{f}_\omega|^2\,\mathrm{d}\mathbb{P}(\omega)\right)^{A/2} \\
&= \left(\mathrm{E}(|\hat{f}_\omega|^2)\right)^{A/2}=1.
\end{aligned}
$$

On the set Y_2, by using Lemma 4.2, we have

$$
\begin{aligned}
\int_{Y_2}\mathrm{e}^{A|\log V_\omega(\varphi,r)|}\,\mathrm{d}\mathbb{P}(\omega) &= \int_0^\infty \mathbb{P}\left(\left\{\omega\in Y_2:\mathrm{e}^{A|\log V_\omega(\varphi,r)|}\geq\lambda\right\}\right)\mathrm{d}\lambda \\
&= \int_0^\infty \mathbb{P}\left(\left\{\omega\in Y_2:\mathrm{e}^{-A\ \log|\mathrm{Re}(\hat{f_\omega})\cos\varphi+\mathrm{Im}(\hat{f_\omega})\sin\varphi|}\geq\lambda\right\}\right)\mathrm{d}\lambda \\
&= \int_0^\infty \mathbb{P}\left(\left\{\omega\in Y_2:|\mathrm{Re}(\hat{f}_\omega)\cos\varphi+\mathrm{Im}(\hat{f}_\omega)\sin\varphi|\leq(1/\lambda)^{1/A}\right\}\right)\mathrm{d}\lambda \\
&\leq 1+\int_1^\infty \mathbb{P}\left(\left\{\omega\in \Omega:|\mathrm{Re}(\hat{f}_\omega)\cos\varphi+\mathrm{Im}(\hat{f}_\omega)\sin\varphi|\leq(1/\lambda)^{1/A}\right\}\right)\mathrm{d}\lambda \\
&\leq 1+C\int_1^\infty\frac{d\lambda}{\lambda^{1/3A}}.
\end{aligned}
$$

Thus, when $0<A<1/3$, we have

$$\int_{Y_2}\mathrm{e}^{A|\log V_\omega(\varphi,r)|}\,\mathrm{d}\mathbb{P}(\omega)\leq 1+C\int_1^\infty\frac{\mathrm{d}\lambda}{\lambda^{1/3A}}<+\infty.$$

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Therefore, setting $A = 1/3(1+\tau)$ as before, we obtain

$$\mathrm{E}\left(\mathrm{e}^{\frac{1}{3(1+\tau)}X_r}\right) \leq C_0 \int_{Y_1} \mathrm{e}^{A\,\log V_\omega(\varphi,r)}\,\mathrm{d}\mathbb{P}(\omega) + C_0 \int_{Y_2} \mathrm{e}^{A|\log V_\omega(\varphi,r)|}\,\mathrm{d}\mathbb{P}(\omega) = C_1 < +\infty. \tag{4.4}$$

It follows that

$$\mathbb{P}\left(X_r \geq 3(1+\tau)^2 \log x\right) \leq \frac{C_1}{x^{1+\tau}}.$$

This completes the proof of the lemma. □

Lemma 4.4. *Let f and f_ω be entire functions of the forms (1.1) and (1.2), respectively. Then, there is a constant $r_1 > 0$ such that, for $r > r_1$,*

$$T(r,f) \leq \log\sigma(r,f) + \frac{1}{2}\log 2 \quad \textit{and} \quad T(r,f_\omega) \leq \log\sigma(r,f_\omega) + \frac{1}{2}\log 2.$$

Proof. By Parseval identity and Jensen inequality, we obtain

$$\begin{aligned} T(r,f_\omega) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f_\omega\left(r\,\mathrm{e}^{i\theta}\right)|\mathrm{d}\theta \leq \frac{1}{4\pi}\int_0^{2\pi} \log\left(|f_\omega\left(r\,\mathrm{e}^{i\theta}\right)|^2 + 1\right)\mathrm{d}\theta \\ &\leq \log\sigma(r,f_\omega) + \frac{1}{2}\log 2. \end{aligned}$$

The other inequality in the lemma can be proved in the same manner. □

Lemma 4.5. (Plane Growth Lemma, e.g. [2], p. 100). *Let $F(r)$ be a positive, non-decreasing continuous function satisfying $F(r) \geq e$ for $e < r_0 < r < \infty$. Let $\psi(r) \geq 1$ be a real-valued, continuous, non-decreasing function on the interval $[e,\infty)$ and $\int_e^\infty (\mathrm{d}r/(r\psi(r))) < \infty$. Let $\phi(r)$ be a positive, non-decreasing function defined for $r_0 \leq r < \infty$. Set $R = r + \phi(r)/\psi(F(r))$. If $\phi(r) \leq r$ for all $r \geq r_0$, then there exists a closed set $E \subset [r_0,\infty)$ with $\int_E (\mathrm{d}r/\phi(r)) < \infty$ such that for all $r > r_0$, $r \notin E$, we have*

$$\log F(R) < \log F(r) + 1,$$

and

$$\log\frac{R}{r(R-r)} \leq \log\frac{\psi(F(r))}{\phi(r)} + \log 2.$$

Lemma 4.6. *Let f be an entire function defined as in Equation (1.1). There is a set E of finite logarithmic measure such that, for all large $r \notin E$,*

$$\log M(r,f) \leq \log\sigma(r,f) + \log\log\sigma(r,f) + O(1).$$

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Proof. For any $R > r$, by Cauchy–Schwarz inequality, we get

$$\begin{aligned}
M(r,f) &\le \sum_{j=0}^{\infty} |a_j| r^j = \sum_{j=0}^{\infty} |a_j| R^j \frac{r^j}{R^j} \\
&\le \left(\sum_{j=0}^{\infty} |a_j|^2 R^{2j} \right)^{1/2} \left(\sum_{j=0}^{\infty} \frac{r^{2j}}{R^{2j}} \right)^{1/2} \\
&\le \left(\sum_{j=0}^{\infty} |a_j|^2 R^{2j} \right)^{1/2} \left(\sum_{j=0}^{\infty} \frac{r^{j}}{R^{j}} \right)^{1/2} \\
&= \sigma(R,f) \left(\frac{R}{R-r} \right)^{1/2}.
\end{aligned}$$

Applying Lemma 4.5 to $F(r) = \sigma(r,f)$, $\phi(r) = r$, $\psi(x) = (\log x)^2$ and $R = r + \frac{r}{\psi(F(r))}$ gives

$$\log \sigma(R,f) \le \log \sigma(r,f) + 1 \quad \text{and} \quad \log \frac{R}{R-r} \le 2 \log \log \sigma(r,f) + \log 2$$

for all large $r \notin E$. The lemma is proved. □

Remark 5. It is straightforward to show that $\sigma(r,f) \le M(r,f)$ for all $r > 0$.

Now we recall a generalized logarithmic derivative estimates of Gol'dberg–Grinshtein type. To state this result, we introduce some notation. Given a non-constant meromorphic function g and $a \in \mathbb{C}$, we can always write $g(z) = (z-a)^m h(z)$, where integer m is called the order of g at the point a and is denoted by $ord_a g$. And, the first non-zero coefficient of Laurent series of $g(z)$ in the neighbourhood of the point $z = a$ is denoted by $c_g(a)$.

Lemma 4.7. ([2], p. 96). *Let g be a meromorphic function in the complex plane and let $0 < \alpha < 1$. There exists a constant r_0 such that, for all $r_0 < r < R < \infty$,*

$$\int_0^{2\pi} \left| \frac{r g'(r\,\mathrm{e}^{i\theta})}{g(r\,\mathrm{e}^{i\theta})} \right|^{\alpha} \frac{\mathrm{d}\theta}{2\pi} \le C(\alpha) \left(\frac{R}{R-r} \right)^{\alpha} (2T(R,g) + \beta_1)^{\alpha},$$

where

$$\beta_1 = \beta_1(g, r_0) = |ord_0 g| \log^+ \frac{1}{r_0} + |\log|c_g(0)|| + \log 2$$

and

$$C(\alpha) = 2^{\alpha} + (8 + 2^{\alpha+1}) \sec \frac{\alpha\pi}{2}.$$

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5. Proofs of our main theorems

5.1. Proof of Theorem 3.1

Let f_ω be a random entire function on $(\Omega, \mathcal{F}, \mu)$ of the form (1.2). For $\omega \in \Omega$, by Jensen formula,

$$\begin{aligned} N(r,0,f_\omega) &= \int_0^{2\pi} \log|f_\omega(r\,\mathrm{e}^{i\theta})|\frac{\mathrm{d}\theta}{2\pi} - \log|c_{f_\omega}(0)| \\ &= \log\sigma(r,f) + \int_0^{2\pi} \log|\hat{f}_\omega(r\,\mathrm{e}^{i\theta})|\frac{\mathrm{d}\theta}{2\pi} - \log|c_{f_\omega}(0)|. \end{aligned}$$

It follows that, for any $r>0$,

$$|N(r,0,f_\omega) - \log\sigma(r,f) + \log|c_{f_\omega}(0)|| \le \int_0^{2\pi} |\log|\hat{f}_\omega(r\,\mathrm{e}^{i\theta})||\frac{\mathrm{d}\theta}{2\pi} = X_r.$$

Since $\log\sigma(r,f)$ is increasing and unbounded, for any positive integer n, there is r_n such that $\log\sigma(r_n,f)=n$ and the sequence $\{r_n\}$ is increasing. Since $f_\omega \in \mathcal{Y}$, there are positive constants A and B such that $\mathrm{E}(\exp(AX_r^B)) = C_1 < +\infty$. For any $C>1$, set

$$A_n = \left\{\omega\in\Omega : |N(r_n,0,f_\omega) - \log\sigma(r_n,f) + \log|c_{f_\omega}(0)|| \ge \left(\frac{C}{A}\log n\right)^{1/B}\right\}.$$

Therefore, by Equation (4.1) in Lemma 4.3,

$$\mathbb{P}(A_n) \le \mathbb{P}\left(X_r \ge \left(\frac{C}{A}\log n\right)^{1/B}\right) \le \frac{C_1}{n^C}.$$

Consequently, $\sum\mathbb{P}(A_n) < \infty$ and by Borel–Cantelli lemma,

$$\mu(A) := \mu(\cap_{j=1}^\infty \cup_{n=j}^\infty A_n) = 0.$$

Thus, for $\omega \in \Omega\setminus A$, there exist j_0 such that for all $n > j_0$, we have

$$|N(r_n,0,f_\omega) - \log\sigma(r_n,f) + \log|c_{f_\omega}(0)|| < \left(\frac{C}{A}\log n\right)^{1/B}.$$

It follows that, for $r\in(r_n, r_{n+1}]$ with $n>j_0$, for almost all $\omega\in\Omega$, we have

$$\begin{aligned} \log\sigma(r,f) &\le \log\sigma(r_{n+1},f) = \log\sigma(r_n,f)+1 \\ &\le N(r_n,0,f_\omega) + (C/A)^{1/B}(\log\,\log\sigma(r_n,f))^{1/B} + 1 + \log|c_{f_\omega}(0)| \\ &\le N(r,0,f_\omega) + (C/A)^{1/B}(\log\,\log\sigma(r,f))^{1/B} + 1 + \log|c_{f_\omega}(0)|, \end{aligned} \tag{5.1}$$

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and

$$N(r,0,f_\omega) \le N(r_{n+1},0,f_\omega) \le \log\sigma(r_{n+1},f) + (C/A)^{1/B}(\log\log\sigma(r_{n+1},f))^{1/B} + \log|c_{f_\omega}(0)| \tag{5.2}$$

$$\le \log\sigma(r_n,f) + 1 + (C/A)^{1/B}(\log\log\sigma(r_n,f))^{1/B} + O(1) + \log|c_{f_\omega}(0)|$$

$$\le \log\sigma(r,f) + (C/A)^{1/B}(\log\log\sigma(r,f))^{1/B} + O(1) + \log|c_{f_\omega}(0)|. \tag{5.2}$$

Now we estimate the term $\log|c_{f_\omega}(0)|$. Since $f_\omega(z) = \sum_{j=0}^{\infty} a_j\chi_j(\omega)z^j$, we denote all the j satisfying $a_j \neq 0$ by the non-decreasing sequence $\{j_k\}_{k=0}^{\infty}$. It suffices to estimate $|\chi_{j_k}(\omega)|$. Define

$$B_k = \{\omega \in \Omega : \chi_{j_0}(\omega) = 0, \ldots, \chi_{j_{k-1}}(\omega) = 0, \chi_{j_k}(\omega) \neq 0\},$$

and

$$B'_{km} = \{\omega \in B_k : m < |\chi_{j_k}(\omega)| \le m+1\}.$$

It is trivial to see that

$$\mathbb{P}(\cup_{k=0}^{\infty} B_k) = 1 \quad \text{and} \quad B_k = \cup_{m=0}^{\infty} B'_{km}.$$

Thus, for almost every $\omega \in \Omega$, there exist unique k and m such that $\omega \in B'_{km}$. Therefore,

$$\log|c_{f_\omega}(0)| = \log|a_{j_k}| + \log|\chi_{j_k}(\omega)| \le \log|a_{j_k}| + \log(m+1).$$

This together with Equations (5.1) and (5.2) gives, for r sufficiently large,

$$|\log\sigma(r,f) - N(r,0,f_\omega)| \le (C'/A)^{1/B}(\log\log\sigma(r,f))^{1/B}.$$

The proof of Theorem 3.1 is complete.

5.2. Proof of Theorem 3.2

To prove Theorem 3.2, we need the following lemma, whose proof is based on the result of our Theorem 3.1.

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Lemma 5.1. *Let $f_\omega(z) \in \mathcal{Y}$. Then there exists a constant $r_0 = r_0(\omega)$ such that, for $r > r_0$, we have*

$$\log \sigma(r, f_\omega) \le \log \sigma(r, f) + \log\, \log \sigma(r, f) + 2 \qquad a.s.$$

and

$$\log \sigma(r, f) - (C/A)^{1/B}(\log\, \log \sigma(r, f))^{1/B} \le \log \sigma(r, f_\omega), \qquad a.s.,$$

where $C > 1$ is any constant, and constants A and B are from Condition Y.

Proof. Let φ be a non-negative increasing function. Since

$$\mathbb{E}(\sigma^2(r, f_\omega)) = \sum_{j=0}^{\infty} \mathbb{E}(|\chi_j(\omega)|^2)|a_n|^2 r^{2n} = \sigma^2(r, f),$$

by Markov's inequality, we have

$$\mathbb{P}\left(\sigma^2(r, f_\omega) > \sigma^2(r, f)\varphi(\sigma(r, f))\right) \le \frac{\mathbb{E}(\sigma^2(r, f_\omega))}{\sigma^2(r, f)\varphi(\sigma(r, f))} = \frac{1}{\varphi(\sigma(r, f))}.$$

For any positive integer n, there is r_n such that $\sigma(r_n, f) = e^n$ and the sequence $\{r_n\}$ is increasing. Set

$$B_n = \{\omega \in \Omega : \sigma^2(r_n, f_\omega) > \sigma^2(r_n, f)\varphi(\sigma(r_n, f))\}.$$

Thus, by taking $\varphi(x) = (\log x)^2$, we have

$$\mathbb{P}(B_n) \le \frac{1}{\varphi(\sigma(r_n, f))} = \frac{1}{n^2}.$$

Consequently, $\sum \mathbb{P}(B_n) < +\infty$. Thus, by Borel–Cantelli lemma, for almost all $\omega \in \Omega$, there is $j_1 = j_1(\omega)$, when $n > j_1$, $r \in (r_n, r_{n+1}]$, we have

$$\begin{aligned}\sigma^2(r, f_\omega) &\le \sigma^2(r_{n+1}, f_\omega) \le \sigma^2(r_{n+1}, f)\varphi(\sigma(r_{n+1}, f)) \\ &= (\mathrm{e}^{n+1})^2(n+1)^2 = \mathrm{e}^2\sigma^2(r_n, f)(\log \sigma(r_n, f) + 1)^2 \\ &\le \mathrm{e}^2\sigma^2(r, f)(\log \sigma(r, f) + 1)^2.\end{aligned}$$

For $r > r_0$ sufficiently large, we get

$$\log \sigma(r, f_\omega) \le \log \sigma(r, f) + \log\log \sigma(r, f) + 2 \qquad \text{a.s.}$$

On the other hand, by Theorems 3.1 and 2.2 and Lemma 4.4, for any $C > 1$, there is a constant $r_0 = r_0(\omega)$, for $r > r_0(\omega)$,

$$\log \sigma(r, f) \le N(r, 0, f_\omega) + (C/A)^{1/B}(\log\, \log \sigma(r, f))^{1/B}$$

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$$\leq T(r, f_\omega) + (C/A)^{1/B}(\log\log\sigma(r, f))^{1/B}$$
$$\leq \log\sigma(r, f_\omega) + (C'/A)^{1/B}(\log\log\sigma(r, f))^{1/B}.$$

This completes the proof of this lemma. □

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. By applying Theorem 3.1 to f^*_ω, we obtain two positive constants A, B such that for any positive constant $C > 1$, there exists a constant $r_0 = r_0(\omega) > 0$, so that for $r > r_0$,

$$\log\sigma(r, f^*) \leq N(r, 0, f^*_\omega) + (C/A)^{1/B}(\log\log\sigma(r, f^*))^{1/B} \qquad \text{a.s.}$$

Since $\sigma(r, f^*) \geq \sigma(r, f) \geq e$ for all large r, say, $r > r_0$, and the function $y(x) = x - C_0 \log x$ is increasing on $[x_0, +\infty)$, we have

$$\log\sigma(r, f) \leq N(r, 0, f^*_\omega) + (C/A)^{1/B}(\log\log\sigma(r, f))^{1/B} \qquad \text{a.s.} \tag{5.3}$$

By Jensen formula and Theorem 2.2, we have, for any $r < R$ and $0 < \alpha < 1$,

$$\begin{aligned} N(r, 0, f^*_\omega) - N(r, a, f_\omega) &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f^*_\omega(r\,e^{i\theta})}{f_\omega(r\,e^{i\theta}) - a}\right| \mathrm{d}\theta - \log\frac{|c_{f^*_\omega}(0)|}{|c_{f_\omega}(a)|} \\ &= \frac{1}{\alpha}\int_0^{2\pi} \log\left|\frac{f^*_\omega(r\,e^{i\theta})}{f_\omega(r\,e^{i\theta}) - a}\right|^\alpha \frac{\mathrm{d}\theta}{2\pi} - \log\frac{|c_{f^*_\omega}(0)|}{|c_{f_\omega}(a)|} \\ &\leq \frac{1}{\alpha}\log\left(\int_0^{2\pi} \left|\frac{f^*_\omega(r\,e^{i\theta})}{f_\omega(r\,e^{i\theta}) - a}\right|^\alpha \frac{\mathrm{d}\theta}{2\pi}\right) - \log\frac{|c_{f^*_\omega}(0)|}{|c_{f_\omega}(a)|}. \end{aligned}$$

Thus, by using Lemma 4.7 and the estimate of $\log|c_{f_\omega}(0)|$ as in the proof of Theorem 3.1, we obtain

$$\begin{aligned} N(r, 0, f^*_\omega) - N(r, a, f_\omega) &\leq \frac{1}{\alpha}\log\left(C(\alpha)\left(\frac{R}{R-r}\right)^\alpha (2T(R, f_\omega - a) + \beta_1)^\alpha\right) + O(1) \\ &\leq \frac{1}{\alpha}\log\left(C_1^\alpha C(\alpha)\left(\frac{R}{R-r}\right)^\alpha T^\alpha(R, f_\omega - a)\right) + O(1) \\ &\leq \log T(R, f_\omega) + \log\frac{R}{R-r} + O(1), \end{aligned}$$

where β_1 is a random constant related to $\log|c_{f_\omega}(0)|$, and C_1 is an absolute constant.

It follows from Lemma 4.4 and Lemma 5.1 that, for any $r < R$,

$$\begin{aligned} N(r, 0, f^*_\omega) &\leq N(r, a, f_\omega) + \log\log\sigma(R, f_\omega) + \log\frac{R}{R-r} + O(1) \qquad (5.4) \\ &\leq N(r, a, f_\omega) + \log\log\sigma(R, f) + \log\log\log\sigma(R, f) + \log\frac{R}{R-r} + O(1). \end{aligned}$$

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Applying Lemma 4.5 to the functions $F(r) = \log \sigma(r, f)$, $\phi(r) = r$, $\psi(r) = \log^2 r$ and $R = r + \frac{r}{\psi(F(r))}$, we get a set $E \subset [r_0, +\infty)$ of finite logarithmic measure, so that for all large r, say, $r > r_0$, and $r \notin E$,

$$\log\,\log \sigma(R, f) < \log\,\log \sigma(r, f) + 1,$$

and

$$\log \frac{R}{R-r} \leq 2 \log\,\log\,\log \sigma(r, f) + \log 2.$$

Thus, plugging the above two estimates to Equation (5.4) gives

$$N(r, 0, f_\omega^*) \leq N(r, a, f_\omega) + \log\,\log \sigma(r, f) + 3\,\log\,\log\,\log \sigma(r, f) + O(1), \tag{5.5}$$

for $r > r_0$ and $r \notin E$.

It follows from Equations (5.3) and (5.5) that there is $r_1 = r_1(a, r_0)$ such that

$$\log \sigma(r, f) \leq N(r, a, f_\omega) + (C/A)^{1/B} (\log\,\log \sigma(r, f))^{1/B} + (1 + o(1)) \log\,\log \sigma(r, f), \tag{5.6}$$

for $r > r_1$ and $r \notin E$.

On the other hand, by Nevanlinna's first main theorem, Lemma 4.4 and Lemma 5.1,

$$\begin{aligned} N(r, a, f_\omega) &\leq T(r, a, f_\omega) = T(r, f_\omega) + O(1) \\ &\leq \log \sigma(r, f_\omega) + O(1) \\ &\leq \log \sigma(r, f) + (1 + o(1)) \log\,\log \sigma(r, f). \end{aligned}$$

Combining this with Equation (5.6) completes the proof of the theorem. □

5.3. Proof of Corollaries

Proof of Corollary 3.2. Since the function $x - (C/A \log x)^{1/B}$ is an increasing function for all large x and using Lemma 4.4, we have

$$T(r, f) - (C/A)^{1/B} (\log T(r, f))^{1/B} \leq \log \sigma(r, f) - (C/A)^{1/B} (\log\,\log \sigma(r, f))^{1/B}.$$

The rest is a straightforward consequence of Theorem 3.1. □

Proof of Corollary 3.3. Let f_ω be a Gaussian entire function. By Equation (4.2) in the proof of Lemma 4.3, we have, for any $\tau > 0$, $r > 0$,

$$\mathbb{E}\left(\mathrm{e}^{\frac{2}{1+\tau} X_r}\right) < \infty.$$

Thus, $f_\omega \in \mathcal{Y}$ by choosing $A = 2/(1 + \tau)$ and $B = 1$. Applying Corollary 3.2 for $A = 2/(1 + \tau)$, $B = 1$ and $C = 1 + \tau$, we can finish the proof of the corollary in this case.

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Similarly, if f_ω is a Steinhaus entire function or Rademacher entire function, we can choose $A = 1/3(1+\tau)$, $B = 1$, $C = 1+\tau$ and $A = \epsilon \in (0, (6/(eC_0))^6)$, $B = 1/6$, $C = 1+\tau$, respectively. This completes the proof of Corollary 3.3. □

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