NOTE ON P.P. RINGS

(A SUPPLEMENT TO HATTORI'S PAPER)

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A ring R is called, according to [2], a left p.p. ring if any principal left ideal of R is projective. A ring which is left and right p.p. is called a p.p. ring.

In this short note we shall give some additional remarks to A. Hattori [2]. In Proposition 1 we shall give a characterization of commutative p.p. rings, and in Proposition 3 we shall give a generalization of Proposition 17 and 18 in [2], which shows also that the modified torsion theory over commutative p.p. rings coincides with the usual torsion theory.

Our notations and terminologies are the same as those in [2].

We begin with

LEMMA 1. *A commutative ring R is regular (in Neumann's sense) if and only if the quotient ring $R_{\mathfrak{m}}$ of R with respect to any maximal ideal $\mathfrak{m}$ of R is a field, or if and only if any element of R is expressible as the product of a unit and an idempotent.*

Proof. The first part: If R is regular, then any $R_{\mathfrak{m}}$ is obviously regular. Since a local ring is regular when and only when it is a field, any $R_{\mathfrak{m}}$ is a field. Hence we have only to show the if part. Let a be an element of R and set $\mathfrak{b} = \{b\ ;\ ba = 0,\ b \in R\}$. Since any $R_{\mathfrak{m}}$ is a field, $\mathfrak{b}$ is not contained in any maximal ideal of R containing a. Setting $\mathfrak{c} = (a, \mathfrak{b})$, $\mathfrak{c}$ is not contained in any maximal ideal of R, and so we have $R = (a, \mathfrak{b})$. Since $(a)\mathfrak{b} = 0$, (a) is a direct summand of R. Accordingly we have $(a) = (e)$ for a suitable idempotent e of R and also have $\mathfrak{b} = (1-e)$. Furthermore, if we set $d = (1-e) + a$, then d is clearly a unit of R and we have $de = ae = a$. So we obtain $ad^{-1}a = a$. This proves that R is regular.

The second part: This follows directly from the above proof of the first part.

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PROPOSITION 1. *A commutative ring R is a p.p. ring if and only if the total quotient ring K of R is regular and the quotient ring $R_{\mathfrak{m}}$ of R with respect to any maximal ideal $\mathfrak{m}$ of R is an integral domain.*

Proof. The if part: Set $\mathfrak{p}_{\mathfrak{m}}=\{a\ ;\ as=0$ for some $s\in R-\mathfrak{m},\ a\in R\}$ for any maximal ideal $\mathfrak{m}$ of R. Since $R_{\mathfrak{m}}$ is an integral domain, $\mathfrak{p}_{\mathfrak{m}}$ is a prime ideal of R. Further, since K is regular, the set of all $\mathfrak{p}_{\mathfrak{m}}K$ coincides with the set of all maximal ideals of K. Let e' be an idempotent of K. If $\mathfrak{p}_{\mathfrak{m}}K\ni e'$, we have $se'=0$ for some $s\in R-\mathfrak{m}$. On the other hand, if $\mathfrak{p}_{\mathfrak{m}}K\not\ni e'$, then $\mathfrak{p}_{\mathfrak{m}}K\ni 1-e'$, and therefore we have $s'(1-e')=0$ for some $s'\in R-\mathfrak{m}$. Hence $s'e'=s'$. If we set $\mathfrak{a}=\{a\ ;\ ae'\in R,\ a\in R\}$, then $\mathfrak{a}$ is an ideal of R which is not contained in any maximal ideal of R. Consequently $\mathfrak{a}=R$. This shows $e'\in R$. Now let a be an element of R. Since K is regular, we have $a=\alpha e'$ for a suitable idempotent e' of K and a unit α of K. Then we have $(a)\cong(e')$ as R-modules. Since $e'\in R$, this shows that (a) is projective.

The only if part: If R is a p.p. ring, then K and $R_{\mathfrak{m}}$ are obviously p.p. rings. If a local ring is a p.p. ring, then it is an integral domain, for any local ring has no idempotent except a unit element. Therefore we have only to show that K is regular. If K is not regular, then, by Lemma 1, there exists at least one maximal ideal $\mathfrak{m}'$ of K such that $K_{\mathfrak{m}'}$ is not a field. If we set $\mathfrak{p}'_{\mathfrak{m}'}=\{a'\ ;\ a's'=0$ for some $s'\in K-\mathfrak{m}',\ a'\in K\}$, $\mathfrak{p}'_{\mathfrak{m}'}$ is a prime ideal of K strictly contained in $\mathfrak{m}'$ since $K_{\mathfrak{m}'}$ is not a field but an integral domain. Now we choose an element c' of K which is not contained in $\mathfrak{p}'_{\mathfrak{m}'}$ but contained in $\mathfrak{m}'$. A principal ideal (c') of K is projective by our assumption. Hence we have $c'=a'e'$ for a unit a' and an idempotent e', by Lemma 1. Then e' is not contained in $\mathfrak{p}'_{\mathfrak{m}'}$ but contained in $\mathfrak{m}'$. Since $e'(1-e')=0$, we have $1-e'\in\mathfrak{p}'_{\mathfrak{m}'}$, i.e., $\in\mathfrak{m}'$. This is obviously a contradiction. This shows that K must be regular.

A ring R is called a *normal* ring if any idempotent of R lies in the center of R.

LEMMA 2. *Let R be a normal right p.p. ring. Then for every element c of R there exist an idempotent a and a left non zero divisor a of R such that $c=ae$.*

Proof. Put $\varphi(r)=cr$ for each $r\in R$. Then φ is a homomorphism of R onto $(c)_r$. Since R is a right p.p. ring, we have $\mathfrak{r}(c)=(e')$ for an idempotent e' of R. If we set $e=1-e'$, then e is also an idempotent. Now set $a=e'+c$. If

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$ad=0$, $d\in R$, then $ad=e'd+cd=0$, and so $ead=ecd=ced=cd=0$ and $e'ad=e'd+e'cd=e'd+ce'd=e'd=0$. This shows $d\in(e)_r\cap(e')_r=(0)_r$. Therefore a is a left non zero divisor. Since $ae=e'e+ce=ce=c$, a and e are elements as required.

LEMMA 3. *Let R be a normal right p.p. ring. Then every left or right non zero divisor of R is a non zero divisor of R.*

Proof. Let b be a left non zero divisor of R. Suppose that $cb=0$, $c\in R$. By Lemma 2 we have $c=ae$ for an idempotent e of R and a left non zero divisor a of R. Then we have $aeb=0$. Since a is a left non zero divisor, we have $eb=be=0$, and so $e=0$, as b is a left non zero divisor. This shows $c=0$. Thus b is a (right) non zero divisor of R.

Now let b be a right non zero divisor of R. Again, by Lemma 2, $b=ae$ for an idempotent e and a left non zero divisor a of R. Then $eb=ea=ae=b$. Hence $(1-e)b=0$. Since b is a right non zero divisor, we have $1=e$, so $b=a$. Thus b is a (left) non zero divisor of R.

PROPOSITION 2. *A normal ring R is a left p.p. ring if and only if it is a right p.p. ring.*

Proof. Obvious by Lemmas 2 and 3.

In the following proposition we shall denote by "Torsion-modules and Torsion-Free modules" torsion modules and torsion-free modules in the modified torsion theory of Hattori [2].

PROPOSITION 3. *For any normal p.p. ring R, the following conditions are equivalent;*

1) *R has the left quotient ring.*

2) *For any left R-module A, we have* $\mathrm{t}(A)=\mathrm{T}(A)$.

3) *For any non zero divisor a of R, R/Ra is a torsion left R-module.*

Proof. The implications 2) → 3) → 1) can be seen easily. Hence it suffices to show the implication 1) → 2). By assumption $\mathrm{t}(A)$ is clearly a left R-submodule of $\mathrm{T}(A)$. Since $A/\mathrm{t}(A)$ is torsion-free, $\mathrm{T}(A)/\mathrm{t}(A)$ is also torsion-free. On the other hand, by [2] Proposition 12, $\mathrm{T}(A)/\mathrm{t}(A)$ is a Torsion module. If $\mathrm{T}(A)/\mathrm{t}(A)$ is Torsion-Free, then $\mathrm{T}(A)/\mathrm{t}(A)=0$, i.e., $\mathrm{T}(A)=\mathrm{t}(A)$. Hence it suffices to show that any torsion-free R-module M is Torsion-Free. Suppose that $cu=0$

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for a non zero element c of R and an element u of M. Since R is a normal p.p. ring, we have, by Lemmas 2 and 3, $c = ae = ea$, for an idempotent e of R and a non zero divisor a of R. Hence we have $aeu = 0$. As M is torsion-free, we have $eu = 0$, so $u = (1 - e)u$. Since $1 - e \in \mathrm{r}(c)$, this shows $u \in \mathrm{r}(c)M$. Thus M is Torsion-Free.

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