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ON NONLOCAL NONLINEAR ELLIPTIC PROBLEMS WITH THE FRACTIONAL LAPLACIAN

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Abstract. In this paper, we study the existence of positive solutions to a semilinear nonlocal elliptic problem with the fractional α-Laplacian on R^n, $0 < \alpha < n$. We show that the problem has infinitely many positive solutions in $C^\tau(R^n) \bigcap H^{\alpha/2}_{loc}(R^n)$. Moreover, each of these solutions tends to some positive constant limit at infinity. We can extend our previous result about sub-elliptic problem to the nonlocal problem on R^n. We also show for $\alpha \in (0, 2)$ that in some cases, by the use of Hardy's inequality, there is a nontrivial non-negative $H^{\alpha/2}_{loc}(R^n)$ weak solution to the problem

$$(-\Delta)^{\alpha/2}u(x) = K(x)u^p \quad \text{in } R^n,$$

where $K(x) = K(|x|)$ is a non-negative non-increasing continuous radial function in R^n and $p > 1$.

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1. Introduction. In their recent work, Chen et al. [4] further developed the method of moving planes and proved that the following fractional semilinear elliptic equation:

$$(-\Delta)^{\alpha/2}u = u^p, \quad u \geq 0, \quad \text{in } R^n,$$

where $\alpha \in (0, 2)$ only has trivial solution when $p \in \left(1, \frac{n+\alpha}{n-\alpha}\right)$ and all solutions are radially symmetric about some points when $p = \frac{n+\alpha}{n-\alpha}$. Since there are many articles concerning the method of moving planes for nonlocal equations, mainly for integral equations, we cannot mention all but refer the readers to the reference [8] for one case with enjoyable detail. In this paper, we consider the existence result of positive solutions to the following nonlocal problem:

$$(-\Delta)^{\alpha/2}u + k(x)u = K(x)u^p, \quad u \geq 0, \quad \text{in } R^n, \tag{1.1}$$

where $\alpha \in (0, n)$, $(-\Delta)^{\alpha/2}u$ is defined as usual by Fourier transform, $k(x)$ and $K(x)$ are given regular functions in R^n, and $p > 1$ is arbitrary. This equation comes from the stationary version of fractional space diffusion.

When $\alpha = 2$ and $p = \frac{n+2}{n-2}$ in equation (1.1), this problem is the prescribed scalar curvature problem and there is also other deep scientific background. Li and Ni [7] used the variational method to prove the existence of a positive solution with decay at infinity and finite Dirichlet energy to a large class of semilinear elliptic problem in R^n, which includes the Matukuma equation in R^3 from the models of globular clusters of stars, namely,

$$\Delta U + \frac{U^p}{1+|x|^2} = 0 \ \text{ in } R^3,$$

where $p > 1$ is arbitrary. Note that the power $p > 1$ can be bigger than the Sobolev critical exponent. To achieve the existence of a positive solution, they use the symmetric rearrangement process and a calculus lemma derived from the integration by part. Their method cannot be directly used in our problem (1.1).

We remark that in our previous work [10], we have considered the following general uniformly elliptic semilinear equation:

$$\Delta_H U - k(x)U + K(x)U^p = 0 \ \text{ in } R^n,$$

where Δ_H is the sub-elliptic Hormander–Laplacian operator defined by a group of vector fields $\{X_j\}_{j=1}^m$ in R^n with $n \geq 3, p > 1, k(x)$, and $K(x)$ as measurable functions. Under different assumptions about $k(x)$ and $K(x)$, we can obtain an existence result of positive solutions by the Perron method (also called the monotone method) and a non-existence result of positive solutions by the maximal norm growth argument. When $\Delta_H U = \Delta U$ with the Laplacian operator Δ, $k(x) = 0$, and $K(x) = K(|x|)$ is a radial function with suitable decay assumption, the equation also includes the scalar curvature problem in R^n. These results extend some results of Gidas and Spruck [6] in the case when $1 < p < \frac{n+2}{n-2}$ and Gidas et al. [5] when $p = \frac{n+2}{n-2}$. See also reference [3] for a beautiful proof of the latter result.

We prove the following result:

THEOREM 1.1. *Assume $0 < \alpha < n$ and $p > 1$. Let $\omega : R_+ \to R_+$ be the monotone non-increasing function such that $\omega(r/2) \leq C\omega(r)$ for any $r > 1$ and for some uniform constant $C > 0$ and it holds that*

$$\int_1^\infty \frac{\omega(r)}{r} dr = A < \infty. \tag{1.2}$$

Then there are positive constants $\beta \in (0, 1)$ and θ such that for smooth functions $k(x)$ and $K(x)$ with $|K(x)| \leq \theta\omega(|x|)(1+|x|)^{-\tau}$ and $0 \leq k(x) \leq \theta\omega(|x|)(1+|x|)^{-\tau}$ on R^n for some $\tau > \alpha$ and $\tau \neq n$, the problem (1.1) has infinitely many positive solutions in $C^\beta(R^n) \bigcap H^{\alpha/2}_{loc}(R^n)$. Moreover, each of these solutions tends to some positive constant limit at infinity.

The novelty of the result above is $\alpha \in (0, n)$ about the range of the power α and the precise condition about the function $\omega(\cdot)$. In the previous study, the condition that $\omega(r/2) \leq C\omega(r)$ for any $r > 1$ is not mentioned. We prove the above result by using the classical Perron method argument and the Perron method is based on the comparison principle. Similar argument had been carried out in reference [10] for nonlinear sub-elliptic equations on R^n.

Let $\alpha \in (0, 2)$, and we use the variational method to consider the existence of non-negative $H^{\alpha/2}$ weak solutions to the following nonlinear nonlocal superlinear elliptic problem:

$$(-\Delta)^{\alpha/2} u(x) = K(x)u^p \quad \text{in } R^n, \tag{1.3}$$

where $K(x) = K(|x|)$ is a non-negative non-increasing continuous radial function in R^n and $p > 1$. We have the following result:

THEOREM 1.2. *Assume $\alpha \in (0,2)$ and $p > 1$. Assume that $K(x) = K(|x|)$ is a non-negative continuous radial function in R^n such that it is non-increasing in $r = |x|$ and it satisfies the condition*

$$K(|x|)|x|^{(\alpha-n)(p+1)/2} \in L^1(R^n). \tag{1.4}$$

Then there is a positive radial $H^{\alpha/2}_{loc}$ weak solution to (1.3).

We remark that related regularity result has been developed in reference [1] (in case when $\alpha \in (0,2)$). We point out that in the case when $\alpha = 2$, the corresponding result (even a stronger result than the statement above) has been obtained as in reference [7]. However, their method cannot be directly used to prove the result here. We use the Hardy inequality and symmetrization re-arrangement argument to obtain the existence of the ground state. This idea may be useful for other problems such as the stationary nonlinear nonlocal Schrodinger systems. Note that solutions to the fractional Ginzburg–Landau model have different behavior [9].

We denote by C the uniform constants, which may vary in different inequalities or formulae. We also denote by $B_R = B_R(0)$ the ball of radius R in R^n.

We present the potential analysis about nonlocal Poisson equation in Section 2 and also we prove the main results, Theorems 1.1 and 1.2, in Sections 3 and 4, respectively.

2. Preliminary. To prove Theorem 1.1, we need to prepare some lemmata about the fractional Poisson equation in R^n. As we have used them in reference [9] and they may be useful to other situations, we present the proofs in great details.

Let $\omega : R^n \to R$ be a radially symmetric monotone non-increasing function as in Theorem 1.1. Recall that the function $\omega = \omega(|x|)$ is a non-increasing non-negative radial function on R^n such that there is an uniform constant $C > 0$ such that $\omega\left(\frac{|x|}{2}\right) \le C\omega(|x|)$ and $\int_1^\infty \frac{\omega(r)}{r} dr < \infty$. By the latter condition, we know that there exists a sequence (r_j), $r_j \to \infty$ such that $\omega(r_j) \lg r_j \to 0$.

Let $f : R^n \to R$ be a locally Holder continuous function with the decay growth as below

$$|f(x)| \le C\omega(|x|)(1 + |x|)^{-\tau}, \tag{2.1}$$

where $C > 0$ and $n \neq \tau > \alpha$ are uniform constants.

We study the nonlocal Poisson equation:

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), \text{ in } R^n, \tag{2.2}$$

where $0 < \alpha < n$.

We consider the Riesz potential of f:

$$u(x) = c_{n,\alpha} \int_{R^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \tag{2.3}$$

where $c_{n,\alpha}$ is an universal constant depending only on α and n. Recall that $c_{n,\alpha}|x-y|^{\alpha-n}$ is the Green function of the α-Laplace operator $(-\Delta)^{\frac{\alpha}{2}}$ in R^n. Then $u(x)$ solves the Poisson equation (2.2) provided f satisfies some decay condition, saying, for example, $f \in C_0^\infty(R^n)$.

LEMMA 2.1. *Assume that τ is a positive number such that $n \neq \tau > \alpha$. Let $\omega : R^n \to R$ be a radially symmetric monotone non-increasing function as in Theorem 1.1 and let f be the function used in equation (2.1). Then the function $u(x)$ defined above is well defined with the following decay estimate at ∞*

$$|u(x)| \le \begin{cases} C|x|^{\alpha-n}\omega(|x|) & \text{if } \tau > n, \\ C|x|^{\alpha-n}\lg|x|\omega(|x|) & \text{if } \tau = n, \\ C|x|^{\alpha-\tau}\omega(|x|) & \text{if } \alpha < \tau < n, \\ C\lg|x|\omega(|x|) & \text{if } \alpha = \tau < n. \end{cases}$$

Proof. We show that $u(x)$ is well defined with the desired decay at infinity. Note that

$$|u(x)| \le C\int_{R^n} \frac{1}{|x-y|^{n-\alpha}} \cdot \frac{\omega(y)}{(1+|y|)^\tau}dy.$$

The later integral can be estimated by dividing the domain into three parts as below. Assume $|x| > 1$ and set

$$D_1 = \left\{y \in R^n;\ |y-x| \le \frac{|x|}{2}\right\},$$
$$D_2 = \left\{y \in R^n;\ \frac{|x|}{2} \le |y-x| \le 2|x|\right\},$$

and

$$D_3 = \{y \in R^n;\ 2|x| \le |y-x|\}.$$

Then, $R^n = D_1 \cup D_2 \cup D_3$.

Let

$$I_i = C\int_{D_i} \frac{\omega(y)}{|x-y|^{n-\alpha}(1+|y|^\tau)}dy, \text{ for } i = 1, 2, 3.$$

Then,

$$|u(x)| \le I_1 + I_2 + I_3.$$

The term I_1 may be bounded as follows. For $y \in D_1$, we have $|x| - |y| \le |x-y| \le \frac{|x|}{2}$ and $|y| \ge \frac{|x|}{2} \ge |x-y|$. Then for $\tau > \alpha$,

$$\begin{aligned} I_1 &\le C\int_{|y-x|\le\frac{|x|}{2}} \frac{\omega\left(\frac{|x|}{2}\right)}{|x-y|^{n-\alpha}\left[1+\frac{|x|}{2}\right]^\tau}dy \\ &\le C\frac{\omega\left(\frac{|x|}{2}\right)}{|x|^\tau}\int_0^{\frac{|x|}{2}} \frac{1}{r^{n-\alpha}} \cdot r^{n-1}dr \\ &= C\frac{\omega\left(\frac{|x|}{2}\right)}{|x|^\tau}\int_0^{\frac{|x|}{2}} r^{\alpha-1}dr \\ &= C\frac{1}{\alpha}\frac{\omega\left(\frac{|x|}{2}\right)}{|x|^\tau}\left(\frac{|x|}{2}\right)^\alpha \\ &= C|x|^{\alpha-\tau}\omega\left(\frac{|x|}{2}\right). \end{aligned}$$

For $\tau = \alpha$, we have $I \le C\omega(\frac{|x|}{2}) \lg|x|$.

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The term I_3 can be estimated similarly. For $y \in D_3$, we have $|y-x| \le |y| + |x| \le |y| + \frac{|y-x|}{2}$ and $|x| \le \frac{|y-x|}{2} \le |y|$. Thus,

$$\begin{aligned} I_3 &\le C \int_{D_3} \frac{\omega\left(\frac{|x-y|}{2}\right)}{|x-y|^{n-\alpha}\left[1+\left(\frac{|x-y|}{2}\right)\right]^{\tau}} dy \\ &\le C \int_{2|x|}^{\infty} \frac{\omega\left(\frac{r}{2}\right)}{r^{n-\alpha} \cdot r^{\tau}} \cdot r^{n-1} dr \\ &\le C|x|^{\alpha-\tau} \omega(|x|/2). \end{aligned}$$

Finally, we estimate the term I_2. Note that for $y \in D_2$, from $|y| - |x| \le |y-x| \le 2|x|$ we have $|y| \le 3|x|$. Thus,

$$\begin{aligned} I_2 &\le \frac{C\omega\left(\frac{|x|}{2}\right)}{|x|^{n-\alpha}} \int_{D_2} \frac{1}{(1+|y|)^{\tau}} dy \\ &\le C|x|^{\alpha-n} \omega\left(\frac{|x|}{2}\right)\left(\int_{|y|\le 1}^{\infty} \frac{1}{(1+|y|)^{\tau}} + \int_{1\le|y|\le 3|x|} \frac{1}{(1+|y|)^{\tau}}\right) \\ &= C|x|^{\alpha-n} \omega\left(\frac{|x|}{2}\right)\left(1 + \int_{1}^{3|x|} \frac{1}{r^{\tau}} \cdot r^{n-1} dr\right). \end{aligned}$$

Note that the last term (denoted by B) above has the following estimate:

$$B \begin{cases} \le 1 & \tau > n, \\ \simeq \lg |x| & \tau = n, \\ \lesssim |x|^{n-\tau} & \alpha < \tau < n. \end{cases}$$

In conclusion, we have

$$|u(x)| \le \begin{cases} C|x|^{\alpha-n} \omega\left(\frac{|x|}{2}\right) & \text{if } \tau > n, \\ C|x|^{\alpha-n} \lg |x| \omega\left(\frac{|x|}{2}\right) & \text{if } \tau = n, \\ C|x|^{\alpha-\tau} \omega\left(\frac{|x|}{2}\right) & \text{if } \alpha < \tau < n, \\ C \lg |x| \omega\left(\frac{|x|}{2}\right) & \text{if } \alpha = \tau < n. \end{cases}$$

This completes the proof. □

We now give the lower about the Riesz potential.

LEMMA 2.2. *Let $\omega : R^n \to R$ be a radially symmetric monotone non-increasing function as in Theorem 1.1. Assume $f \ge 0$ in R^n and $f(x) \ge C(1+|x|)^{-\tau}\omega(|x|)$ for some $\tau > \alpha$ and $C > 0$. Then the Riesz potential u defined above has the following lower bounds at ∞:*

$$u(x) \ge \begin{cases} C|x|^{\alpha-n} \omega(|x|) & \text{if } \tau > n, \\ C \lg |x| \omega(|x|) & \text{if } \tau = n, \\ C|x|^{\alpha-\tau} \omega(|x|) & \text{if } \tau < n. \end{cases}$$

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Proof. As in the proof of Lemma 2.1 above, we have

$$u(x) = J_1 + J_2 + J_3 \geq J_2,$$

where $J_i = \int_{\tilde{D}_i} \frac{Cf(y)}{|x-y|^{n-\alpha}} dy$ and

$$\tilde{D}_2 = \left\{ y \in R^n;\ \frac{|x|}{2} \leq |y-x| \leq |x| \right\}.$$

Note that for $y \in \tilde{D}_2$, we have $|y| \leq 2|x|$.

Take $R > 0$ large such that f is nontrivial on $B_R(0)$. Then for $|x| > 1$ large,

$$\begin{aligned} J_2 &= \int_{B_R(0)} \frac{Cf(y)}{|x-y|^{n-\alpha}} dy + \int_{\tilde{D}_2 \setminus B_R(0)} \frac{Cf(y)}{|x-y|^{n-\alpha}} dy \\ &\geq \frac{C}{|x|^{n-\alpha}} \left(\int_{B_R(0)} f(y) dy + \int_{\tilde{D}_2 \setminus B_R(0)} f(y) dy \right) \\ &\geq \frac{C}{|x|^{n-\alpha}} \left(1 + \int_{\tilde{D}_2 \setminus B_R(0)} \frac{\omega(y)}{|y|^\tau} dy \right) \\ &\geq \frac{C}{|x|^{n-\alpha}} \left(1 + \int_R^{2|x|} \frac{\omega(r)}{r} r^{n-\tau} dr \right). \end{aligned}$$

This completes the proof. □

We then have the following existence result about the nonlocal Poisson equation (2.2).

PROPOSITION 2.3. *Assume f is given as above with $\tau \neq n$. Then for any $a \in R$, the equation (2.2) has a unique solution $u_a \in C^\beta(R^n) \cap H^\alpha_{loc}$ for some $\beta \in (0, 1)$ for which*

$$\lim_{|x| \to \infty} u_a(x) = a.$$

Proof. Note that u defined above is bounded in R^n. The Holder continuity comes from the direct computations (see Theorem 12.1 in the reference [1]). Define

$$u_a(x) = a - u(x) = a - C \int_{R^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

By Lemmata 2.1 and 2.2, we know that

$$\lim_{|x| \to \infty} u_a(x) = a$$

and

$$u_a \in C^\beta(R^n) \cap H^\alpha_{loc}.$$

The solution is unique by applying the maximum principle. □

3. Proof of Theorem 1.1. The argument presented below is based on the potential analysis in Section 2.

We now prove Theorem 1.1. We claim that there is some constants $\theta_1 > 0$ such that the following two assertions are true:

Assertion 1: For any $a \in (1/3, 1/2)$ and for any $C \in (0, \theta_1)$, we can define the function $U_a(x)$ on R^n by adding the constant a to the expression (2.3) and then $U_a(x)$ solves the nonlocal equation:

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$$(-\Delta)^{\alpha/2}u = -\frac{C\omega(x)}{(1+|x|)^{\tau}}, \quad \text{on } R^n$$

with $0 < U_a(x) \leq a$ and $U_a(x) \to a$ at infinity.

Assertion 2: For any $a \in (1/3, 1/2)$ and for any $C \in (0, \theta_1)$, we can define the function $U^a(x)$ on R^n via the formula (2.3) such that $U^a(x)$ solves the nonlocal equation:

$$(-\Delta)^{\alpha/2}u = \frac{C\omega(x)}{(1+|x|)^{\tau}}, \quad \text{on } R^n$$

with $a \leq U^a(x) < 1$ and $U^a(x) \to a$ at infinity.

Choose $\theta = \theta_1/3$ and $C = 2\theta$ in both the above equations. We can verify that U_a is the lower solution to (1.1) and U^a is the upper solution to (1.1) for any $a \in (1/3, 1/2)$. Clearly, we have $0 < U_a \leq U^a < 1$. Compute

$$\begin{aligned}(-\Delta)^{\alpha/2}U^a + &k(x)U^a - K(x)(U^a)^p \\ &= 2\theta_1\omega(|x|)/3(1+|x|)^{\tau} + k(x)U^a - K(x)(U^a)^p \\ &\geq 2\theta_1\omega(|x|)/3(1+|x|)^{\tau} + |K(x)| + |k(x)| \\ &\geq 0 \ \text{ in } R^n,\end{aligned}$$

and, similarly, we have

$$(-\Delta)^{\alpha/2}U_a + k(x)U_a - K(x)(U_a)^p \leq 0, \quad \text{in } R^n.$$

Then, we can use the Perron method (e.g. [**1**] or [**10**]) to get the desired solution $u(x)$ to (1.1) such that $U_a(x) \leq u(x) \leq U^a(x)$ on R^n.

This then completes the proof of Theorem 1.1.

4. Existence of $H^{\alpha/2}_{loc}$ weak solutions. Formally, it is clear that equation (1.3) is the Euler–Lagrange equation of the functional

$$I(u) = \frac{1}{2}\int_{R^n}((-\Delta)^{\alpha/2}u(x), u(x)) - \frac{1}{p+1}\int_{R^n} K(x)|u(x)|^{p+1}$$

on the space

$$H = \left\{u \in L^1_{loc}(R^n);\ \int_{R^n}[((-\Delta)^{\alpha/2}u(x), u(x)) + K(x)|u(x)|^{p+1}]dx < \infty\right\}.$$

We denote by

$$|u|_H := \left[\int_{R^n}((-\Delta)^{\alpha/2}u(x), u(x))\right]^{1/2}$$

the norm on H. We may look for solutions in the class L_r of non-increasing radially symmetric functions in R^n. Define $H_r = H \bigcap L_r$.

Recall the fractional Hardy type inequality (Theorem 1.1 in reference [**2**]). There is an uniform constant $C = C_{n,\alpha} > 0$ such that for any $u \in H_r$, we have

$$\int_{R^n}\int_{R^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}} \geq C\int_{R^n}\frac{|u|^2}{|x|^{\alpha}}.$$

Note that

$$\int_{R^n}\int_{R^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}=\int_{R^n}((-\Delta)^{\alpha/2}u(x),u(x)).$$

For $u\in H_r$, we have

$$u(r)^2r^{n-\alpha}\le c\int_{B_r(0)}\frac{|u(x)^2}{|x|^\alpha}\le c\int_{R^n}\int_{R^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}}. \tag{4.1}$$

With this understanding, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Note that assumption (1.4) implies that for any $u\in H_r$, we automatically have

$$K(|x|)|u(x)|^{p+1}\le cK(|x|)|x|^{(\alpha-n)(p+1)/2}|u|_H^{p+1}\in L^1(R^n).$$

This estimate allows us to use the Lebesgue dominated convergence theorem for any minimizing sequence of the functional I on Σ defined below.

We use the Nehari functional trick to obtain a solution. Define the functional in H by

$$N(u)=\int_{R^n}((-\Delta)^{\alpha/2}u(x),u(x))-\int_{R^n}K(x)|u(x)|^{p+1}$$

and define the Nehari manifold as

$$\Sigma=\{u\in H;\,u\neq 0,\ N(u)=0\}.$$

For $u\in\Sigma$, we have

$$I(u)=\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{R^n}((-\Delta)^{\alpha/2}u(x),u(x))$$

and we may take this functional as the new definition of $I(u)$ on Σ.

Define

$$d=\inf_{u\in\Sigma}I(u).$$

We want to take a minimizing sequence $(u_j)\subset\Sigma$ such that $I(u_j)\to d$ and we want to replace each u_j by a radial function in H_r.

Claim: $d>0$.

In fact, for $u\in\Sigma$, we let u^* be the radial arrangement of u. As

$$I(u)\ge I(u^*)$$

and

$$\int_{R^n}K(x)|u(x)|^{p+1}\le\int_{R^n}K(x)|u^*(x)|^{p+1},$$

we have $N(u^*)\le 0$.

Note that $u^*\in H_r$. If $N(u^*)=0$, we have $u^*\in\Sigma$. From the estimate above, we have

$$\int_{R^n}K(x)|u(x)|^{p+1}\le c|u|_H^{p+1}$$

and then,

$$|u|_H^2 \leq c|u|_H^{p+1},$$

which implies that $|u|_H^2 \geq c > 0$ for some uniform constant $c > 0$. Then, we have

$$d \geq \left(\frac{1}{2} - \frac{1}{p+1}\right) c > 0.$$

Otherwise, we have $N(u^*) < 0$ and we may choose $\lambda \in (0, 1)$ such that $\hat{u} := \lambda u^*$, $N(\hat{u}) = 0$ and doing the same thing as above we get $|\hat{u}|_H^2 \geq c > 0$. So,

$$\left(\frac{1}{2} - \frac{1}{p+1}\right) c \leq I(\lambda u^*) = \lambda^2 I(u^*) \leq I(u^*) \leq I(u) \tag{4.2}$$

and again we have $d \geq \left(\frac{1}{2} - \frac{1}{p+1}\right)c$.

We now choose a minimizing sequence $(u_j) \subset \Sigma$ such that $I(u_j) \to d$. It is clear that u_j is uniformly bounded in H and we may choose an H-weakly convergent subsequence of u_j, still denoted by (u_j) and its limit is denoted by u. Using the relation (4.2), we may further assume that all $u_j \in H_r$. Otherwise, we may take $\hat{u}_j$ as the minimizing sequence. Applying the Lebesgue dominated convergence theorem to the sequence $u_j \in \Sigma$ with the estimate (4.1), we know that $d = I(u)$, which implies that $u \neq 0$. By the weakly semi-continuity of the H norm, we know that $N(u) \leq 0$. If u does not in Σ, i.e., $I(u) < 0$, then we may find a real number $\lambda \in (0, 1)$ such that $\lambda u \in \Sigma$ and then $I(\lambda u) \geq d$. However, we have

$$I(\lambda u) = \lambda^2 I(u) = \lambda^2 d < d,$$

which is impossible. Hence, we have $u \in \Sigma$ and $I(u) = d$. Therefore, using Lagrange's multiplier method, we know that u satisfies (1.3) in H-weak sense.

This completes the proof of Theorem 1.2. □

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