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Semifield metric spaces

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Extending the results of Antonovskiĭ, Boltjanskiĭ, and Sarymsakov on semifield metric spaces, the authors define a regular semifield metric to be one in which the distance in the standard Tychonoff product representation of a point from a disjoint closed set is nonzero. It is shown that every completely regular topological space possesses a completely regular semifield metric and that there is an equivalent completely regular semifield metric for every semifield metric space. A normal semifield metric is defined to be one in which the distance between two disjoint closed sets is nonzero and it is shown that possessing a normal semifield metric is equivalent to being a normal topological space. Finally, Cauchy nets in semifield metric spaces are introduced leading to the notion of completeness. It is shown that a semifield metric space is complete iff every Cauchy net with the property that its directed set has cardinality less than or equal to the cardinality of the indexing set of the Tychonoff product representation of the semifield converges.

1. Introduction

The theory of topological semifields was developed by Antonovskiĭ, Boltjanskiĭ, and Sarymsakov (see [*1*] and [*2*]) who showed that every semifield is a subsemifield of a Tychonoff product of reals under coordinatewise addition and multiplication. Semifields which are products of reals are called "Tychonoff" by those authors and there is no restriction in our work to consider only that type.

In a semifield R^I , we say that x is weakly less than y , written

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$x \ll y$, iff $\pi_i(x) \leqq \pi_i(y)$ for all $i \in I$. This partial ordering is used in the characterization of semifields. A function d_I from $X \times X$ into the non-negative members of R^I (in the above ordering with zero being the element 0 all of whose coordinates are zero) is a semifield metric on X iff $\pi_i \circ d_I$ is a pseudometric on X for each $i \in I$. Explicitly, a semifield metric d_I has the properties:

$$d_I(x , y) \gg 0$$
$$d_I(x , y) = d_I(y , x)$$
$$d_I(x , z) \ll d_I(x , y) + d_I(y , z)$$
$$d_I(x , y) = 0 \text{ iff } x = y$$

for every $x , y , z \in X$.

A semifield metric d_I gives rise to a topology on X in the following way. A subset G of X is open in X iff for every $x \in G$ there exists a positive real number ε and a finite subset $J \subset I$ such that

$$\{y \in X : \pi_i \circ d_I(x , y) < \varepsilon , \quad i \in J\}$$

is a subset of G . A topological space X is semifield metrizable iff there exists a semifield metric on X which induces the topology. It has been shown (see [1]) that a topological space is semifield metrizable iff it is a Tychonoff (completely regular Hausdorff) space.

A natural semifield metric on the semifield R^I is the function $d_I : R^I \times R^I \to R^I$ such that $\pi_i \circ d_I(x , y) = |\pi_i(x - y)|$ for every $i \in I$ and $x , y \in R^I$. This semifield metric on R^I induces the product space topology on R^I . We shall now point out one fundamental difference between semifield metrics and real metrics. If A is a nonempty subset of a metric space (X , d) , then $x \in \mathrm{cl}(A)$ iff $d(x , A) = 0$. This is not the case for semifield metrics. If d_I is a semifield metric on X , we let $d_I(x , A) = \inf \{d_I(x , y) : y \in A\}$ where this infimum is taken with

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respect to the above mentioned order, and hence coordinatewise.

EXAMPLE. Let R^I be a semifield and let $A = \{x \in R^I :$ for some $i \in I$, $x(i) = 1$ while $x(j) = 0$ if $j \neq i\}$. Now $\mathrm{cl}(A) = A \cup \{0\}$ since that set is compact. Let d_I be the natural semifield metric of R^I over itself. Then $d_I(1 , A) = 0$ even though $1 \notin \mathrm{cl}(A)$.

On the other hand, we have

THEOREM A. $\mathrm{cl}(A) \subset \{x \in X : d_I(x , A) = 0\}$.

Proof. If $d_I(x , A) \neq 0$, then there exists an $i \in I$ such that $\varepsilon = \inf \{\pi_i \circ d_I(x , y) : y \in A\} \neq 0$. Hence $U = \{y \in X : \pi_i \circ d_I(x , y) < \varepsilon\}$ is a neighborhood of x which is disjoint from A . Therefore, $x \notin \mathrm{cl}(A)$.

2. Completely regular semifield metrics

A semifield metric d_I on X is said to be completely regular if for every nonempty closed subset F of X and point $x \notin F$, $d_I(x , F) \neq 0$. Clearly, every metric is completely regular.

THEOREM B. *Every completely regular space possesses a completely regular semifield metric.*

Proof. Let X be a completely regular space, Φ be the set of all real valued continuous mappings on X , θ be the evaluation function from X to R^Φ such that $\pi_f \circ \theta(x) = f(x)$ for every $x \in X$ and $f \in \Phi$. If d is the natural semifield metric on R^Φ restricted to $\theta(X)$, then d is a semifield metric on $\theta(X)$. If we let $d^*(x , y) = d(\theta(x) , \theta(y))$ for every $x , y \in X$, then d^* is a semifield metric on X and θ is an isometry from X semifield metrized by d^* to $\theta(X)$ semifield metrized by d . We shall show that d^* is completely regular. Let F be a closed subset of X and let $x \in X - F$. Then there exists an $f \in \Phi$ such that $f(x) = 0$ and $f(F) = 1$. But then $\pi_f \circ d^*(x , F) = 1$ since

$$\begin{aligned}\pi_f \circ d^*(x , y) &= \pi_f \circ d(\theta(x) , \theta(y)) \\ &= |\pi_f(\theta(x) - \theta(y))| \\ &= |\pi_f(\theta(x)) - \pi_f(\theta(y))| \\ &= |f(x) - f(y)| \\ &= |0 - 1| \\ &= 1\end{aligned}$$

if $y \in F$. This shows that $d^*(x , F) \neq 0$ and so d^* is completely regular.

If (X , d) is a pseudometric space then d is completely regular in the above sense. Although not every semifield metric on a space X is completely regular, we can strengthen the last theorem as follows:

THEOREM C. *If (X , d_I) is a semifield metric space, then there exists an equivalent completely regular semifield metric d_I^* for X.*

Proof. Suppose β is a one-to-one mapping from the finite subsets of I onto I. Then we define $d_I^*(x , y)$ in the following way. For each $i \in I$ there exists a finite subset J of I such that $\beta(J) = i$ and we let $\pi_i \circ d_I^*(x , y) = \sup \{\pi_j \circ d_I(x , y) : j \in J\}$. Now d_I^* is a semifield metric on X to R^I since $\pi_i \circ d_I$ is a pseudometric for every $i \in I$. Now a d_I-basic neighborhood of a point x of the form $\{y \in X : \pi_j \circ d_I(x , y) < \varepsilon \text{ for } j \in J\}$ with $\varepsilon > 0$ and J a finite subset of I clearly contains the d_I^*-basic neighborhood $\{y \in X : \pi_i \circ d_I^*(x , y) < \varepsilon\}$ where $\beta(J) = i$. Thus d_I induces a weaker topology than d_I^*. Conversely, $\pi_i \circ d_I^*$ is a d_I-continuous pseudometric on X for every $i \in I$ which shows that d_I^* induces a weaker topology than d_I, hence the two are equivalent. Finally, let F be a closed subset of X and let $x \notin F$. There exists a finite subset $J \subset I$ and positive real number ε such that the neighborhood $\{y \in X : \pi_j \circ d_I(x , y) < \varepsilon , j \in J\}$ is disjoint from F. Let $\beta(J) = i$

and we have the smaller neighborhood $\{y \in X : \pi_i \circ d_I^*(x , y) < \varepsilon\}$ which shows that $d_I^*(x , F) \nleqq 0$.

For each subset $J \subset I$ we will denote by $d_{I|J}$ the semifield metric defined by setting $\pi_i \circ d_{I|J}(x , y) = \pi_i \circ d_I(x , y)$ if $i \in J$ and zero otherwise. It is easy to show that if $d_{I|J}$ is completely regular for some $J \subset I$, then $d_{I|J}$ is equivalent to d_I .

3. Normal semifield metrics

For closed subsets F and G of (X , d_I) we set $d_I(F , G) = \inf \{d_I(x , y) : x \in F , \ y \in G\}$. Even if F and G are disjoint it may happen that $d_I(F , G) = 0$. A semifield metric d_I on X is said to be normal if $d_I(F , G) \nleqq 0$ for every pair of nonempty disjoint closed subsets F and G of X . It is evident that a normal semifield is completely regular. However, a completely regular semifield metric need not be normal since metrics need not be normal.

THEOREM D. *A semifield metrizable space is normal iff it possesses a normal semifield metric.*

Proof. Let X be a normal semifield metrizable space. We shall show that the semifield metric d_I^* given in Theorem B is normal. Let F and G be disjoint nonempty closed subsets of X . Since X is normal there exists an $f \in \Phi$ such that $f(F) = 0$ and $f(G) = 1$. As in Theorem B we have $\pi_f \circ d_I^*(F , G) = 1$ which shows that $d_I^*(F , G) \nleqq 0$, so d_I^* is normal. Next suppose that d_I is a normal semifield metric on X . Then if F and G are disjoint nonempty closed subsets of X , we wish to find a continuous function $f : X \to [0 , 1]$ such that $f(F) = 0$ and $f(G) = 1$. Since d_I is normal, $\inf \{d_I(x , F) : x \in G\} \nleqq 0$, so there exists an $i \in I$ such that

$$\delta = \pi_i(\inf \{d_I(x , F) : x \in G\}) = \inf \{\pi_i \circ d_I(x , F) : x \in G\} > 0 .$$

Since $d_I(x , F)$ is a continuous function of x ,

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$f(x) = \inf \{1 , \pi_i \circ d_I(x , F) / \delta\}$ is the desired function.

If (X , d_I) is a compact semifield metric space, then so is the weaker space $(X , d_i) = (X , \pi_i \circ d_I)$. The converse is not true, however, as we saw in our earlier example. There the space $(A , d_I|A)$ is not compact for A is not a closed subset of R^I , even though, for every $i \in I$ the space $(A , \pi_i \circ d_I|A)$ is compact. We notice that if (X , d_I) is compact, then $(X , d_{I|J})$ is also compact for every subset $J \in I$. It is easy to show that if $d_{I|J}$ separates the points of X , then it is equivalent to d_I .

THEOREM E. *If* (X , d_I) *is a compact semifield metric space then there exists an equivalent normal semifield metric* d_I^* .

Proof. Without loss of generality we can assume that d_I is completely regular. Let β be a one-to-one mapping of the finite subsets of I onto I . For each $i \in I$ we have $i = \beta(J)$ for some subset $J \subset I$ and we let $d_i^* = \sup \{d_j : j \in J\}$. Defining d_I^* to be the product of the d_i^* over $i \in I$, we obtain an equivalent normal semifield metric for X .

4. Complete spaces

We shall denote by N a general directed set so that nets will be written $\{s_n : n \in N\}$, usually without explicit mention of the ordering. The following characterizes the convergence of nets in semifield metric spaces.

THEOREM F. *If* $\{s_n : n \in N\}$ *is a net in the semifield metric space* (X , d_I) *, then* $s_n \to s \in X$ *with respect to* d_I *iff* $s_n \to s$ *with respect to each* d_i .

Proof. One implication is obvious since the topology induced by d_i is weaker then that induced by d_I . On the other hand, suppose that $s_n \to s$ with respect to each d_i . Let ε be a positive real number and let J be a finite subset of I. For each $j \in J$ there exists an element $n_j \in N$ such that $d_j(s, s_n) < \varepsilon$ whenever $n > n_j$. Since N is directed, there exists some $n^* \in N$ such that $n_j < n^*$ for all $j \in J$. Then we have $d_j(s, s_n) < \varepsilon$ for all $n > n^*$ and $j \in J$ which shows that $s_n \to s$ with respect to d_I.

A net $\{s_n : n \in N\}$ in a semifield metric space (X, d_I) is said to be Cauchy iff for every positive real number ε and finite subset J of I, there exists an $n^* \in N$ such that $d_j(s_m, s_n) < \varepsilon$ for each $j \in J$ whenever $n^* < m, n \in N$.

THEOREM G. *A net $\{s_n : n \in N\}$ in (X, d_I) is Cauchy with respect to d_I iff it is Cauchy with respect to each d_i.*

Proof. Essentially the same argument as in the previous theorem will work here.

Just as with metrics, it is easy to show that a net $\{s_n : n \in N\}$ is Cauchy in (X, d_I) iff $\{d_I(s_m, s_n) : (m, n) \in N \times N\} \to 0 \in R^I$.

A semifield metric space (X, d_I) is said to be complete iff every Cauchy net in X converges with respect to d_I. It would seem that (X, d_I) would be complete iff each pseudometric space (X, d_i) were complete, but neither of these implications holds. For example, if $(X, d_{\{1,2\}})$ is the unit square in the plane with the point (1,1) deleted using the restricted natural semifield metric for the plane, then both (X, d_1) and (X, d_2) are complete and yet X is not complete. For a counter-example to the converse, we need only consider the subset $\{(x, y) : xy = 1\}$ of the plane.

THEOREM H. *A semifield metric space (X, d_I) is complete iff every Cauchy net $\{s_n : n \in N\}$, where $\mathrm{card}(N) \leq \mathrm{card}(I)$, converges.*

Proof. Let us suppose that (X, d_I) satisfies the obviously necessary condition of the theorem and let $\{s_n : n \in N\}$ be a Cauchy net.

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Consider the natural number 1 . For any set $\{i\}$ with $i \in I$, there exists an $n^1_{\{i\}} \in N$ such that $d_i(s_m , s_n) < 1$ whenever $n^1_{\{i\}} < m , n \in N$. Suppose that whenever A and B are nonempty subsets of I with fewer than k elements, where k is a natural number, we have already picked $n^1_A , n^1_B \in N$ such that (1) if A is a proper subset of B , then $n^1_A < n^1_B$, and (2) if $m , n > \max[n^1_A , n^1_B]$, then $d_i(s_m , s_n) < 1$ for all $i \in A \cup B$. Next suppose that C is any subset of I with k elements. For each such C we choose an $n^1_C \in N$ such that (1) if A is a nonempty proper subset of C , then $n^1_A < n^1_C$, and (2) if $m , n > n^1_C$, then $d_i(s_m , s_n) < 1$ for every $i \in C$. Suppose that t is a natural number such that if p and q are natural numbers such that $p < q < t$, and if C is a nonempty proper subset of the finite subset D of I , then we have chosen $n^p_C < n^q_C < n^p_D < n^q_D \in N$ with the properties that

$$n^p_C < m , n \in N \text{ implies } d_i(s_m , s_n) < 1/p \text{ for } i \in C ,$$

$$n^q_C < m , n \in N \text{ implies } d_i(s_m , s_n) < 1/q \text{ for } i \in C ,$$

$$n^p_D < m , n \in N \text{ implies } d_i(s_m , s_n) < 1/p \text{ for } i \in D ,$$

$$n^q_D < m , n \in N \text{ implies } d_i(s_m , s_n) < 1/q \text{ for } i \in D .$$

Then for every nonempty finite subset E of I it is possible to choose an $n^t_E \in N$ such that

$$n^t_E < m , n \in N \text{ implies } d_i(s_m , s_n) < 1/t \text{ for } i \in E .$$

This n^t_E can be chosen large enough so that $n^t_E > n^p_C$ for every nonempty subset C of E and natural number $p < t$. Let N^* be the set of all such n^t_E chosen, so that $\mathrm{card}(N^*) \leqq \mathrm{card}(I)$. We note that N^* is clearly directed since $n^{p+q}_{A\cup B}$ is larger than any elements $n^p_A , n^q_B \in N^*$. Thus $\{s_n : n \in N^*\}$ is a net; in fact, it is a Cauchy net which converges to some point s by hypothesis. Now let E be a finite subset of I and

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let ε be a positive real number. Let t be a natural number such that $2/t < \varepsilon$. By convergence, there exists an $n^* \in N^*$ such that $n^* < n \in N^*$ implies $d_i(s_n , s) < 1/t$ for $i \in E$. Also, $n_E^t < m , n \in N$ implies $d_i(s_m , s_n) < 1/t$ for $i \in E$. Now choose m^* such that $n^* , n_E^t < m^* \in N^*$ and fix $n \in N^*$ with $n > m^*$. We have

$$d_i(s , s_m) \leqq d_i(s , s_n) + d_i(s_n , s_m) < 1/t + 1/t = 2/t < \varepsilon$$

for all $i \in E$ and all $m^* < m \in N$. This shows that $\{s_n : n \in N\}$ converges to s and the space is complete.

We note that the previous theorem could be rephrased to require that $\mathrm{card}(N) \leqq \aleph_0 \cdot \mathrm{card}(I)$ and then the usual theorem that sequences are adequate to describe completeness in metric spaces would be a corollary.

With the framework of complete semifield metric spaces we can introduce and study contractive mappings. Standard results on the continuity of such maps and the existence of fixed points in complete spaces can be obtained. We will not, however, list these results here.

By an obvious generalization of the standard proof for metric spaces using equivalence classes of Cauchy nets (see [3], pp. 123-124) we can construct a unique completion of a semifield metric space, where one need only be careful to always choose as directed sets the $\{n_E^t\}$ of Theorem H with cardinality equal to $\mathrm{card}(I)$.

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