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The attracting centre of a continuous self-map of the interval

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Abstract. Let f denote a continuous map of a compact interval I to itself. A point $x \in I$ is called a γ-limit point of f if it is both an ω-limit point and an α-limit point of some point $y \in I$. Let Γ denote the set of γ-limit points. In the present paper, we show that (1) $\bar{P} - \Gamma$ is either empty or countably infinite, where $\bar{P}$ denotes the closure of the set P of periodic points, (2) $x \in I$ is a γ-limit point if and only if there exist y_1 and y_2 in I such that x is an ω-limit point of y_1, and y_1 is an ω-limit point of y_2, and if and only if there exists a sequence $y_1, y_2, \ldots$ of points in I such that x is an ω-limit point of y_1, and y_i is an ω-limit point of y_{i+1} for every $i \geq 1$, and (3) the period of each periodic point of f is a power of 2 if and only if every γ-limit point is recurrent.

1. *Introduction*

Throughout this paper f will be a continuous map of the interval $I = [0, 1]$ to itself, P the set of periodic points of f, R the set of recurrent points of f, and Ω the set of nonwandering points of f.

For a subset Y of I, define $\Lambda(Y) = \bigcup_{x \in Y} \omega(x)$, where $\omega(x)$ is the set of ω-limit points of x. Let $\Lambda^1 = \Lambda(I)$ and for any $n > 1$, inductively define $\Lambda^n = \Lambda(\Lambda^{n-1})$. Obviously, $\Lambda^1 \supset \Lambda^2 \supset \Lambda^3 \supset \cdots$. The set $\Lambda^\infty = \bigcap_{n=1}^{\infty} \Lambda^n$ will be called the *attracting centre* of f.

We will say that a point y is a γ-limit point of $x \in I$ if $y \in \omega(x) \cap \alpha(x)$, where $\alpha(x)$ is the set of α-limit points of x. Let $\gamma(x) = \omega(x) \cap \alpha(x)$ and $\Gamma = \bigcup_{x \in I} \gamma(x)$.

In [**8**] the author investigated the set $\Omega - \bar{P}$ and showed that it is always countable. In this paper we show

THEOREM 1. *Suppose that f is a continuous map of the interval I. Then*

(1) $\Omega - \Gamma$ *is countable.*

(2) $\Lambda^1 - \Gamma$ *and* $\bar{P} - \Gamma$ *are either empty or countably infinite.*

A. N. Sharkovskii [**6**] has shown that Λ^1 is closed and hence that $\bar{P} \subset \Lambda^1$. L. Block and E. Coven [**1**] have shown that $\omega(x)$ is an infinite minimal set for any $x \in \Lambda^1 - \bar{P}$. It follows that $\Lambda^2 \subset \bar{P}$, because each minimal set is contained in R and $\bar{R} = \bar{P}$ (see [**7**], for example). In [**1**] and [**2**], one can find examples in which $\Lambda^1 \neq \bar{P}$. Therefore,

$\Lambda^1 = \Lambda^2$ does not hold in general. However, we will prove

THEOREM 2. *Suppose that f is a continuous map of the interval I. Then*

$$\Lambda^\infty = \cdots = \Lambda^3 = \Lambda^2 = \Lambda(\bar{P}) = \Lambda(\Omega) = \Gamma.$$

In particular, Γ is the attracting centre of f.

Remark 1. Theorem 2 shows that the following conditions are equivalent.

(1) $y \in I$ is an ω-limit point of a nonwandering point of f.

(2) $y \in I$ is an ω-limit point of a point in the closure of the set of periodic points of f.

(3) There is a point $x \in I$ such that $y \in I$ is both an ω-limit point and an α-limit point of x.

(4) For each $n \geq 2$, there are n points $x_1, x_2, \ldots, x_n \in I$ such that y is an ω-limit point of x_1, x_1 is an ω-limit point of $x_2, \ldots$, and x_{n-1} is an ω-limit point of x_n.

(5) There is a sequence $x_1, x_2, \ldots$ of points in I such that y is an ω-limit point of x_1, and x_i is an ω-limit point of x_{i+1} for every $i \geq 1$.

The continuous maps of the interval I into itself can be divided into two disjoint classes, determined by whether or not the period of each periodic point is a power of 2. It has been shown that maps in different classes have quite different dynamical properties. (See [**3**] for a survey, and [**9**] and [**10**] for some new results.) In the following theorem it will be shown that the class to which an interval map belongs is determined by whether or not $\Gamma - R$ is empty.

THEOREM 3. *Suppose that f is a continuous map of the interval I to itself. Then the following conditions are equivalent.*

(1) *The period of each periodic point of f is a power of* 2.

(2) *Every γ-limit point of f is recurrent* (*i.e.* $\Gamma = R$).

2. *Preliminaries*

Recall that f is a continuous map of the interval $I = [0, 1]$ to itself. Let $x \in I$.

A point $y \in I$ is called an ω-limit point of x if there exist $n_i \to \infty$ such that $f^{n_i}(x) \to y$. Let $\omega(x)$ denote the set of ω-limit points of x. We will use the symbols $\omega_+(x)$ (*resp.* $\omega_-(x)$) to denote the set of all points y such that there exist $n_i \to \infty$ such that $f^{n_i}(x) \to y$ and $y < f^{n_i}(x)$ (*resp.* $f^{n_i}(x) < y$) for every $i > 0$. Clearly, $y \in \omega_+(x)$ (*resp.* $y \in \omega_-(x)$) if and only if there exist $n_i \to \infty$ such that $f^{n_i}(x) \to y$ and $y < \cdots < f^{n_2}(x) < f^{n_1}(x)$ (*resp.* $f^{n_1}(x) < f^{n_2}(x) < \cdots < y$). It is clear that if $x \notin P$, then $\omega(x) = \omega_+(x) \cup \omega_-(x)$. Define $\Lambda_+ = \bigcup_{x \in I} \omega_+(x)$ and $\Lambda_- = \bigcup_{x \in I} \omega_-(x)$.

A point $y \in I$ is called an α-limit point of x if there exist $n_i \to \infty$ and $x_i \to y$ such that $f^{n_i}(x_i) = x$ for every $i > 0$. We will use the symbols $\alpha_+(x)$ (*resp.* $\alpha_-(x)$) to denote the set of all points y such that there exist $n_i \to \infty$ and $x_i \to y$ such that $f^{n_i}(x_i) = x$ and $y < x_i$ (*resp.* $x_i < y$) for every $i > 0$. It is clear that if $x \notin P$, then $\alpha(x) = \alpha_+(x) \cup \alpha_-(x)$.

A point is called a γ-limit point of x if it is both an ω-limit point of x and an α-limit point of x. The symbol $\gamma(x)$ denotes the set of γ-limit points of x and

$\Gamma = \bigcup_{x \in I} \gamma(x)$. Define $\gamma_+(x) = \omega_+(x) \cap \alpha_+(x)$ and $\gamma_-(x) = \omega_-(x) \cap \alpha_-(x)$. Then $\Gamma_+ = \bigcup_{x \in I} \gamma_+(x)$ and $\Gamma_- = \bigcup_{x \in I} \gamma_-(x)$.

The forward orbit $O_P(x)$ of $x \in I$ is the set $\{f(x), f^2(x), \ldots\}$ and the reverse orbit $O_N(x)$ of $x \in I$ is the set $\bigcup_{n=1}^{\infty} f^{-n}(x)$.

Let Y be a subset of I. $\bar{Y}$ denotes the closure of Y as usual. A point $y \in I$ is called a right-sided (*resp.* left-sided) accumulation point of Y if for any $\varepsilon > 0$, $(y, y+\varepsilon) \cap Y \neq \emptyset$ (*resp.* $(y-\varepsilon, y) \cap Y \neq \emptyset$). The right-sided closure $\bar{Y}_+$ (*resp.*, the left-sided closure $\bar{Y}_-$) is the union of Y and the set of right-sided (*resp.* left-sided) accumulation points of Y. A point which is both a right-sided and a left-sided accumulation point of Y is called a two-sided accumulation point of Y. It is easy to see that $\bar{Y} = \bar{Y}_+ \cup \bar{Y}_-$.

We need the following known results.

PROPOSITION A [**5**]. *$x \in \Omega$ if and only if $x \in \alpha(x)$.*

An interval (i.e. a connected subset of the real line) $J \subset I$ is said to be of positive (*resp.* negative) type if there exist $x' \in J$ and $n' > 0$ such that $f^{n'}(x) \in J$, and for any $x \in J$ and any $n > 0$, $x < f^n(x)$ (*resp.* $f^n(x) < x$) provided $f^n(x) \in J$. An interval $J \subset I$ is said to be of free type if $f^n(x) \notin J$ for any $x \in J$ and any $n > 0$.

PROPOSITION B [**4**], [7]. *If $J \subset I$ is an interval such that $J \cap P = \emptyset$, then one and only one of the following conditions holds:*

(1) *J is of positive type;*

(2) *J is of negative type;*

(3) *J is of free type.*

The following proposition is a slightly stronger version of a theorem of Sharkovskii [**6**]. (See also [**4**].)

PROPOSITION C. *$\bar{P}_+ - P \subset \Lambda_+$ and $\bar{P}_- - P \subset \Lambda_-$.*

Proof. Let $x \in \bar{P}_- - P$. Choose a sequence $z_1, z_2, \ldots$ of periodic points of f such that $z_i \to x$ and $z_1 < z_2 < \cdots < x$. Let p_i denote the period of z_i with respect to f.

Fix $i > 0$, and let $g = f^{p_i}$. Then

$$K = L_i \cup g(L_i) \cup g^2(L_i) \cup \ldots$$

is an interval, where $L_i = [z_i, x]$. Let n_j denote the period of z_j with respect to g. For $k = 1, 2$, or 3, suppose a subsequence of $g^{n_{j+1}-k}(z_{j+1}), g^{n_{j+2}-k}(z_{j+2}), \ldots$ converges to $u_k \in \bar{K}$. It is clear that $g^k(u_k) = x$. If $u_{k'} = u_{k''}$ for some k' and k'' with $k' < k''$, then

$$g^{k''-k'}(x) = g^{k''}(u_{k'}) = g^{k''}(u_{k''}) = x,$$

and so x is periodic. Thus u_1, u_2, and u_3 are distinct points, and $u_{\tilde{k}} \in K$, where $u_{\tilde{k}}$ is the one which lies between the other two. Choose $v_i \in L_i$ and $\tilde{m}_i > 0$ so that $u_{\tilde{k}} = g^{\tilde{m}_i}(v_i)$. Let $m_i = \tilde{k} + \tilde{m}_i$. Then $g^{m_i}(v_i) = x$.

Summarizing, for each $i > 0$, we have $v_i \in L_i$ and $m_i > 0$ such that $f^{m_i p_i}(v_i) = x$.

Let $q_i = p_i m_i$. Since $f^{q_i}(z_i) = z_i$ and $f^{q_i}(v_i) = x$, it follows that $f^{q_i}(L_i) \supset L_i$. Let $F_0 = I$ and inductively define $F_n = F_{n-1} \cap f^{-t_n}(L_n)$, $n > 0$, where $t_n = \sum_{i=1}^{n} q_i$. Obviously, for any $n \geq 0$, F_n is closed. Note that $F_n \neq \emptyset$ for any $n \geq 0$. On the other hand, it is clear that $F_0 \supset F_1 \supset F_2 \supset \cdots$. Hence, $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. Let $y \in \bigcap_{n=1}^{\infty} F_n$. Then $f^{t_n}(y) \in L_n$ for any $n > 0$. Therefore $f^{t_n}(y) \to x$, and since $x \notin P$, $f^{t_n}(y) < x$ for every $n > 0$. Thus $x \in \omega_-(y)$, completing the proof of the proposition.

PROPOSITION D [9]. *The following conditions are equivalent.*

(1) *The period of each periodic point is a power of* 2.

(2) *The set* $\bar{P} - R$ *is countable.*

PROPOSITION E [10]. *If the period of each periodic point is a power of* 2, *then for any* $x \in \bar{P} - P$, *any* $n \geq 0$, *and any odd integer* $m > 0$, *between* x *and* $f^{m2^n}(x)$ *there is a periodic point with period* 2^n, *and there is no periodic point with period* $2^{n'}$ *for every* $n' < n$.

3. *Proof of theorem* 1

LEMMA 1. *If* $y \in \Omega$, *then*

(1) $\omega_+(y) = \gamma_+(y)$ *and* $\omega_-(y) = \gamma_-(y)$,

(2) $\omega(y) = \gamma(y)$.

Therefore $\Gamma \supset \Lambda(\Omega)$.

Proof. (1) Without loss of generality, we prove only that $\omega_+(y) = \gamma_+(y)$. Let $x \in \omega_+(y)$. There exist $n_i \to \infty$ such that $f^{n_i}(y) \to x$ and $x < f^{n_i}(y)$ for every $i > 0$. It follows from Proposition A that $y \in \alpha(y)$. It is easy to see that $f^{n_i}(y) \in \alpha(y)$ for any $i > 0$. Hence, it follows immediately that $x \in \alpha_+(y)$, and so $x \in \gamma_+(y)$. This shows that $\omega_+(y) \subset \gamma_+(y)$. On the other hand, it is trivial that $\omega_+(y) \supset \gamma_+(y)$.

(2) If $y \in P$, it is clear that $\omega(y) = \gamma(y)$. If $y \notin P$, then $\omega(y) = \omega_+(y) \cup \omega_-(y) = \gamma_+(y) \cup \gamma_-(y) \subset \gamma(y)$, and hence $\omega(y) = \gamma(y)$.

LEMMA 2. *For* $y \in I$,

(1) $\overline{(O_N(y))_+} = O_N(y) \cup \alpha_+(y)$, *and*

(2) $\overline{(O_N(y))_-} = O_N(y) \cup \alpha_-(y)$.

Proof. Without loss of generality, we prove only (1). Obviously, $\overline{(O_N(y))_+} \supset O_N(y) \cup \alpha_+(y)$. On the other hand, if x is a right-sided accumulation point of $O_N(y)$, then we may choose a sequence $v_i \to x$ of points in $O_N(y)$ such that $x < v_i$ for every $i \geq 1$. Let $m_i > 0$ be such that $f^{m_i}(v_i) = y$. If the sequence m_i has a constant subsequence $m_{i(j)} = m$, then $f^m(v_{i(j)}) = y$ and $f^m(x) = y$, i.e. $x \in O_N(y)$. If the sequence m_i has a subsequence $m_{i(j)} \to \infty$, then $x \in \alpha_+(y)$ may be shown by verifying that the sequence $m_{i(j)}$ and the sequence $v_{i(j)}$ satisfy the conditions of the definition of $\alpha_+(y)$. Therefore, $\overline{(O_N(y))_+} \subset O_N(y) \cup \alpha_+(y)$.

LEMMA 3. *For* $y \in I$,

(1) $\omega_+(y) \cap \overline{(O_N(y))_+} = \gamma_+(y)$, *and*

(2) $\omega_-(y) \cap \overline{(O_N(y))_-} = \gamma_-(y)$.

Proof. Without loss of generality, we prove only (1). By lemma 2,

$$\omega_+(y) \cap \overline{(O_N(y))_+} = (\omega_+(y) \cap O_N(y)) \cup \gamma_+(y).$$

It is trivial that (1) holds if $\omega_+(y) \cap O_N(y) = \emptyset$. If $\omega_+(y) \cap O_N(y) \neq \emptyset$, choose a point x in this set, and let $m>0$ be such that $f^m(x)=y$. Since $x\in\omega_+(y)$, we know that $y\in\omega_+(y)$. Hence y is recurrent, and so nonwandering. By lemma 1, $\omega_+(y)=\gamma_+(y)$, and hence (1) follows.

LEMMA 4. (1) *If $x\in\Lambda_+$, and if for every $\varepsilon>0$ there exist $v\in(x,x+\varepsilon)$ and $m>0$ such that $f^m(v)\in(x,x+\varepsilon)$ and $f^m(x)>x$, then $x\in\Gamma_+$.*

(2) *If $x\in\Lambda_-$, and if for every $\varepsilon>0$ there exist $v\in(x-\varepsilon,x)$ and $m>0$ such that $f^m(v)\in(x-\varepsilon,x)$ and $f^m(x)<x$, then $x\in\Gamma_-$.*

Proof. Without loss of generality, we prove only (1). Let $x\in\Lambda_+$ and let y be a point such that $x\in\omega_+(y)$. There exist $n_i\to\infty$ such that $f^{n_i}(y)\to x$ and $x<f^{n_i}(y)$ for every $i>0$. For each $i>0$, choose $v_i\in(x,f^{n_i}(y))$ and $m_i>0$ such that $f^{m_i}(v_i)\in(x,f^{n_i}(y))$ and $f^{m_i}(x)>x$. If the sequence m_i has a subsequence $m_{i(j)}$ such that $f^{m_{i(j)}}(x)\to x$, then $x\in R$, and so $x\in\omega_+(x)=\gamma_+(x)$ by lemma 1. If no such subsequence exists, then there exists $i'>0$ such that $f^{m_i}(x)>f^{n_{i'}}(y)$ for every $i>0$. Choose $N>0$ such that $f^{m_i}(v_i)\in(x,f^{n_{i'}}(y))$ whenever $i\geq N$. Let $z=f^{n_{i'}}(y)$. Since $z\in f^{m_i}((x,f^{n_i}(y))$ for every $i\geq N$, we know that $x\in\overline{(O_N(z))}_+$. Clearly, $\omega_+(y)=\omega_+(z)$. Therefore $x\in\omega_+(z)\cap\overline{(O_N(z))}_+=\gamma_+(z)$ by lemma 3.

PROPOSITION 1. *Suppose that f is a continuous map of the interval I. Let D denote a connected component of $I-P$, and let a denote the left end point of D, b the right end point of D. Then*

(1) *If D is of positive (resp. negative) type, then $b\in\Gamma_+\cup P$ (resp. $a\in\Gamma_-\cup P$).*

(2) *If D is of free type, then either $a\in\Gamma_-\cup P$ or $b\in\Gamma_+\cup P$.*

Proof. (1) Suppose D is of positive type. If $b\in P$, there is nothing to prove. Assume then that $b\notin P$. In this case, $b\in D$. Since an interval of positive type is not a singleton, we see that $b\in\bar{P}_+-P$. It follows from Proposition C that $b\in\omega_+(y)$ for some $y\in I$.

Since D is of positive type, we may choose $d\in D$ and $k>0$ such that $f^k(d)\in D$ and $d<f^k(d)$. Since D contains no periodic points, it follows that $b<f^k(b)$.

We verify that b satisfies the hypotheses of lemma 4(1). Let $\varepsilon>0$. Choose a periodic point $u\in(b,f^k(b))\cap(x,x+\varepsilon)$, and let m denote the period of u. Since $b\in\bar{P}$, it follows that $f^m(b)\notin\mathring{D}$. If $f^m(b)\leq a$, then $f^m(b,u))\supset[d,b]$, and hence $f^{m+k}([d,b])\supset[d,b]$, by the fact that $f^k([d,b])\supset(b,u)$. Therefore there is a periodic point in $[d,b]$, which is contained in D, a contradiction. Thus the only possibility is that $f^m(b)>b$. We have verified that the hypotheses of lemma 4(1) are satisfied by b, and hence $b\in\Gamma_+$.

(2) Suppose D is of free type. If either a or b is periodic, there is nothing to prove. Assume then that neither a nor b is periodic. It follows that $a,b\in D$ and D is closed. We divide our discussion into three cases.

Case I. $a=0$. In this case $D=[0,b]$ and $b\in\bar{P}_+-P$. By proposition C, we know that $a\in\Lambda_+$. Let $\varepsilon>0$. Choose a periodic point $v\in(b,b+\varepsilon)$ and let the period of v be m. Then $f^m(v)=v\in(b,b+\varepsilon)$ and $f^m(b)>b$, because $f^m(0)>0$. Therefore it follows from lemma 4 that $b\in\Gamma_+$.

Case II. $b=1$. In this case, an argument similar to the one used in case I leads us to the fact that $a\in\Gamma_-$.

Case III. $0 < a \le b < 1$. In this case, $a \in \bar{P}_- - P$ and $b \in \bar{P}_+ - P$. By proposition C, $a \in \Lambda_-$ and $b \in \Lambda_+$. To prove that either $a \in \Gamma_-$ or $b \in \Gamma_+$, we show that $b \in \Gamma_+$ under the assumption that $a \notin \Gamma_-$. First, there exists $\delta' > 0$ such that $O_P(a) \cap (a - \delta', a) = \varnothing$. (If not, $a \in \omega_-(a)$, a is a recurrent point, and hence $a \in \gamma_-(a)$ by lemma 1, contradicting our assumption.) Then $O_P(a) \cap (a - \delta', b] = \varnothing$ because D is of free type. Second, it follows from lemma 4 that we may choose $\delta > 0$ with $\delta' > \delta > 0$ such that whenever $m > 0$ with $f^m(u) \in (a - \delta, a)$ for some $u \in (a - \delta, a)$, then $f^m(a) > a$.

We verify that b satisfies the hypotheses of lemma 4(1). We have shown that $b \in \Lambda_+$. Let $\varepsilon > 0$. Choose a periodic point $u \in (a - \delta, a)$ and let p be its period, choose a periodic point $v \in (b, b + \varepsilon)$ and let q be its period. Clearly, $f^{pq}(v) = v \in (b, b + \varepsilon)$. On the other hand, since $f^{pq}(u) = u \in (a - \delta, a)$, it follows that $f^{pq}(a) > a$, and hence that $f^{pq}(b) > b$. Therefore $b \in \Gamma_+$.

COROLLARY 1. *Suppose that f is a continuous map of the interval I. Then*

(1) $\min \bar{P} \in \Gamma_+ \cup P$ *and* $\max \bar{P} \in \Gamma_- \cup P$. *In particular,* $\min \bar{P}$ *and* $\max \bar{P}$ *are γ-limit points.*

(2) *No end point of I is in $\Omega - \Gamma$.*

(3) *If a two-sided accumulation point of periodic points is not periodic, then it is either in Γ_+ or in Γ_-. Therefore $\bar{P}_+ \cap \bar{P}_- \subset \Gamma$.*

Proof. (1) If $\min \bar{P}$ is not periodic, then $[0, \min \bar{P}]$ is a connected component of $I - P$ which is not of negative type. If $[0, \min \bar{P}]$ is of positive type, then its right end point $\min P$ is in $\Gamma_+ \cup P$ by proposition 1. In the case that $[0, \min \bar{P}]$ is of free type, it follows from proposition 1 that $\min \bar{P} \in \Gamma_+ \cup P$, because $0 \notin \Gamma_-$.

By a similar argument, we also see that $\max \bar{P} \in \Gamma_- \cup P$.

(2) If $0 \in \Omega$, then $0 \in \bar{P}$ by lemma 2.7 in [**5**], and hence it follows by (1) of this corollary that $0 \in \Gamma$. Similarly, if $1 \in \Omega$, then $1 \in \Gamma$.

(3) Let x be a two-sided accumulation point of P. If x is not periodic, then the singleton $\{x\}$ is a connected component of $I - P$ which is of free type, and hence its unique end point x is either in Γ_+ or in Γ_-, by proposition 1.

Proof of theorem 1. (1) For any subset Y of I, each point of $\bar{Y} - (\bar{Y}_+ \cap \bar{Y}_-)$ is an end point of a connected component of $I - \bar{Y}$. Since $I - \bar{Y}$ has only countably many connected components, $\bar{Y} - (\bar{Y}_+ \cap \bar{Y}_-)$ is a countable set. It follows that $\bar{P} - (\bar{P}_+ \cap \bar{P}_-)$ is countable. By corollary 1, $\bar{P} - \Gamma$ is also countable. It follows from [**8**] that $\Omega - \bar{P}$ is countable, and so is $\Omega - \Gamma$.

(2) We claim that if Y is a strictly invariant subset of I, and Z an invariant subset of I containing P, then $Y - Z$ is either empty or infinite. To show this, note that if $Y - Z \neq \varnothing$, then we may choose by induction a sequence $y_1, y_2, \ldots$ such that $y_n \in Y - Z$ and $y_n = f(y_{n+1})$ for every $n \ge 1$. Since there is no periodic point in $Y - Z$, the points $y_1, y_2, \ldots$ are pairwise distinct, and hence $Y - Z$ is infinite. The proof of the claim is complete.

It follows from the claim above that $\Lambda^1 - \Gamma$ and $\bar{P} - \Gamma$ are either empty or infinite, because Λ^1 and $\bar{P}$ are strictly invariant and Γ is invariant. On the other hand, $\Lambda^1 - \Gamma$ and $\bar{P} - \Gamma$, as subsets of the countable set $\Omega - \Gamma$, are countable.

4. *Proof of theorem* 2

LEMMA 5. $\Omega \supset \Lambda^1 \supset \bar{P} \supset \Gamma$.

Proof. The inclusion $\Omega \supset \Lambda^1$ is obvious, and the inclusion $\Lambda^1 \supset \bar{P}$ is an immediate consequence of the theorem of Sharkovskii mentioned in § 1. (See also proposition C.)

It remains to prove that $\bar{P} \supset \Gamma$. To do this, assume that $x \in \Gamma - \bar{P}$. Let y be such that $x \in \omega(y) \cap \alpha(y)$, and D be the connected component of $I - \bar{P}$ containing x. Clearly D is not of free type. By proposition B, we may assume, without loss of generality, that D is of positive type. Since $x \in \omega(y)$, there exist $n_i \to \infty$ such that $f^{n_i}(y) \to x$. Let i be an integer such that $f^{n_i}(y) \in D$. Since D is of positive type, we have that $f^{n_i}(y) < x$. Since $x \in \alpha(y)$, there exists $u \in D \cap (f^{n_i}(y), 1]$ such that $f^m(u) = y$ for some $m > 0$. Then $f^{m+n_i}(u) = f^{n_i}(y)$. Hence D fails to be of positive type, a contradiction.

LEMMA 6. For $y \in I$,

(1) $\omega_+(y) \cap \alpha_-(y) \subset \Gamma_+ \cup P$, *and*

(2) $\omega_-(y) \cap \alpha_+(y) \subset \Gamma_- \cup P$.

Proof. Without loss of generality, we prove only (1). Let $x \in \omega_+(y) \cap \alpha_-(y)$. If $x \notin \Gamma_+$, then it follows from lemma 3 and lemma 4 that there exists $\varepsilon > 0$ such that

(a) $(x, x+\varepsilon) \cap O_N(y) = \varnothing$, and

(b) if $m > 0$ and $f^m(u) \in (x, x+\varepsilon)$ for some $u \in (x, x+\varepsilon)$, then $f^m(x) \le x$.

Since $x \in \omega_+(y)$, we may choose m and n with $m > n > 0$ such that $f^m(y), f^n(y) \in (x, x+\varepsilon)$. Then by condition (b), $f^{m-n}(x) \le x$. On the other hand, it follows from the condition (a) that $f^{m-n}(x) \ge x$. For if $f^{m-n}(x) < x$, then $f^{m-n}((x, x+\varepsilon)) \supset (f^{m-n}(x), x)$, which contains points of $O_N(y)$ because $x \in \alpha_-(y)$. Therefore $x = f^{m-n}(x)$ and x is a periodic point.

PROPOSITION 2. *Suppose that f is a continuous map of the interval I. Then* $\Gamma = \Gamma_+ \cup \Gamma_- \cup P$.

Proof. Obviously, $\Gamma \supset \Gamma_+ \cup \Gamma_- \cup P$. On the other hand, it is easily seen that $\omega(y)$ is a periodic orbit if y is periodic. If y is not periodic, then it follows that $\omega(y) = \omega_+(y) \cup \omega_-(y)$ and $\alpha(y) = \alpha_+(y) \cup \alpha_-(y)$. Therefore

$$\begin{aligned}\gamma(y) &= \gamma_+(y) \cup \gamma_-(y) \cup (\omega_+(y) \cap \alpha_-(y)) \cup (\omega_-(y) \cap \alpha_+(y)) \\ &\subset \Gamma_+ \cup \Gamma_- \cup P\end{aligned}$$

by lemma 6. Thus $\Gamma = \bigcup_{y \in I} \gamma(y) \subset \Gamma_+ \cup \Gamma_- \cup P$.

LEMMA 7. *Let $y \in I - P$, and for each $n \ge 0$, let C_n denote the connected component of $I - O_N(y)$ containing $f^n(y)$. Then*

(1) $f^n(\bar{C}_0) \subset \bar{C}_n$, *and*

(2) $\bar{C}_0 \cap \Gamma \neq \varnothing$.

Proof. (1) $f^n(C_0)$ is a connected subset of $I - O_N(y)$ containing $f^n(y)$. Therefore $f^n(C_0) \subset C_n$, and so $f^n(\bar{C}_0) \subset \bar{C}_n$.

(2) If $\bar{C}_0 \cap P \neq \emptyset$, there is nothing to prove. Assume then that $\bar{C}_0 \cap P = \emptyset$. In this case, $\bar{C}_0$ is contained in some connected component D of $I - P$. Let $\bar{C}_0 = [a, b]$, and let a' denote the left end point of D, b' the right end point of D.

We claim that $\bar{C}_0$ and D have at least one end point in common. If not, then $a' < a \leq y \leq b < b'$. Since $(a', a] \cap O_N(y) \neq \emptyset$ and $(b, b'] \cap O_N(y) \neq \emptyset$, D is not of positive type, of negative type, or of free type. This contradicts proposition B.

If the end points of $\bar{C}_0$ and of D coincide, then $\bar{C}_0 \cap \Gamma \neq \emptyset$ because at least one of the endpoints of D is in Γ by proposition 1.

Assume then, without loss of generality, that $a' < a$. In this case, it follows that D is of positive type, because $(a', a] \cap O_N(y) \neq \emptyset$. Then the right end point $b = b'$ of D is in $\Gamma_+ \cup P$.

The proof is complete.

Proof of theorem 2. Lemma 1 shows that $\Gamma \supset \Lambda(\Omega)$, and the inclusions $\Lambda(\Omega) \supset \Lambda^2 \supset \Lambda(\bar{P}) \supset \Lambda(\Gamma)$ are immediate consequences of lemma 5.

We prove that $\Lambda(\Gamma) \supset \Gamma$ as follows. Let $x \in \Gamma$. Obviously $x \in \Lambda(\Gamma)$ if x is periodic. Assume then that $x \notin P$. Then either $x \in \Gamma_+$ or $x \in \Gamma_-$ by proposition 2. Assume, without loss of generality, that $x \in \Gamma_+$. Let y be such that $x \in \gamma_+(y)$, and C_n the connected component of $I - O_N(y)$ containing $f^n(y)$. Since $x \in \gamma_+(y)$, there exist $n_i \to \infty$ such that $f^{n_i}(y) \to x$, $x < \cdots < f^{n_2}(y) < f^{n_1}(y)$, and for every $i > 0$ the interval $(f^{n_{i+1}}(y), f^{n_i}(y))$ contains at least two distinct points of $O_N(y)$. Then the intervals $\bar{C}_{n_1}, \bar{C}_{n_2}, \ldots$ are pairwise disjoint and $L_i \to 0$, where L_i denotes the length of $\bar{C}_{n_i}$. By lemma 7(2), there exists $u \in \bar{C}_0 \cap \Gamma$, and it follows from lemma 7(1) that $|f^{n_i}(u) - f^{n_i}(y)| \leq L_i$. Therefore $f^{n_i}(u) \to x$, and so $x \in \Lambda(\Gamma)$.

Up to now, we have shown that

$$\Gamma = \Lambda(\Omega) = \Lambda^2 = \Lambda(\bar{P}) = \Lambda(\Gamma).$$

Then it follows by induction that for every $n \geq 2$,

$$\Lambda^{n+1} = \Lambda(\Lambda^n) = \Lambda(\Gamma) = \Gamma.$$

The proof of theorem 2 is completed.

5. *Proof of theorem* 3

(1)$\Rightarrow$(2). Suppose that condition (1) holds. Let $x \in \Gamma$. We show that $x \in R$ as follows. Obviously, $x \in R$ if x is periodic. Assume then that $x \notin P$.

$\Gamma = \Lambda(\bar{P})$ by theorem 2, so there exists $y \in \bar{P}$ such that $x \in \omega(y)$. Since x is not periodic, y is not either. Therefore $x \in \omega_+(y) \cup \omega_-(y)$. We may assume, without loss of generality, that $x \in \omega_+(y)$. Then there exist $n_i \to \infty$ such that $f^{n_i}(y) \to x$ and $x < f^{n_i}(y)$ for every $i > 0$.

Given $\varepsilon > 0$, let $p > 0$ be the least integer such that there is a periodic point in $(x, x+\varepsilon)$ with period 2^p, and let u be the smallest periodic point in $(x, x+\varepsilon)$ with period 2^p. Choose $j > 0$ such that $x < f^{n_j}(y) < u$. Let $q > 0$ be the least integer such that there is a periodic point in $(x, f^{n_j}(y))$ with period 2^q, and v a periodic point in $(x, f^{n_j}(y))$ with period 2^q. Clearly, $q > p$. Then, let $i' > 0$ be such that $n_i > n_j$ and $f^{n_i}(y) \in (x, v)$ whenever $i \geq i'$.

Fix $i \geq i'$. Let $n_i - n_j = m_i 2^{t_i}$, where $t_i > 0$ and $m_i > 0$ is odd. We claim that $t_i = q$. For if $t_i < q$, then by proposition E, there would be a periodic point between $f^{n_j}(y)$ and $f^{n_i}(y)$ $(= f^{m_i 2^{t_i}}(f^{n_j}(y)))$ with period 2^{t_i}, contradicting the definition of q, and if $t_i > q$, then by the same proposition, there would be no periodic point with period 2^q between the two points above.

Therefore if we write $n_i - n_j + 2^q = \tilde{m}_i 2^{\tilde{t}_i}$, where $\tilde{t}_i > 0$ and $\tilde{m}_i > 0$ is odd, then $\tilde{t}_i > q$. By proposition E, between $f^{n_i+2^q}(y)$ and $f^{n_j}(y)$ there is no periodic point with period either 2^p or 2^q. Thus, $f^{n_i+2^q}(y) \in (v, u)$.

Since $f^{n_i+2^q}(y) \to f^{2^q}(x)$, we have that $f^{2^q}(x) \in (x, x+\varepsilon)$. This shows that $x \in R$ and the proof of the implication (1)⇒(2) is complete.

(2)⇒(1). If $\Gamma = R$, then $\bar{P} - R$ is countable by theorem 1. Therefore, it follows from proposition D that (1) holds.

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