CATEGORICAL FIBRATIONS

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Introduction. If one considers the theories of Hurewicz, - Serre - or other fibrations in the categories of topological or pointed topological spaces, one can see that many of the fundamental theorems can be formulated and proven in the general case of categories for which certain functors and natural transformations are given. And, since fibrations may be defined either by a cylinder or a path space construction, we shall give in an analogous way two different definitions of fibrations in the general case. One we call a (Z, e)-fibration, where $Z:\mathcal{L}\to\mathcal{L}$ is a functor, $e:1\to Z$ a natural transformation; the other is called a (P, s)-fibration, where $P:\mathcal{L}\to\mathcal{L}$ and $s:P\to 1$. It will be shown in section 1 that, if Z is adjoint to P and e adjoint to s, these definitions coincide. In § 2 we will then prove the fundamental structure theorems on induced fibrations, path lifting property, liftings, cross sections, factoring through the fiber, irreducibility and strong deformation retractions. To prove all these theorems we need only the following: two functors Z, $P:\mathcal{L}\to\mathcal{L}$ with Z adjoint to P and three natural transformations e, $\bar{e}:1\to Z$, $r:Z\to 1$ such that $r\,e = 1 = r\,\bar{e}$. Our concepts of fibration and homotopy are modelled upon these concepts in topology in the same way as done by Kan [3].

We wish to make the remark here that we do not state our theorems on fibrations relative to a full subcategory, which is nevertheless necessary e.g. in case of Serre fibrations. But the reader will see himself that the theorems hold as well in the relative case when modified in the usual way. In addition, we shall assume that our category has a null object (like the category of

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pointed topological spaces). If this is not the case, one has to put instead of null object a point object, i.e. an object for which hom (X, pt) contains exactly one element for each object X, and to modify the theorems again in an obvious way. We exclude in our statements both cases since we want to state the theorems and proofs in a short and suggestive way.

Section 1. Preliminary remarks on categories. In the following we will assume that our category $\mathcal{L}$ has a null object and that products and cartesian squares always exist. We will use the following notations: $\overset{\alpha}{\rightarrowtail}$ indicates that the morphism α is monic, $\overset{\alpha}{\twoheadrightarrow}$ that α is epic and $\overset{\alpha}{\rightarrowtail\!\!\!\rightarrow}$ that α is an equivalence. The morphisms which factor through a null object are denoted by 0. A product of two objects C_1, C_2 is denoted by $C_1 \sqcap C_2$, the associated projections by $\pi_i : C_1 \sqcap C_2 \to C_i$ $(i=1,2)$. Consider now a diagram

$$\begin{array}{ccc} & & C_1 \\ & & \downarrow \varphi \\ C_2 & \overset{\rightarrow}{\psi} & C \end{array} \tag{1.1}$$

Then the associated cartesian square

$$\begin{array}{ccc} I & \overset{\psi'}{\rightarrow} & C_1 \\ \varphi' \downarrow & & \downarrow \varphi \\ C_2 & \overset{\rightarrow}{\psi} & C \end{array} \tag{1.2}$$

is denoted by $(\varphi', \psi') = I(\varphi, \psi)$. By the universal properties associated with the notions of product and cartesian square it is clear that they are determined up to canonical equivalences. If $\varphi_1 : C \to C_1$, $\varphi_2 : C \to C_2$ are any two morphisms, we will denote by $\{\varphi_1, \varphi_2\} : C \to C_1 \sqcap C_2$ the uniquely determined morphism commuting with the projections and, in case of $\varphi_1 : C_1 \to C_1'$, $\varphi_2 : C_2 \to C_2'$ we define $\varphi_1 \sqcap \varphi_2 = \{\varphi_1 \pi_1, \varphi_2 \pi_2\}$.

Let us also recall the definition of adjoint functors and state a fundamental theorem. Let $S : \mathcal{L} \to \mathcal{L}'$ and $T : \mathcal{L}' \to \mathcal{L}$ be functors. Then S is called adjoint to T and T coadjoint to S if there exists a natural equivalence $\phi : \hom_{\mathcal{L}'}(SC, C') \to \hom_{\mathcal{L}}(C, TC')$.

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Now the fundamental lemma:

PROPOSITION 1.3. Let $S_1, S_2 : \mathcal{L} \to \mathcal{L}'$ and $T_1, T_2 : \mathcal{L}' \to \mathcal{L}$ be functors and assume that S_i is adjoint to T_i by $\phi_i : \hom(S_i C, C') \to \hom(C, T_i C')$, $(i = 1, 2)$. Then, if $s : S_1 \to S_2$ is a natural transformation, there exists a unique natural transformation $t : T_2 \to T_1$ such that the square

$$(1.4) \qquad \begin{array}{ccc} \hom(S_2 C, C') & \overset{\phi_2}{\to} & \hom(C, T_2 C') \\ s^* \downarrow & & \downarrow t_* \\ \hom(S_1 C, C) & \overset{\phi_1}{\to} & \hom(C, T_1 C') \end{array}$$

always commutes. And conversely, if $t : T_2 \to T_1$ is a natural transformation there exists a unique natural transformation $s : S_1 \to S_2$ rendering (1.4) commutative.

The proof is straightforward and can be found in [4]. We will only notice here that given $s : S_1 \to S_2$, t is defined by $tC' = \phi_1(\phi_2^{-1} 1_{T_2} C' \circ sT_2 C') : T_2 C' \to T_1 C'$ and given $t : T_2 \to T_1$, s is defined by $sC = \phi_1^{-1}(tS_2 C \circ \phi_2 1_{S_2 C}) : S_1 C \to S_2 C$. If the diagram (1.4) commutes, we will call t coadjoint to s and s adjoint to t with respect to ϕ_1, ϕ_2.

<u>Section 2.</u> Z- <u>and</u> P- <u>Fibrations</u>. We will now define - analogously to the cylinder and path space constructions - two types of fibrations which we will call Z- and P- fibrations.

DEFINITION 2.1. Let $Z : \mathcal{L} \to \mathcal{L}$ be a functor and $e : 1 \to Z$ a natural transformation from the identity functor to Z. Then we call a morphism $\varphi : C_1 \to C$ a (Z, e)- fibration if to every object X and each commutative square of the form

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$$(2.2) \qquad \begin{array}{ccc} X & \overset{\alpha}{\rightarrow} & C_1 \\ {\scriptstyle eX}\downarrow & \overset{\omega}{\nearrow} & \downarrow{\scriptstyle \varphi} \\ ZX & \underset{\beta}{\rightarrow} & C \end{array}$$

there exists a morphism $\omega : ZX \rightarrow C$ as indicated such that $\omega \circ eX = \alpha$, $\varphi\omega = \beta$.

DEFINITION 2.3. Let $P : \mathcal{L} \rightarrow \mathcal{L}$ be a functor and $s : P \rightarrow 1$ a natural transformation. Then we call a morphism $\varphi : C_1 \rightarrow C$ a (P, s)- fibration if to each object X and each commutative square of the form

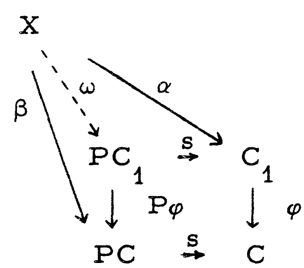
$$(2.4) \qquad \begin{array}{ccc} X & \overset{\alpha}{\rightarrow} & C_1 \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \varphi} \\ PC & \underset{sC}{\rightarrow} & C \end{array}$$

there exists a morphism $\omega : X \rightarrow PC_1$ such that $P\varphi \circ \omega = \beta$ and $sC_1 \circ \omega = \alpha$.

From now on we shall use the following abbreviations: Instead of $eX : X \rightarrow ZX$, $sX : PX \rightarrow X$ we will write e, s respectively and instead of (Z, e)- and (P, s)- fibrations we will speak of Z- and P-fibrations.

Observe further that, if φ is a P-fibration, we can derive from (2.4) a commutative diagram of the form

(2.5)



Now we are interested in "natural" relations between Z and P, e and s such that the notions of Z- and P-fibrations coincide. For this purpose we prove the following theorem:

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THEOREM 2.6. If Z is adjoint to P by ϕ: hom $(ZX, Y) \to$ hom (X, PY) and if e is adjoint to s such that

$$
\begin{array}{ccc}
\hom(ZX, Y) & \overset{\phi}{\to} & \hom(X, PY) \\
e^* \downarrow & & \downarrow s_* \\
\hom(X, Y) & \underset{1}{\to} & \hom(X, Y)
\end{array}
\tag{2.6}
$$

commutes, then Z- and P-fibrations coincide.

The proof is straightforward since the commutativity of (2.6) yields that for each $\beta: ZX \to Y$ we have $\beta e = s\phi\beta$. Hence if φ is a Z-fibration and if a commutative square like (2.4) is given then $\varphi\alpha = \phi^{-1}\beta e$ which gives us an ω such that $\omega e = \alpha$ and $\varphi\omega = \phi^{-1}\beta$. Now it is obvious that for the morphism $\phi\omega: X \to PC_1$ the equations $s\phi\omega = \alpha$ and $P\varphi\phi\omega = \beta$ hold, the latter by the naturality of ϕ. It is plain that the converse holds too.

Now proposition 1.3 shows us that given Z and P with Z adjoint to P and given $e: 1 \to Z$ (or $s: P \to 1$), there exists a natural transformation $s: P \to 1$ $(e: 1 \to Z)$ such that Z- and P-fibrations coincide. This and our notion of homotopy, which will be introduced later, leads us to the following assumptions which we shall make from now on: we assume that we are given functors $Z, P: \mathcal{L} \to \mathcal{L}$ with Z adjoint to P by ϕ: hom $(ZX, Y) \to$ hom (X, PY) and, further, that we have natural transformations $e, \bar{e}: 1 \to Z$ and $r: Z \to 1$ such that $re = 1 = r\bar{e}$. Then there are unique natural transformations $s, \bar{s}: P \to 1$, $f: 1 \to P$ coadjoint to $e, \bar{e}, r$ with respect to ϕ and 1, where 1 stands for the trivial adjointness of the identity functor to itself. Given all this, we know that Z- and P-fibrations coincide; hence we will speak shortly of fibrations. A (Z) or (P) at the end of the statement of a theorem will indicate that the proof will be given entirely by using only the definition of Z-, or P-fibration respectively.

<u>Section 3.</u> <u>Structure theorems for fibrations.</u> From the definition it is plain that compositions of fibrations yield fibrations. The first important theorem is that on induced fibrations:

THEOREM 3.1. Let $\varphi: C_1 \to C$ be a fibration, $\psi: C_2 \to C$ any morphism. Then φ' in the cartesian square $(\varphi', \psi') = I(\varphi, \psi)$

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is a fibration. (Z)

Proof. Consider any commutative square of the form

$$\begin{array}{ccc} X & \overset{\alpha}{\to} & I \\ e \downarrow & & \downarrow \varphi' \\ ZX & \underset{\beta}{\to} & C_2 \end{array}$$

Since (1.2) commutes and φ is a fibration there exists an $\bar{\omega}: ZX \to C_1$ with $\bar{\omega} e = \psi' \alpha$ and $\varphi \bar{\omega} = \psi\beta$. The latter equation gives us a uniquely determined $\omega: ZX \to I$ with $\psi'\omega = \bar{\omega}$ and $\varphi'\omega = \beta$. Since the square

$$\begin{array}{ccc} X & \overset{\psi'\alpha}{\to} & C_1 \\ \varphi'\alpha \downarrow & & \downarrow \varphi \\ C_2 & \underset{\psi}{\to} & C \end{array}$$

also commutes there exists a uniquely determined $\nu: X \to I$ with $\psi'\nu = \psi'\alpha = \bar{\omega} e$ and $\varphi'\nu = \varphi'\alpha = \beta e$. This gives us by uniqueness $\nu = \alpha = \omega e$. Hence φ' is a fibration.

For completeness we mention here that if φ is a retraction then φ' is also a retraction. But this is so in general and does not depend on the fact that φ is a fibration. The same is true of the statement that ψ can be factored through φ if and only if φ' is a retraction (often such theorems are related to fibrations in the literature; compare [1], [2]).

In the case that ψ is a retraction the converse of theorem 3.1 holds:

THEOREM 3.2. If, in addition to the assumptions of 3.1 ψ is a retraction then φ' is a fibration if and only if φ is a fibration. (Z)

Proof. Assume that φ' is a fibration and let $\sigma: C \to C_2$ be a morphism with $\psi\sigma = 1$. Consider any commutative square like (2.2). Since $\psi\sigma\beta e = \varphi\alpha$ there exists $\lambda: X \to I$ with $\psi'\lambda = \alpha$ and $\varphi'\lambda = \sigma\beta e$. Hence the square

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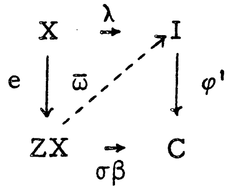


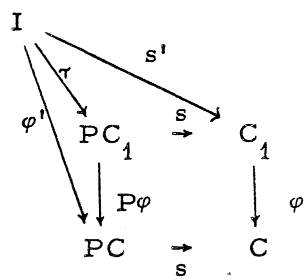
commutes and since φ' is a fibration there exists an ϖ as indicated rendering both triangles commutative. Setting $\omega = \psi'\varpi$ yields the theorem.

Another important theorem states that the path lifting property is equivalent to the fibration property. This is expressed as follows:

THEOREM 3.3. Let $\varphi : C_1 \to C$ be any morphism and let $(\varphi', s') = I(\varphi, sC)$. Denote by $\rho : PC_1 \to I$ the uniquely determined morphism satisfying $s'\rho = sC_1$ and $\varphi'\rho = P\varphi$. Then φ is a fibration if and only if ρ is a retraction, (P)

Proof: Necessity. In (2.4) let $X = I$, $\alpha = s'$, $\beta = \varphi'$. This gives us an $\omega : I \to PC_1$ rendering the corresponding diagram (2.5) commutative. Now the uniqueness of the universal property of a cartesian square shows $\rho\omega = 1$.

Sufficiency. Let $\tau : I \to PC_1$ be a morphism with $\rho\tau = 1$. Since $\varphi' = \varphi'\rho\tau = P\varphi \cdot \tau$ and $s' = s'\rho\tau = s\tau$ the diagram



commutes and in the general case of a commutative square like (2.4) there exists certainly an $\varpi : X \to I$ with $\alpha = s'\varpi$ and $\beta = \varphi'\varpi$. Define $\omega = \tau\varpi$ and the theorem is proven.

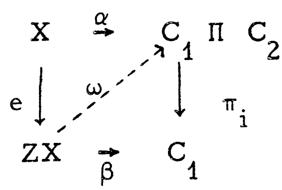
PROPOSITION 3.4. Let $\varphi_i : C_i \to C_i'$ $(i = 1,2)$ be fibrations. Then $\varphi_1 \Pi \varphi_2$ is a fibration. (Z)

This is plain since if $(\varphi_1 \Pi \varphi_2)\alpha = \beta e$ and $\pi_i \beta$ factors through φ_i by ω_i with respect to $\pi_i \alpha$ then β factors through $\varphi_1 \Pi \varphi_2$ by $\{\omega_1, \omega_2\}$ with respect to α.

Looking for types of morphisms which are always fibrations we will prove:

PROPOSITION 3.5. Equivalences, projections and $0:C \to 0$ are fibrations. (Z)

<u>Proof.</u> If the square



commutes, choose ω as $\{\beta, \pi_2 \alpha r\}$. This proves the theorem for projections and by choosing $C_2 = 0$ it follows for equivalences, by $C_1 = 0$ for the null morphism to a null object.

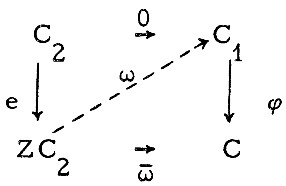
To proceed in the theory of fibrations let us introduce a concept of homotopy: given two morphisms $\alpha, \bar{\alpha} : X \to Y$, we say that $\bar{\alpha}$ is homotopic to α if there exists a morphism $\omega : ZX \to Y$ such that $\omega e = \alpha$ and $\omega \bar{e} = \bar{\alpha}$. ω is then called a homotopy from ωe to $\omega \bar{e}$. Turning to P, s, $\bar{s}$ we can clearly express homotopy as $\bar{\alpha}$ is homotopic to α iff there exists an $\omega : X \to PY$ such that $s\omega = \alpha$ and $\bar{s}\omega = \bar{\alpha}$. This notion of homotopy is not necessarily an equivalence relation. But it has the following two properties. (i) If $\bar{\alpha}$ is homotopic to α, and β is any morphism such that $\beta\alpha$ is defined, then $\beta\bar{\alpha}$ is homotopic to $\beta\alpha$, and (ii) if $\bar{\alpha}$ is homotopic to α, β a morphism such that $\alpha\beta$ is defined then $\bar{\alpha}\beta$ is homotopic to $\alpha\beta$. In addition our homotopy relation is reflexive since $r:Z \to 1$ is a retraction for e and $\bar{e}$ and to have symmetry it is necessary and sufficient to have morphisms $\iota : ZX \to ZX$ for every object X satisfying $\iota e = \bar{e}$ and $\iota \bar{e} = e$.

We can now state a theorem on liftings:

THEOREM 3.6. Let $\varphi : C_1 \to C$ be a fibration, and let $\psi : C_2 \to C$ be homotopic to 0. Then ψ can be factored through φ

by a morphism which is itself homotopic to 0 .

Proof. Let $\bar{\omega} : ZC_2 \to C$ be a homotopy with $\bar{\omega} e = 0$, $\bar{\omega}\bar{e} = \psi$. Hence the square



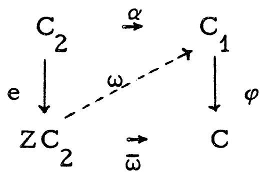
commutes, which gives us ω as indicated. Now $\omega\bar{e}$ lifts ψ and $\omega e = 0$ shows $\omega\bar{e}$ homotopic to 0 .

COROLLARY 3.7. If $\varphi : C_1 \to C$ is a fibration and $1 : C \to C$ is homotopic to 0 then φ is a retraction.

We introduce now the notion of fiber by defining fiber φ = ker φ. Then we have the following

THEOREM 3.8. Let $\varphi: C_1 \to C$ be a fibration, $\alpha: C_2 \to C_1$ any morphism. Then there exists $\bar{\alpha} : C_2 \to C_1$ homotopic to α such that $\bar{\alpha}$ factors through the fiber if and only if 0 is homotopic to $\varphi\alpha$. (Z)

Proof. (i) Let 0 be homotopic to $\varphi\alpha$ by $\bar{\omega} : ZC_2 \to C$. Since the square



commutes we can find ω as indicated. Hence $\omega\bar{e}$ is homotopic to α and $\varphi\omega\bar{e} = \bar{\omega}\bar{e} = 0$ shows that $\omega\bar{e}$ factors through the fiber.

(ii) If there exists $\bar{\alpha}$ homotopic to α and factoring through the fiber, then $\varphi\bar{\alpha} = 0$ is homotopic to $\varphi\alpha$.

Next we will introduce the notion of irreducibility: an epic morphism α is called irreducible if each morphism homotopic

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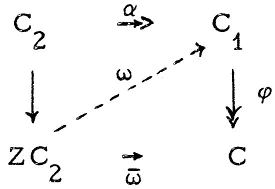
to α is again epic. Irreducible morphisms are of interest because if $\alpha : C_1 \twoheadrightarrow C_2$ is irreducible and C_2 is not a null object, then 0 is not homotopic to α. For the proof we need only to observe that if $0 : C_1 \to C_2$ is epic then $0 = 1 : C_2 \to C_2$, hence $C_2 = 0$.

PROPOSITION 3.9. Let $\varphi : C_1 \twoheadrightarrow C$ be an epic fibration. If $\alpha : C_2 \twoheadrightarrow C_1$ is irreducible then $\varphi\alpha$ is irreducible. (Z)

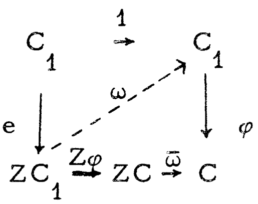
Proof. Let $\bar{\omega} : ZC_2 \to C$ be any morphism with $\bar{\omega} e = \varphi\alpha$. Then



commutes for a certain ω. Now $\bar{\omega}\bar{e} = \varphi\omega\bar{e}$ with $\omega\bar{e}$ epic since α is irreducible. Thus $\bar{\omega}\bar{e}$ is epic.

COROLLARY 3.10. If φ is an epic fibration and $1 : C_1 \to C_1$ is irreducible then φ and $1 : C \to C$ are irreducible. (Z)

Proof. φ is clearly irreducible since $\varphi = \varphi 1$. If $\bar{\omega} : ZC \to C$ is any morphism with $\bar{\omega} e = 1$ we get a commutative diagram of the form



Now $\bar{\omega}\bar{e}\varphi = \bar{\omega} Z\varphi\bar{e} = \varphi\omega\bar{e}$ which is epic. Hence $\bar{\omega}\bar{e}$ is epic which proves 1_C irreducible.

As a special case we get from 3.10 that if φ is an epic fibration and C is not a null object and 1_{C_1} is irreducible then 0 is neither homotopic to φ nor to 1_C.

Let us now turn to the notion of regular fibrations. For that purpose we give the definition of a stationary covering homotopy. We start from the following situation: let

$$\begin{array}{ccc} & & C_1 \\ & \overset{\omega}{\nearrow} & \\ ZX & & \downarrow \varphi \\ & \underset{\bar{\omega}}{\searrow} & \\ & & C \end{array}$$

be a commutative triangle. Then ω is called stationary with $\bar{\omega}$ if for any subobject $\mu : X_1 \rightarrowtail X$ such that

$$\begin{array}{ccc} ZX_1 & \overset{Z\mu}{\rightarrow} & ZX \\ r \downarrow & & \downarrow \bar{\omega} \\ X_1 & \underset{\bar{\tau}}{\rightarrow} & C \end{array}$$

commutes for a certain $\bar{\tau}$ there exists a morphism $\tau : X_1 \rightarrow C_1$ such that

$$\begin{array}{ccc} ZX_1 & \overset{Z\mu}{\rightarrow} & ZX \\ r \downarrow & & \downarrow \omega \\ X_1 & \underset{\tau}{\rightarrow} & C_1 \end{array}$$

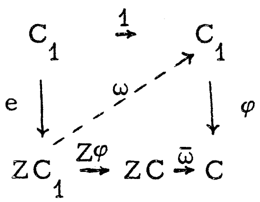
commutes too. A (Z-) fibration φ is then called regular if in any commutative square (2.2) ω can be chosen stationary with β. It is left to the reader to transfer the definition of stationarity to P, f. Further it may be verified by the reader which propositions hold with regularity for the fibration φ presumed and how one has to modify theorems in this special case (e.g. projections are regular fibrations). Our purpose for introducing regularity is a theorem on strong deformation coretractions. To formulate this theorem we define $\kappa : C' \rightarrow C$ to be a deformation coretraction if there exists an $\omega : ZC \rightarrow C$ such that $\omega e = 1$ and $\omega \bar{e} = \kappa\rho$ with $\rho : C \rightarrow C'$ and $\rho\kappa = 1_{C'}$. If one can further choose ω in such a way that

(3.11)
$$\begin{array}{ccc} ZC' & \overset{Z\kappa}{\rightarrow} & ZC \\ r \downarrow & & \downarrow \omega \\ C' & \underset{\kappa}{\rightarrow} & C \end{array}$$

commutes, κ is called a strong deformation coretraction. It is obvious that if κ is a deformation coretraction then κ is monic.

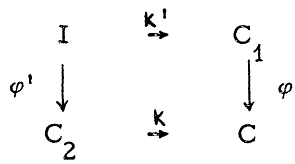
THEOREM 3.12. Let $\varphi : C_1 \to C$ be a regular fibration and $\kappa : C_2 \to C$ a strong deformation coretraction. Then κ' in the cartesian square $(\varphi', \kappa') = I(\varphi, \kappa)$ is a strong deformation coretraction. (Z)

Proof. Let $\bar{\omega} : ZC \to C$ be chosen such that $\bar{\omega} e = 1$, $\bar{\omega}\bar{e} = \kappa\rho$ where $\rho : C \to C_2$, $\rho\kappa = 1$ and the corresponding square (3.11) commutes. Consider



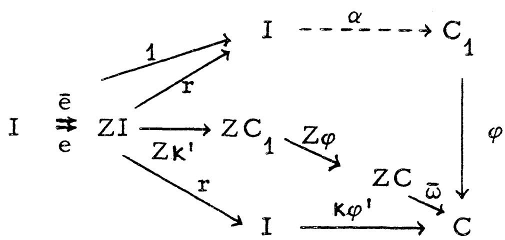
with $\bar{\omega} Z\varphi e = \bar{\omega} e \varphi = \varphi$ showing the square commutative. Insert ω stationary with $\bar{\omega} Z\varphi$ as indicated rendering in addition the two triangles commutative. Since $\varphi\omega\bar{e} = \bar{\omega} Z\varphi\bar{e} = \omega\bar{e}\varphi = \kappa\rho\varphi$, there exists a unique $\rho' : C_1 \to I$ with $\varphi'\rho' = \rho\varphi$ and $\kappa'\rho' = \omega\bar{e}$ where

(3.13)



shows the cartesian square $(\varphi', \kappa') = I(\varphi, \kappa)$. Turning to the diagram

(3.14)



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the calculation $\bar{\omega} Z\varphi Z\kappa' = \bar{\omega} Z(\varphi\kappa') = \bar{\omega} Z(\kappa\varphi') = \bar{\omega} Z\kappa Z\varphi' = \kappa r Z\varphi'$ $= \kappa\varphi' r$ shows the lower part commutative. Hence there exists α rendering the upper part commutative. But then $\alpha = \omega Z\kappa' e =$ $\omega e \kappa' = \kappa'$ and $\alpha = \omega Z\kappa'\bar{e} = \kappa'\rho'\kappa'$. Hence $\varphi'\rho'\kappa' = \varphi'$ and $\kappa'\rho'\kappa' = \kappa'$ which gives by the uniqueness of the universal property of the cartesian square (3.13) $\nu'\kappa' = 1$. In addition (3.14) shows that κ' is a strong deformation coretraction.

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