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ON THE JOINT SURVIVAL PROBABILITY OF TWO COLLABORATING FIRMS

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Abstract

We consider the problem of controlling the drift and diffusion rate of the endowment processes of two firms such that the joint survival probability is maximized. We assume that the endowment processes are continuous diffusions, driven by independent Brownian motions, and that the aggregate endowment is a Brownian motion with constant drift and diffusion rate. Our results reveal that the maximal joint survival probability depends only on the aggregate risk-adjusted return and on the maximal risk-adjusted return that can be implemented in each firm. Here the risk-adjusted return is understood as the drift rate divided by the squared diffusion rate.

Keywords: Ruin probability; optimal control; two-dimensional Brownian motion

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Secondary 91B70

1. Introduction

Consider two insurance companies that aim to collaborate so as to maximize their joint survival probability, or equivalently, to minimize the probability that one of the two companies gets ruined. Assume that the two companies can commit themselves to helping the other in case of financial distress. To assess the benefit of a collaboration, Grandits [5] has set up a model where the endowment processes, also called surplus processes, of the two companies are given by two independent Brownian motions with drift, and the companies can collaborate by transfer payments. These payments are assumed to be absolutely continuous with respect to the Lebesgue measure and to be bounded in such a way that each company keeps a minimal positive drift rate.

The collaborations considered in [5] are assumed to have an impact only on the drift rates of the companies' endowment processes. There are types of collaboration, however, that also entail a change of the diffusion rates; for example, think of mutual reinsurance agreements or agreements to transfer high-risk subsidiaries. In this paper we address the question of how to quantify the maximal benefit if a collaboration also has an impact on the diffusion rate of the two endowment processes.

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To measure the benefit of collaboration we introduce a control problem, where an agent can continuously allocate a drift and diffusion rate to two diffusion processes representing the endowment processes of the two companies. The aggregate drift and diffusion rates are assumed to be constant and independent of the allocation plan. Moreover, we assume that the set of implementable drift rates is bounded, and the set of implementable diffusion rates is bounded and bounded away from zero. The agent aims at choosing an allocation plan that maximizes the joint survival probability of the two companies. One can think of the agent as a mediator between the companies suggesting a mutual help contract.

As in [5], the optimal control turns out to be of bang-bang type: it is optimal that the agent implements the highest possible risk-adjusted return, defined as the ratio of the drift rate and the volatility squared, in the endowment dynamics of the company behind. Besides, the formula for the value function reveals that the maximal joint survival probability only depends on the maximal implementable risk-adjusted return and on the risk-adjusted return of the aggregate endowment process. Our assumptions entail that the latter does not depend on the allocation strategy.

We solve the control problem via a classical verification technique. To this end we construct an explicit solution of the associated Hamilton–Jacobi–Bellman (HJB) equation. We use the fact that the optimal control can be characterized as a bang-bang feedback function jumping at the line bisecting the first quadrant, where the first quadrant is interpreted as the set of non-negative endowment pairs. Since the optimal control is of bang-bang type, the HJB is linear below the bisector and above the bisector. The boundary conditions and a smooth fit condition along the bisector lead to a specific solution of the HJB equation, which can be verified to coincide with the value function. We remark that our construction of the solution of the HJB equation and also the verification bear some similarities to the approach used in [5].

McKean and Shepp [10] and Grandits [5] both consider the problem of maximizing the joint survival probability of two firms whose endowment processes are given by independent Brownian motions with drift and which are allowed to collaborate by transfer payments. In [10] these transfer payments are at most as high as the drift rates, whereas in [5] each company keeps a given positive minimal drift rate. In both cases the value function is derived and turns out to be a classical solution to the associated HJB equation. We emphasize that we allow for negative drift rates in our model. Grandits and Klein [6] extend the model of [5] and [10] to endowment processes driven by Brownian motions that are *correlated*. In all three articles [5, 6, 10] the derived optimal strategy is of bang-bang type and implements the highest possible risk-adjusted return for the company behind.

Schmidli [11] deals with maximizing the survival probability of one company by choosing an optimal dynamic proportional reinsurance strategy in the diffusion model. Also in this model the optimal strategy maximizes the risk-adjusted return among all admissible strategies.

Finally, the literature also comprises several articles analyzing the ruin probability within multidimensional Brownian risk models with *non-controllable* dynamics; see e.g. [4] and [7].

The paper is organized as follows. In Section 2 we introduce our model and provide the value function and an optimal strategy. We explain how to derive the formula for the value function in Section 3. Finally, we prove our results in Section 4.

2. Model and main results

Let $\underline{\sigma}, \overline{\sigma} \in (0, \infty)$ with $\underline{\sigma} \le \overline{\sigma}$ and $\underline{\mu}, \overline{\mu} \in \mathbb{R}$ such that $\underline{\mu} \le \overline{\mu}$. We define

$$M := \underline{\mu} + \overline{\mu} \quad \text{and} \quad \Sigma := \underline{\sigma} + \overline{\sigma}$$

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and assume that

$$M > 0. \tag{2.1}$$

Let D be a measurable, non-empty subset of $[\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]$ such that

$$(\mu, \sigma) \in D \Longrightarrow (M - \mu, \Sigma - \sigma) \in D. \tag{2.2}$$

We interpret an element $(\mu, \sigma) \in D$ as an implementable pair of drift and diffusion rate for the endowment process of each company. Condition (2.2) means that the sets of implementable drift and diffusion rate pairs for the two companies coincide. At the end of the section we provide an explicit example for D.

The set of admissible controls consists of all measurable functions $(\mu, \sigma)\colon \mathbb{R}^2 \to D$ and is denoted by $\mathcal{M}$.

We denote the endowment processes of the two companies by $X = (X_t)_{t\in[0,\infty)}$ and $Y = (Y_t)_{t\in[0,\infty)}$, respectively. Given a control (μ, σ), we assume that the dynamics of the pair (X, Y) satisfy the stochastic differential equation (SDE)

$$\begin{aligned} \mathrm{d}X_t &= \mu(X_t, Y_t)\,\mathrm{d}t + \sqrt{\sigma(X_t, Y_t)}\,\mathrm{d}W_t^1, && X_0 = x, \\ \mathrm{d}Y_t &= (M - \mu(X_t, Y_t))\,\mathrm{d}t + \sqrt{\Sigma - \sigma(X_t, Y_t)}\,\mathrm{d}W_t^2, && Y_0 = y, \end{aligned} \tag{2.3}$$

where $W = (W^1, W^2)$ denotes a two-dimensional Brownian motion and $(x, y) \in \mathbb{R}^2$. Note that we use the notation $\sqrt{\sigma(X_t, Y_t)}$ for the volatility instead of the more commonly used $\sigma(X_t, Y_t)$ in (2.3), as this provides some advantages in later considerations. For every $(\mu, \sigma) \in \mathcal{M}$ and $(x, y) \in \mathbb{R}^2$, there exists a weak solution of (2.3) satisfying the initial condition $(X_0, Y_0) = (x, y)$, and we have uniqueness in law for (2.3) (see Theorem 3 and the following comment in [9]). Recall that a weak solution of (2.3) consists of a tuple $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P}, W, X, Y)$, where the first four components build a filtered probability space, W is a two-dimensional Brownian motion with respect to the filtration $(\mathcal{F}_t)$, and the processes X, Y, W satisfy the SDE (2.3) (see e.g. Section 5.3 in [8]).

Now let $x, y \in [0, \infty)$ and $(\mu, \sigma) \in \mathcal{M}$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P}, W, X, Y)$ be a weak solution of (2.3) with initial condition $(X_0, Y_0) = (x, y)$. The probability that both companies survive is given by

$$J(x, y, \mu, \sigma) := \mathbb{P}\left[\inf_{t\in[0,\infty)} X_t \geq 0,\ \inf_{t\in[0,\infty)} Y_t \geq 0\right]. \tag{2.4}$$

We refer to $J(x, y, \mu, \sigma)$ as the *joint survival probability* of the two companies, given initial endowments (x, y) and a collaboration control (μ, σ). The maximal joint survival probability for an initial endowment $(x, y) \in [0, \infty)^2$ is given by

$$V(x, y) := \sup_{(\mu,\sigma)\in\mathcal{M}} J(x, y, \mu, \sigma). \tag{2.5}$$

We comment further on the model assumptions. Notice that we allow for Markov controls only. The time-homogeneous dynamics (2.3) entail that there exists an optimal control that is a Markov control. To simplify the outline of the model, we restrict the control set to Markov controls upfront.

Notice that the volatilities of both processes X and Y are bounded away from zero. Hence the probability in (2.4) does not change if we replace $\geq$ with the strict inequality symbol $>$.

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Assumption (2.1) means that the drift rate of the aggregate endowment process $X+Y$ is positive. If M is non-positive, then with probability one the aggregate process hits zero. This further implies that at least one of the two companies gets ruined, and hence the value function (2.5) is constant equal to zero. Thus the only interesting case is where (2.1) is satisfied.

The symmetry (2.2) of D facilitates the search for the optimal strategy and a closed-form formula of the value function that turns out to be symmetric around the line bisecting the first quadrant.

It turns out that the maximal joint survival probability essentially depends only on the two ratios

$$L := L(D) = \sup_{(\mu,\sigma)\in D} \frac{\mu}{\sigma} \quad \text{and} \quad S := \frac{\underline{\mu}+\overline{\mu}}{\underline{\sigma}+\overline{\sigma}} = \frac{M}{\Sigma}.$$

Notice that $L \le \overline{\mu}/\underline{\sigma} < \infty$, because $D \subseteq [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]$.

Our main result is as follows.

Theorem 2.1. *The value function of the optimal control problem* (2.5) *is given by*

$$V(x,y) = \begin{cases} 1 - \mathrm{e}^{-2L\min\{x,y\}} - 2L\min\{x,y\}\,\mathrm{e}^{-L(x+y)}, & L = 2S, \\ 1 - \mathrm{e}^{-2L\min\{x,y\}} - \dfrac{L}{L-2S}\,\mathrm{e}^{-2S(x+y)}\left(1 - \mathrm{e}^{-2(L-2S)\min\{x,y\}}\right), & L \neq 2S. \end{cases} \tag{2.6}$$

If L is attained in D, say by $(\hat{\mu}, \hat{\sigma})$, then an optimal control is given by

$$(\mu^*(x,y), \sigma^*(x,y)) = \mathbb{1}_{\{x\le y\}}(\hat{\mu}, \hat{\sigma}) + \mathbb{1}_{\{x>y\}}(M-\hat{\mu}, \Sigma-\hat{\sigma}). \tag{2.7}$$

Remark 2.1.

(a) The value function V only depends on the ratios L – the maximal implementable risk-adjusted return – and S – the risk-adjusted return of the aggregate endowment process.

(b) One can show that the value function V is continuous and strictly increasing in L and S. This fact is supported by the following observations. Since L is the maximal possible risk-adjusted return which is assigned to the company behind, increasing L implies that the joint survival probability increases, too. In addition, the higher S, the smaller is the ruin probability of the aggregated endowment process.

Remark 2.2. Observe that we can change the definition of the optimal control (μ^*, σ^*) on the set $\{x=y\}$ and obtain indistinguishable processes (X_t^*, Y_t^*), $t \in [0,\infty)$, because with probability one the set $\{t \in [0,\infty) : X_t^* - Y_t^* = 0\}$ has Lebesgue measure zero; for details see Appendix C in [2].

Since we have an explicit formula for the value function, we can quantify the gain of collaboration. To this end, we assume that in the case of no collaboration both endowment processes have a constant drift rate of $M/2$ and a constant diffusion rate of $\sqrt{\Sigma/2}$.

The probability for a Brownian motion with drift rate $M/2$ and diffusion rate $\sqrt{\Sigma/2}$ and starting in $z \in (0,\infty)$ to hit zero is given by e^{-2Sz} (see e.g. [1, Chapter V.5, equation (5.6)]). Thus, in the case of no collaboration, the joint survival probability is given by

$$V_{nc}(x,y) = \left(1 - \mathrm{e}^{-2Sx}\right)\left(1 - \mathrm{e}^{-2Sy}\right). \tag{2.8}$$

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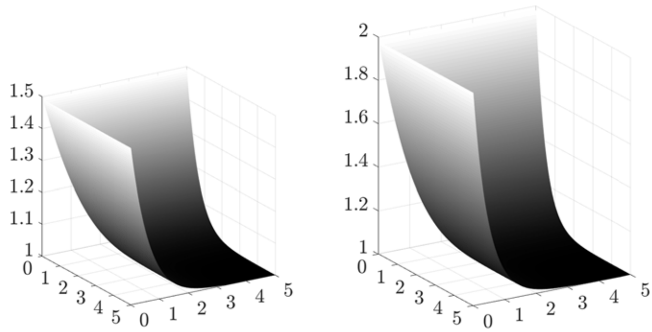

$L = \frac{3}{2}$, $S = 1$. $L = 2$, $S = 1$.

FIGURE 1. The function R for different L and $S = 1$.

Notice that (2.8) also follows from (2.6) by restricting D to the set containing only the element $(M/2, \Sigma/2)$.

In order to quantify the gain of collaboration we introduce

$$R(x, y) := \frac{V(x, y)}{V_{nc}(x, y)}, \qquad x, y > 0.$$

Note that R is the relative increase of the maximal joint survival probability due to a collaboration.

Corollary 2.1. *R is non-increasing in both x and y,*

$$\lim_{x \downarrow 0} R(x, y) = \frac{L}{S} \quad \textit{and} \quad \lim_{x \to \infty} R(x, y) = \frac{1 - e^{-2 L y}}{1 - e^{-2 S y}}.$$

Moreover, for every $a > 0$ we have

$$\lim_{x \downarrow 0} R(x, a\,x) = \frac{L}{S} \quad \textit{and} \quad \lim_{x \to \infty} R(x, a\,x) = 1.$$

Remark 2.3. For a set D of implementable drift and diffusion rate satisfying (2.2) and $L > S$, the relative increase of the maximal joint survival probability also only depends on L and S. Corollary 2.1 implies that a risk transfer is of particular interest if one company is (or both companies are) close to ruin.

See Figure 1 for the function R for different L and $S = 1$.

Remark 2.4. Observe that $L = L(D) \geq S > 0$. Moreover, we have $L > S$ if and only if there exists $(\mu, \sigma) \in D$ with $\mu/\sigma \neq S$. To show the claim we distinguish three cases.

- If D contains an element (μ, σ) with $\mu/\sigma > S$, then also $L = \sup_{(\mu,\sigma)\in D} \mu/\sigma > S$.
- If there exists $(\mu, \sigma) \in D$ with $\mu/\sigma < S$, then $(M - \mu, \Sigma - \sigma) \in D$ by assumption (2.2) and

$$S = \frac{M}{\Sigma} < \frac{M - \mu}{\Sigma - \sigma} \leq L.$$

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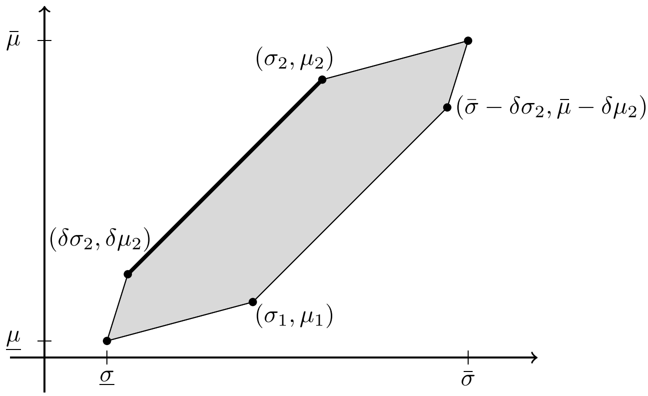

FIGURE 2. The set D for $\mu_1/\sigma_1 < \mu_2/\sigma_2$ in Example 2.1.

- Finally, if $\mu/\sigma = S$ for all $(\mu, \sigma) \in D$, then $L = S$ holds true.

We close the section with an example for the set D of implementable drift and diffusion rate.

Example 2.1. Suppose that the firms can divide among them two assets, where the first asset has a return with drift rate $\mu_1 \in \mathbb{R}$ and diffusion rate $\sigma_1 \in (0, \infty)$ and the second asset has drift rate $\mu_2 \in \mathbb{R}$ and diffusion rate $\sigma_2 \in (0, \infty)$. Suppose that $\mu_1 + \mu_2 > 0$ with $\mu_1 < \mu_2$, $\sigma_1 < \sigma_2$ and that each firm wants to possess at least $\delta \in (0, 1)$ assets shares. Then the set of implementable drift and diffusion pairs consists of

$$D = \{\alpha(\mu_1, \sigma_1) + \beta(\mu_2, \sigma_2) \colon \alpha, \beta \in [0, 1], \delta \le \alpha + \beta \le 2 - \delta\}.$$

Notice that the smallest rectangle containing D is $[\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]$ with

$$\underline{\mu} = \delta\mu_1, \quad \overline{\mu} = (1 - \delta)\mu_1 + \mu_2,$$
$$\underline{\sigma} = \delta\sigma_1, \quad \overline{\sigma} = (1 - \delta)\sigma_1 + \sigma_2.$$

In the case $\mu_1/\sigma_1 < \mu_2/\sigma_2$, the set D is a hexagon (see Figure 2). The bold borderline of the hexagon consists of all drift and diffusion pairs with maximal ratio L. Hence any optimal control assigns to the firm with smaller endowment a pair from the bold line, i.e. a share of the second asset but not of the first one. The example reveals, in particular, that the optimal control is not unique in general.

3. Deriving the value function

In this section we explain how one can derive a solution of the Hamilton–Jacobi–Bellman (HJB) equation associated to (2.5) and thus obtain a candidate for the value function V. Our approach is based on [5], where a ruin problem for two independent Brownian motions with controllable drift is considered. In our setting this corresponds to $\underline{\sigma} = \overline{\sigma} = 1$ and $\underline{\mu} > 0$.

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First observe that the HJB equation associated to (2.5) is given by

$$\sup_{(\mu,\sigma)\in D}\left\{\frac{\sigma}{2}\frac{\partial^2}{\partial x^2}v+\frac{\Sigma-\sigma}{2}\frac{\partial^2}{\partial y^2}v+\mu\frac{\partial}{\partial x}v+(M-\mu)\frac{\partial}{\partial y}v\right\}=0,\quad \text{on } (0,\infty)\times(0,\infty) \tag{3.1}$$

with boundary conditions

$$v(x,0)=v(0,y)=0,\qquad x,y\in[0,\infty), \tag{3.2}$$

$$\lim_{y\to\infty} v(x,y)=1-\mathrm{e}^{-2Lx},\quad x\in[0,\infty), \tag{3.3}$$

$$\lim_{x\to\infty} v(x,y)=1-\mathrm{e}^{-2Ly},\quad y\in[0,\infty). \tag{3.4}$$

We now comment on these boundary conditions for the HJB equation (3.1). Condition (3.2) is due to the fact that if one endowment process is already zero, then the joint survival probability equals zero. If the endowment process of one company attains infinity, then this process is assumed to survive forever. The smaller process obtains the highest possible risk-adjusted return to maximize its survival probability, which is given by the right-hand side of equation (3.3) or (3.4), respectively (see e.g. [1, Chapter V.5, equation (5.6)]).

We first consider the case where the set of implementable drift and diffusion rate is given by the rectangle $D=[\underline{\mu},\overline{\mu}]\times[\underline{\sigma},\overline{\sigma}]$. In this case $L=\overline{\mu}/\underline{\sigma}$. Moreover, the supremum over D in (3.1) can be separated and the HJB equation is given by

$$\sup_{\sigma\in[\underline{\sigma},\overline{\sigma}]}\left\{\frac{\sigma}{2}\frac{\partial^2}{\partial x^2}v+\frac{\Sigma-\sigma}{2}\frac{\partial^2}{\partial y^2}v\right\}+\sup_{\mu\in[\underline{\mu},\overline{\mu}]}\left\{\mu\frac{\partial}{\partial x}v+(M-\mu)\frac{\partial}{\partial y}v\right\}=0 \tag{3.5}$$

on $(0,\infty)\times(0,\infty)$.

Note that in the HJB equation (3.5) we maximize a linear function in σ and μ, respectively, over a compact interval. Hence each supremum is attained at the boundary of the corresponding interval. More precisely,

$$\sup_{\sigma\in[\underline{\sigma},\overline{\sigma}]}\left\{\frac{\sigma}{2}\frac{\partial^2}{\partial x^2}v+\frac{\Sigma-\sigma}{2}\frac{\partial^2}{\partial y^2}v\right\}(x,y)$$
$$=\begin{cases}\dfrac{\overline{\sigma}}{2}\dfrac{\partial^2}{\partial x^2}v(x,y)+\dfrac{\underline{\sigma}}{2}\dfrac{\partial^2}{\partial y^2}v(x,y), & \text{if } \dfrac{\partial^2}{\partial x^2}v(x,y)\ge\dfrac{\partial^2}{\partial y^2}v(x,y),\\[2ex] \dfrac{\underline{\sigma}}{2}\dfrac{\partial^2}{\partial x^2}v(x,y)+\dfrac{\overline{\sigma}}{2}\dfrac{\partial^2}{\partial y^2}v(x,y), & \text{otherwise,}\end{cases}$$

and

$$\sup_{\mu\in[\underline{\mu},\overline{\mu}]}\left\{\mu\frac{\partial}{\partial x}v+(M-\mu)\frac{\partial}{\partial y}v\right\}(x,y)$$
$$=\begin{cases}\overline{\mu}\dfrac{\partial}{\partial x}v(x,y)+\underline{\mu}\dfrac{\partial}{\partial y}v(x,y), & \text{if } \dfrac{\partial}{\partial x}v(x,y)\ge\dfrac{\partial}{\partial y}v(x,y),\\[2ex] \underline{\mu}\dfrac{\partial}{\partial x}v(x,y)+\overline{\mu}\dfrac{\partial}{\partial y}v(x,y), & \text{otherwise.}\end{cases}$$

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In the following we make several assumptions on the solution v of the HJB equation. After obtaining the explicit formula given on the right-hand side of (2.6) we can check that all the assumptions are satisfied. Finally, one has to verify that v is indeed the value function of our problem (2.5).

We assume that v is a classical solution of the HJB equation (3.5), that is,

$$v \in C^2((0, \infty) \times (0, \infty)) \cap C([0, \infty) \times [0, \infty))$$

with boundary conditions (3.2), (3.3), and (3.4). Since our control problem (2.5) is symmetric in the initial values of the endowment processes, every candidate v for the value function should satisfy $v(x, y) = v(y, x)$. Due to this symmetry and the monotonicity of the maximization problem (2.5), we impose that

$$\left\{(x, y)\colon \frac{\partial}{\partial x} v(x, y) > \frac{\partial}{\partial y} v(x, y)\right\} = \left\{(x, y)\colon \frac{\partial^2}{\partial x^2} v(x, y) < \frac{\partial^2}{\partial y^2} v(x, y)\right\} = \{(x, y)\colon x < y\}$$

and

$$\left\{(x, y)\colon \frac{\partial}{\partial x} v(x, y) = \frac{\partial}{\partial y} v(x, y)\right\} = \left\{(x, y)\colon \frac{\partial^2}{\partial x^2} v(x, y) = \frac{\partial^2}{\partial y^2} v(x, y)\right\} = \{(x, y)\colon x = y\}.$$

Observe that this implies that the smaller endowment process is assigned the lowest possible volatility and the highest possible drift rate to minimize the risk that this firm is ruined. In other words, the agent chooses the maximal implementable risk-adjusted return for the company behind.

Using $v(x, y) = v(y, x)$, we only focus on the set

$$G = \{(x, y) \in [0, \infty) \times [0, \infty)\colon x \leq y\}.$$

In the interior of G it holds – under our assumptions – that v has to satisfy

$$\frac{\underline{\sigma}}{2} \frac{\partial^2}{\partial x^2} v + \frac{\overline{\sigma}}{2} \frac{\partial^2}{\partial y^2} v + \overline{\mu} \frac{\partial}{\partial x} v + \underline{\mu} \frac{\partial}{\partial y} v = 0 \tag{3.6}$$

with

$$\begin{aligned} v(0, y) &= 0, \\ \lim_{y \to \infty} v(x, y) &= 1 - \mathrm{e}^{-2Lx}, \\ \frac{\partial}{\partial x} v(t, t) &= \frac{\partial}{\partial y} v(t, t), \qquad t \in (0, \infty), \\ \frac{\partial^2}{\partial x^2} v(t, t) &= \frac{\partial^2}{\partial y^2} v(t, t), \qquad t \in (0, \infty). \end{aligned} \tag{3.7}$$

We make the ansatz

$$v(x, y) = 1 - \mathrm{e}^{-2Lx} + f(x) g(y), \qquad (x, y) \in G.$$

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The function $(x, y) \mapsto f(x)g(y)$ fulfills (3.6) in the interior of G. More precisely,

$$\frac{\underline{\sigma}}{2} f''(x)g(y) + \frac{\overline{\sigma}}{2} f(x)g''(y) + \overline{\mu}\, f'(x)g(y) + \underline{\mu}\, f(x)g'(y) = 0 \tag{3.8}$$

with

$$f(0)g(y) = 0, \qquad y \in [0, \infty), \tag{3.9}$$

$$\lim_{y\to\infty} f(x)g(y) = 0, \qquad x \in [0, \infty), \tag{3.10}$$

$$f(t)g'(t) - f'(t)g(t) = 2\,L\,\mathrm{e}^{-2\,L\,t}, \qquad t \in (0, \infty). \tag{3.11}$$

Note that we do not impose an additional assumption on $(x, y) \mapsto f(x)g(y)$ to guarantee (3.7) because it turns out that the solution that we construct for (3.8) satisfying (3.9), (3.10), and (3.11) directly implies that condition (3.7) for v is fulfilled; see (3.18) below.

Provided that $f(x)g(y) \neq 0$ for all (x, y) in the interior of G, equation (3.8) can be reformulated as

$$\left(\frac{\underline{\sigma}}{2}\frac{f''}{f} + \overline{\mu}\frac{f'}{f}\right)(x) + \left(\frac{\overline{\sigma}}{2}\frac{g''}{g} + \underline{\mu}\frac{g'}{g}\right)(y) = 0.$$

The above equation can only hold true for all (x, y) in the interior of G if

$$\left(\frac{\underline{\sigma}}{2}\frac{f''}{f} + \overline{\mu}\frac{f'}{f}\right)(x) = \lambda, \tag{3.12}$$

$$\left(\frac{\overline{\sigma}}{2}\frac{g''}{g} + \underline{\mu}\frac{g'}{g}\right)(y) = -\lambda \tag{3.13}$$

for some $\lambda \in \mathbb{R}$.

First we consider the case $\boxed{L \neq 2S}$. This case is a bit more involved than the case $L = 2S$. We assume that

$$\lambda \in \left(-\frac{\overline{\mu}^2}{2\underline{\sigma}}, \frac{\underline{\mu}^2}{2\overline{\sigma}}\right),$$

which guarantees real-valued solutions to (3.12) and (3.13). Later on we have to choose λ in an appropriate way such that the boundary condition (3.11) is fulfilled. By Theorem 1 and Theorem 5 in [3, Chapter 2], we obtain

$$f(x) = C_1 \exp\left((-L + \vartheta_1)\,x\right) + C_2 \exp\left((-L - \vartheta_1)\,x\right),$$

$$g(y) = C_3 \exp\left(\left(-\frac{\underline{\mu}}{\overline{\sigma}} + \vartheta_2\right) y\right) + C_4 \exp\left(\left(-\frac{\underline{\mu}}{\overline{\sigma}} - \vartheta_2\right) y\right)$$

for some $C_1, C_2, C_3, C_4 \in \mathbb{R}$, where

$$\vartheta_1 = \vartheta_1(\lambda) = \frac{\sqrt{\overline{\mu}^2 + 2\,\underline{\sigma}\,\lambda}}{\underline{\sigma}}, \qquad \vartheta_2 = \vartheta_2(\lambda) = \frac{\sqrt{\underline{\mu}^2 - 2\,\overline{\sigma}\,\lambda}}{\overline{\sigma}}.$$

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From (3.9) we conclude that $f(0)=0$ and hence $C_2=-C_1$. Since we are only interested in the product $f(x)g(y)$, we can assume that $C_1=1$ without loss of generality. Note that for $\lambda\in(-\overline{\mu}^2/(2\underline{\sigma}), 0]$ condition (3.10) yields $C_3=0$. Unfortunately, for $\lambda\in(0, \underline{\mu}^2/(2\overline{\sigma}))$ this does not hold true. Nevertheless, we set $C_3=0$ and hope to obtain a solution. In addition, condition (3.11) on the diagonal results in

$$2\,L\exp\left(-2\,L\,t\right)=C_4\left[L-\vartheta_1-\frac{\underline{\mu}}{\overline{\sigma}}-\vartheta_2\right]\exp\left(\left(-L+\vartheta_1-\frac{\underline{\mu}}{\overline{\sigma}}-\vartheta_2\right)t\right)$$
$$+\,C_4\left[-L-\vartheta_1+\frac{\underline{\mu}}{\overline{\sigma}}+\vartheta_2\right]\exp\left(\left(-L-\vartheta_1-\frac{\underline{\mu}}{\overline{\sigma}}-\vartheta_2\right)t\right), \qquad (3.14)$$

which has to be satisfied for all $t\in(0,\infty)$. Therefore it is necessary that the exponent of one summand coincides with $-2\,L\,t$. This directly implies that the coefficient of the other summand vanishes. More precisely, we determine λ such that

$$L-\vartheta_1+\frac{\underline{\mu}}{\overline{\sigma}}+\vartheta_2=2\,L \qquad (3.15)$$

or

$$L+\vartheta_1+\frac{\underline{\mu}}{\overline{\sigma}}+\vartheta_2=2\,L. \qquad (3.16)$$

Some standard but lengthy computations show that

$$\lambda^*=-2\,S\,\frac{\overline{\mu}\,\overline{\sigma}-\underline{\mu}\,\underline{\sigma}}{\underline{\sigma}+\overline{\sigma}}$$

is the unique

$$\lambda\in\left(-\frac{\overline{\mu}^2}{2\underline{\sigma}}, \frac{\underline{\mu}^2}{2\overline{\sigma}}\right)$$

satisfying either (3.15) or (3.16). More precisely, if $L<2S$ then (3.15) holds, and (3.16) is fulfilled if $L>2S$.

For $\lambda=\lambda^*$ and $L<2S$ equation (3.14) is given by

$$2\,L\exp(-2\,L\,t)=2\,C_4[L-2\,S]\exp(-2\,L\,t).$$

Thus

$$C_4=\frac{L}{L-2\,S}.$$

Similarly, for $L>2\,S$ we conclude that

$$C_4=-\frac{L}{L-2\,S}.$$

To sum up, we have

$$f(x)g(y)=\frac{L}{L-2\,S}\mathrm{e}^{-2\,S\,y}\left(\mathrm{e}^{2\,(S-L)\,x}-\mathrm{e}^{-2\,S\,x}\right)$$
$$=-\frac{L}{L-2\,S}\mathrm{e}^{-2\,S\,(x+y)}\left(1-\mathrm{e}^{-2\,(L-2\,S)\,x}\right). \qquad (3.17)$$

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Now we use the symmetry of our problem and obtain v on $[0, \infty) \times [0, \infty)$ by just mirroring $1 - e^{-2Lx} + f(x)g(y)$, where $(x, y) \mapsto f(x)g(y)$ is given by (3.17) at the line bisecting the first quadrant, which yields that v is given by the right-hand side of (2.6).

It remains to check that the assumptions made on v are satisfied. Indeed, it holds that $f(x)g(y) \neq 0$ for all $x, y \in (0, \infty)$, the function given on the right-hand side of (2.6) is in $C^2((0, \infty) \times (0, \infty)) \cap C([0, \infty) \times [0, \infty))$, and

$$\frac{\partial}{\partial x} v(x, y) - \frac{\partial}{\partial y} v(x, y) = \begin{cases} 2L e^{-2Lx}\big(1 - e^{-2S(y-x)}\big), & x \le y \\ -2L e^{-2Ly}\big(1 - e^{-2S(x-y)}\big), & x > y \end{cases} \begin{cases} > 0, & x < y, \\ = 0, & x = y, \\ < 0, & x > y, \end{cases}$$
$$\frac{\partial^2}{\partial x^2} v(x, y) - \frac{\partial^2}{\partial y^2} v(x, y) = \begin{cases} -4L^2 e^{-2Lx}\big(1 - e^{-2S(y-x)}\big), & x \le y \\ 4L^2 e^{-2Ly}\big(1 - e^{-2S(x-y)}\big), & x > y \end{cases} \begin{cases} < 0, & x < y, \\ = 0, & x = y, \\ > 0, & x > y. \end{cases} \tag{3.18}$$

Hence all assumptions made on v are satisfied.

For the case $\boxed{L = 2S}$ we also use

$$\lambda^* = -2S \frac{\overline{\mu}\,\overline{\sigma} - \underline{\mu}\,\underline{\sigma}}{\underline{\sigma} + \overline{\sigma}},$$

which in this case simplifies to $\lambda^* = -S\overline{\mu}$. Then the solutions of (3.12) and (3.13) are given by

$$f(x) = (C_1 + C_2 x)\exp(-Lx),$$
$$g(y) = C_3 \exp\left(\frac{\overline{\mu}}{\overline{\sigma}}\, y\right) + C_4 \exp(-Ly)$$

for some constants $C_1, C_2, C_3, C_4 \in \mathbb{R}$; again see Theorems 1 and 5 in [3, Chapter 2]. Using (3.9) we conclude that $C_1 = 0$ and (3.10) implies that $C_3 = 0$. Thus

$$f(x)g(y) = C_2 C_4\, x\, \exp(-L(x + y)).$$

Using (3.11) results in $C_2 C_4 = -2L$, and mirroring at the line bisecting the first quadrant yields

$$v(x, y) = 1 - e^{-2L\min\{x,y\}} - 2L\min\{x, y\}\, e^{-L(x+y)}. \tag{3.19}$$

Finally, one can check that the function in (3.19) satisfies all our assumptions made on v.

In the next step we explain how to obtain a solution of the HJB equation (3.1) if D is a proper subset of $[\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]$. As a candidate v for the value function we choose the function on the right-hand side of (2.6), which we derived in the case where D is a rectangle, and adjust the maximal risk-adjusted return L to $L = \sup_{(\mu,\sigma)\in D} \mu/\sigma$. Recall that the risk-adjusted return of the aggregate endowment process equals S and does not have to be changed. Now we want to

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show that our candidate solves the HJB equation (3.1). To this end, observe that for $(\mu, \sigma) \in D$ we have

$$\left(\frac{\sigma}{2}\frac{\partial^2}{\partial x^2}v + \frac{\Sigma-\sigma}{2}\frac{\partial^2}{\partial y^2}v + \mu\,\frac{\partial}{\partial x}v + (M-\mu)\,\frac{\partial}{\partial y}v\right)(x, y)$$
$$= \begin{cases} -2\,\sigma\,L\,\mathrm{e}^{-2Lx}\big(1-\mathrm{e}^{-2S(y-x)}\big)\left[L-\dfrac{\mu}{\sigma}\right], & x \le y, \\ -2\,(\Sigma-\sigma)\,L\,\mathrm{e}^{-2Ly}\big(1-\mathrm{e}^{-2S(x-y)}\big)\left[L-\dfrac{M-\mu}{\Sigma-\sigma}\right], & x > y. \end{cases} \tag{3.20}$$

Since D satisfies (2.2), we have

$$L = \sup_{(\mu,\sigma)\in D} \frac{\mu}{\sigma} = \sup_{(\mu,\sigma)\in D} \frac{M-\mu}{\Sigma-\sigma}.$$

Hence, for all $x, y \in (0, \infty)$,

$$\sup_{(\mu,\sigma)\in D}\left\{\frac{\sigma}{2}\frac{\partial^2}{\partial x^2}v + \frac{\Sigma-\sigma}{2}\frac{\partial^2}{\partial y^2}v + \mu\,\frac{\partial}{\partial x}v + (M-\mu)\,\frac{\partial}{\partial y}v\right\}(x, y) \le 0.$$

For simplicity we assume that L is attained in D, say by $(\hat{\mu}, \hat{\sigma})$. Then $L = \hat{\mu}/\hat{\sigma}$ and (3.20) equals zero for $(\hat{\mu}, \hat{\sigma})$ if $x < y$. If $x > y$ then (3.20) is zero for $(M-\hat{\mu}, \Sigma-\hat{\sigma})$. Therefore the HJB equation (3.1) is fulfilled and v is a candidate for our value function.

Now it remains to verify that the right-hand side of (2.6) is indeed the value function of the optimal control problem (2.5), i.e. to prove Theorem 2.1.

4. Proofs

First, we prove our main result, Theorem 2.1.

Proof of Theorem 2.1. Let v denote the function given by the right-hand side of (2.6). We first show that v is an upper bound for the joint survival probability. For this purpose let $(\mu, \sigma) \in \mathcal{M}$ be an arbitrary admissible control for the drift and diffusion rate. Denote the ruin time of the controlled process $(X_t, Y_t) = \big(X_t^{x,\mu,\sigma}, Y_t^{y,\mu,\sigma}\big)$ by

$$\tau = \inf\{t \in [0, \infty) \colon X_t \le 0 \text{ or } Y_t \le 0\}.$$

Using that $v \in C^2((0, \infty) \times (0, \infty))$, Itô's formula implies

$$\begin{aligned} v(X_t, Y_t) &= v(x, y) + \int_0^t \sqrt{\sigma(X_s, Y_s)}\,\frac{\partial}{\partial x}\,v(X_s, Y_s)\,\mathrm{d}W_s^1 \\ &\quad + \int_0^t \sqrt{\Sigma-\sigma(X_s, Y_s)}\,\frac{\partial}{\partial y}\,v(X_s, Y_s)\,\mathrm{d}W_s^2 \\ &\quad + \int_0^t \left\{\frac{1}{2}\sigma\frac{\partial^2}{\partial x^2}v + \frac{1}{2}(\Sigma-\sigma)\frac{\partial^2}{\partial y^2}v + \mu\frac{\partial}{\partial x}v + (M-\mu)\frac{\partial}{\partial y}v\right\}(X_s, Y_s)\,\mathrm{d}s. \end{aligned} \tag{4.1}$$

Since v solves the HJB equation (3.1), the drift part in (4.1) is non-positive. Hence $(v(X_t, Y_t))_{t\in[0,\infty)}$ and thus $(v(X_{t\wedge\tau}, Y_{t\wedge\tau}))_{t\in[0,\infty)}$ are local supermartingales. Moreover, since

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v is bounded, $(v(X_{t\wedge\tau}, Y_{t\wedge\tau}))_{t\in[0,\infty)}$ is a uniformly integrable supermartingale. Therefore the supermartingale convergence theorem yields that

$$\lim_{t\to\infty} v(X_{t\wedge\tau}, Y_{t\wedge\tau})$$

exists $\mathbb{P}$-a.s. By dominated convergence we conclude that

$$v(x, y) \geq \lim_{t\to\infty} \mathbb{E}[v(X_{t\wedge\tau}, Y_{t\wedge\tau})] = \mathbb{E}\Big[\mathbb{1}_{\{\tau<\infty\}} v(X_\tau, Y_\tau) + \mathbb{1}_{\{\tau=\infty\}} \lim_{t\to\infty} v(X_t, Y_t)\Big]. \tag{4.2}$$

On $\{\tau < \infty\}$ the boundary conditions (3.2) imply that $v(X_\tau, Y_\tau) = 0$. We claim that on $\{\tau = \infty\}$ we have $\lim_{t\to\infty} v(X_t, Y_t) = 1$.

In order to show this, first observe that

$$(X + Y)_t = x + y + Mt + \sqrt{\Sigma}\, W_t, \tag{4.3}$$

where W is a Brownian motion. Thus we know that $\lim_{t\to\infty} (X + Y)_t = \infty$, $\mathbb{P}$-a.s. Moreover, the supermartingale convergence theorem guarantees that on $\{\tau = \infty\}$,

$$\lim_{t\to\infty} v(X_t, Y_t)$$

exists $\mathbb{P}$-a.s. Combining this with the particular form of v yields that on $\{\tau = \infty\}$,

$$\lim_{t\to\infty} \mathrm{e}^{-2L\min\{X_t, Y_t\}}$$

exists, so $\lim_{t\to\infty} \min\{X_t, Y_t\} \in \mathbb{R} \cup \{+\infty\}$ exists $\mathbb{P}$-a.s. We now show that

$$\mathbb{P}\Big[\lim_{t\to\infty} \min\{X_t, Y_t\} < \infty\Big] = 0.$$

By (4.3) and the identity

$$2\min\{X_t, Y_t\} = X_t + Y_t - |X_t - Y_t|,$$

it follows that on $\{\lim_{t\to\infty} \min\{X_t, Y_t\} < \infty\}$ we have $\lim_{t\to\infty} |X_t - Y_t| = \infty$. Hence there exists a time point $t_0 = t_0(\omega)$ beyond which the paths of X and Y do not intersect. Since the paths are continuous this implies that

$$\min\{X_t, Y_t\} = X_t \ \text{ for all } t \geq t_0 \quad \text{or} \quad \min\{X_t, Y_t\} = Y_t \ \text{ for all } t \geq t_0.$$

Thus we have

$$\Big\{\lim_{t\to\infty} \min\{X_t, Y_t\} < \infty\Big\} = \Big\{\lim_{t\to\infty} X_t < \infty\Big\} \cup \Big\{\lim_{t\to\infty} Y_t < \infty\Big\} \quad \mathbb{P}\text{-a.s.}$$

Now, to show that $\mathbb{P}[\lim_{t\to\infty} X_t < \infty] = 0$, recall that

$$X_t = x + \int_0^t \mu(X_s, Y_s)\,\mathrm{d}s + \int_0^t \sqrt{\sigma(X_s, Y_s)}\,\mathrm{d}W_s^1.$$

Let $A(t) := \int_0^t \sigma(X_t, Y_t)\,\mathrm{d}s$. Notice that $A(t)$ is strictly increasing, so we can introduce the time-changed process

$$\widetilde{X}_t := X_{A^{-1}(t)} = x + \int_0^{A^{-1}(t)} \mu(X_s, Y_s)\,\mathrm{d}s + \int_0^{A^{-1}(t)} \sqrt{\sigma(X_s, Y_s)}\,\mathrm{d}W_s^1.$$

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Note that

$$B_t := \int_0^{A^{-1}(t)} \sqrt{\sigma(X_s, Y_s)}\, \mathrm{d}W_s^1, \quad t \in [0, \infty),$$

is a Brownian motion since

$$\langle B, B \rangle_t = \int_0^{A^{-1}(t)} \sigma(X_s, Y_s)\, \mathrm{d}s = A(A^{-1}(t)) = t.$$

Further, a simple substitution in the deterministic integral yields

$$\widetilde{X}_t = x + \int_0^{A^{-1}(t)} \mu(X_s, Y_s)\, \mathrm{d}s + B_t = x + \int_0^t \frac{\mu}{\sigma}(X_s, Y_s)\, \mathrm{d}s + B_t.$$

We know that $(\mu/\sigma)(X_s, Y_s) \leq L$ for all s by the definition of L. For the Brownian motion B it is well known that

$$\mathbb{P}\big[\{B_{n+1} - B_n < -L - 1\} \text{ infinitely often}\big] = 1.$$

This directly implies

$$\mathbb{P}\big[\{\widetilde{X}_{n+1} - \widetilde{X}_n < -1\} \text{ infinitely often}\big] = 1.$$

Consequently, $\mathbb{P}[\lim_{t\to\infty} \widetilde{X}_t < \infty] = 0$. Moreover, $\underline{\sigma} > 0$ yields $\lim_{t\to\infty} A(t) = \infty$. Thus

$$\mathbb{P}\Big[\lim_{t\to\infty} X_t < \infty\Big] = \mathbb{P}\Big[\lim_{t\to\infty} \widetilde{X}_t < \infty\Big] = 0.$$

Similarly, one can show that Y does not converge with probability one. Hence we see that $\mathbb{P}[\lim_{t\to\infty} \min\{X_t, Y_t\} < \infty] = 0$. Therefore it follows that on $\{\tau = \infty\}$ we have

$$\mathbb{P}\Big[\lim_{t\to\infty} \min\{X_t, Y_t\} = \infty\Big] = 1$$

and the particular form of v implies that

$$\mathbb{P}\Big[\lim_{t\to\infty} v(X_t, Y_t) = 1\Big] = 1. \tag{4.4}$$

Thus, plugging (4.4) into (4.2), we see that

$$v(x, y) \geq \mathbb{E}\big[\mathbb{1}_{\{\tau=\infty\}}\big] = \mathbb{P}[\tau = \infty] = J(x, y, \mu, \sigma), \tag{4.5}$$

and hence $v \geq V$.

Now assume that L is attained in D. Then the strategy (μ^*, σ^*) given in (2.7) is admissible, for (μ^*, σ^*) the drift rate in (4.1) vanishes, and therefore the process $(v(X_{t\wedge\tau}, Y_{t\wedge\tau}))_{t\in[0,\infty)}$ is a uniformly integrable martingale. Hence equality holds in (4.5), which implies that v is the value function of the optimal control problem (2.5) and (μ^*, σ^*) is an optimal control.

So far we have shown that the value function V is given by the right-hand side of (2.6) if L is attained in D. Now we consider the case where L is not attained in D, i.e. $\arg\max_{(\mu,\sigma)\in D} \mu/\sigma = \emptyset$. Then there exists a sequence $(\mu_n, \sigma_n)_{n\in\mathbb{N}} \subseteq D$ with $L_n := \mu_n/\sigma_n \nearrow L$ as $n \to \infty$. Without loss of generality we can assume that $L_n \geq S$ (see Remark 2.4) and that

$$\lim_{n\to\infty} \mu_n = \tilde{\mu} \in [\underline{\mu}, \overline{\mu}],$$

$$\lim_{n\to\infty} \sigma_n = \tilde{\sigma} \in [\underline{\sigma}, \overline{\sigma}],$$

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because $D \subseteq [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]$. In particular, we have $\tilde{\mu}/\tilde{\sigma} = L$. Let

$$\widetilde{D} = D \cup \{(\tilde{\mu}, \tilde{\sigma}), (M - \tilde{\mu}, \Sigma - \tilde{\sigma})\}.$$

Then $\widetilde{D} \subseteq [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]$, $\widetilde{D}$ satisfies (2.2) and

$$L(\widetilde{D}) = \sup_{(\mu,\sigma)\in\widetilde{D}} \frac{\mu}{\sigma} = \max\left\{ \sup_{(\mu,\sigma)\in D} \frac{\mu}{\sigma}, \frac{\tilde{\mu}}{\tilde{\sigma}}, \frac{M-\tilde{\mu}}{\Sigma-\tilde{\sigma}} \right\} = \frac{\tilde{\mu}}{\tilde{\sigma}} = L,$$

where we use $L \geq S = M/\Sigma$ by Remark 2.4 and thus $(M - \tilde{\mu})/(\Sigma - \tilde{\sigma}) \leq S \leq L$. In particular, $(\tilde{\mu}, \tilde{\sigma}) \in \arg\max_{(\mu,\sigma)\in\widetilde{D}} \mu/\sigma$. Hence the value function $V^{L(\widetilde{D})}$ for maximizing the joint survival probability over controls taking values in $\widetilde{D}$ is given by (2.6) with $L(\widetilde{D}) = L$. Moreover, $V \leq V^{L(\widetilde{D})} = V^L$.

To derive a lower bound for V, let

$$D_n = \{(\mu_n, \sigma_n), (M - \mu_n, \Sigma - \sigma_n)\}, \quad n \in \mathbb{N}.$$

By definition D_n satisfies (2.2). Since $(\mu_n, \sigma_n) \in D$, it holds that $D_n \subseteq D$. Moreover,

$$L(D_n) = \sup_{(\mu,\sigma)\in D_n} \frac{\mu}{\sigma} = \max\left\{ \frac{\mu_n}{\sigma_n}, \frac{M-\mu_n}{\Sigma-\sigma_n} \right\} = L_n,$$

since $\mu_n/\sigma_n = L_n \geq S$ and therefore

$$\frac{M-\mu_n}{\Sigma-\sigma_n} \leq S \leq L_n.$$

In particular, we have $(\mu_n, \sigma_n) \in \arg\max_{(\mu,\sigma)\in D_n} \mu/\sigma$. Hence the value function V^{L_n} of (2.5) for controls taking values in D_n is given by (2.6) and $V^{L_n} \leq V$. Since the function on the right-hand side of (2.6) is continuous in the parameter L, we conclude that for all $x, y \in [0, \infty)$

$$V^L(x, y) = \lim_{n\to\infty} V^{L_n} \leq V(x, y) \leq V^L(x, y).$$

Therefore, in this case the value function is given by (2.6), too. □

Finally, we prove Corollary 2.1.

Proof of Corollary 2.1. We only show that R is non-increasing. The other results follow by straightforward calculations.

Since R is symmetric, we only need to consider the part of the domain where $x \leq y$. Moreover, we only consider the case $L > 2S$. The cases $L < 2S$ and $L = 2S$ can be proved similarly.

One can show that $\partial R/\partial x$ is non-positive if and only if

$$\begin{aligned} &e^{2Sx}L(L-2S) + e^{2Lx}LS - e^{4Sx}L(L-S) \\ &\quad + e^{2Sy}(L-2S)[e^{2Sx}L - e^{2Lx}S - (L-S)] \leq 0. \end{aligned} \tag{4.6}$$

Since $L \geq S$, one can show by using convexity that $e^{2Sx}L - e^{2Lx}S - (L - S) \leq 0$. Thus the left-hand side of (4.6) is non-increasing in y. Hence (4.6) is fulfilled for all $y \geq x$ if and only if it is fulfilled for $y = x$. Thus we need to verify that

$$S\big[L\big(e^{2Lx} - e^{4Sx}\big) - (L-2S)\big(e^{2(L+S)x} - e^{2Sx}\big)\big] \leq 0. \tag{4.7}$$

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Inequality (4.7) is satisfied if and only if the term in the rectangular bracket on the left-hand side is non-positive, which is equivalent to

$$\frac{e^{2Lx} - e^{4Sx}}{L - 2S} \leq \frac{e^{2(L+S)x} - e^{2Sx}}{L},$$

and hence equivalent to

$$\frac{\sinh((L - 2S)x)}{(L - 2S)x} \leq \frac{\sinh(Lx)}{Lx}. \tag{4.8}$$

Inequality (4.8) holds true, because $z \mapsto \sinh(z)/z$ is strictly increasing for $z \geq 0$. To sum up, we have shown that (4.6) is satisfied and thus $\partial R/\partial x$ is non-positive.

The partial derivative $\partial R/\partial y$ can be shown to be non-positive if and only if

$$L\big(e^{2Lx} - e^{4Sx}\big) - (L - 2S)\big(e^{2(L+S)x} - e^{2Sx}\big) \leq 0.$$

The left-hand side coincides with the bracket terms of (4.7) and is thus non-positive. □

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