



# Equal-Sum-Product problem II

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**Abstract.** In this paper, we present the results related to a problem posed by Andrzej Schinzel. Does the number  $N_1(n)$  of integer solutions of the equation

$$x_1 + x_2 + \cdots + x_n = x_1 x_2 \cdots x_n, \quad x_1 \geq x_2 \geq \cdots \geq x_n \geq 1$$

tend to infinity with  $n$ ? Let  $a$  be a positive integer. We give a lower bound on the number of integer solutions,  $N_a(n)$ , to the equation

$$x_1 + x_2 + \cdots + x_n = a x_1 x_2 \cdots x_n, \quad x_1 \geq x_2 \geq \cdots \geq x_n \geq 1.$$

We show that if  $N_2(n) = 1$ , then the number  $2n - 3$  is prime. The average behavior of  $N_2(n)$  is studied. We prove that the set  $\{n : N_2(n) \leq k, n \geq 2\}$  has zero natural density.

## 1 Introduction

Let  $\mathbb{N} = \{1, 2, 3, \dots\}$  denote the set of all natural numbers (i.e., positive integers). Equal-Sum-Product Problem is relatively easy to formulate but still unresolved (see [4]). Some early research focused on estimating the number of solutions,  $N_1(n)$ , to the equation

$$(1.1) \quad x_1 + x_2 + \cdots + x_n = x_1 x_2 \cdots x_n, \quad x_1 \geq x_2 \geq \cdots \geq x_n \geq 1,$$

which can be found in [3, 8]. Schinzel asked in papers [10, 11] if the number  $N_1(n)$  tends toward infinity with  $n$ . This conjecture is yet to be proven. In [15], it was shown that the set  $\{n : N_1(n) \leq k, n \in \mathbb{Z}, n \geq 2\}$  has zero natural density for all natural  $k$ . It is worth noting that the classical Diophantine equation  $x_1^2 + x_2^2 + x_3^2 = 3x_1 x_2 x_3$  was investigated by Markoff (1879), as mentioned in [1, 7]. Additionally, Hurwitz (see [5]) examined the family of equations  $x_1^2 + x_2^2 + \cdots + x_n^2 = a x_1 x_2 \cdots x_n$ , where  $a, n \in \mathbb{N}, n \geq 3$ . Let us now assume that  $a, n \in \mathbb{N}, n \geq 2$ . In this paper, we provide a lower bound for the number  $N_a(n)$  of integer solutions  $(x_1, x_2, \dots, x_n)$  of the equation

$$(1.2) \quad x_1 + x_2 + \cdots + x_n = a x_1 x_2 \cdots x_n$$

such that  $x_1 \geq x_2 \geq \cdots \geq x_n \geq 1$ . Some of the results presented can be generalized to the case of the equation

$$(1.3) \quad b(x_1 + x_2 + \cdots + x_n) = a x_1 x_2 \cdots x_n,$$

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where  $a, b$  are positive integers. In the case  $a = 1, b = n$ , the equation

$$n(x_1 + \cdots + x_n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

is called Erdős last equation (see [4, 12, 13]). Equation (1.3) is related to the problem of finding numbers divisible by the sum and product of their digits. It is worth noting that if equation (1.2) has solutions, then  $a \leq n$ .

## 2 Basic results

In this section, we discuss the necessary basic results. First, we will show that the number of solutions  $N_a(n)$  is finite for any fixed  $a$  and  $n$ .

**Lemma 2.1** *Let  $n$  be a natural number. If  $x_1, x_2, \dots, x_n$  are any real numbers, then the following formula holds:*

$$(2.1) \quad \left( a \prod_{i=1}^{n-1} x_i - 1 \right) (ax_n - 1) + a \sum_{s=1}^{n-2} \left( \left( \prod_{i=1}^s x_i - 1 \right) (x_{s+1} - 1) \right) = a^2 \prod_{i=1}^n x_i - a \sum_{i=1}^n x_i + a(n-2) + 1.$$

**Proof** Let us denote equation (2.1) as  $T(n)$ . We want to show by induction that  $T(n)$  holds for every natural number  $n$ . The cases  $n = 1$  and  $n = 2$  are trivial:  $(a-1)(ax_1-1) = a^2x_1 - ax_1 - a + 1$ ,  $(ax_1-1)(ax_2-1) = a^2x_1x_2 - a(x_1+x_2) + 1$ . In both cases, equality is true. Therefore, the base step of the induction is satisfied, as  $T(1)$  and  $T(2)$  hold. Let us assume now that  $n \geq 3$  and  $T(n-1)$  holds, i.e., the following equality is true:

$$(2.2) \quad \left( a \prod_{i=1}^{n-2} x_i - 1 \right) (ax_{n-1} - 1) + a \sum_{s=1}^{n-3} \left( \left( \prod_{i=1}^s x_i - 1 \right) (x_{s+1} - 1) \right) = a^2 \prod_{i=1}^{n-1} x_i - a \sum_{i=1}^{n-1} x_i + a(n-3) + 1.$$

In the inductive step, we will be using the equivalent form of equation (2.2):

$$(2.3) \quad - \left( a \prod_{i=1}^{n-2} x_i - 1 \right) (ax_{n-1} - 1) + a \sum_{s=1}^{n-3} \left( \left( \prod_{i=1}^s x_i - 1 \right) (x_{s+1} - 1) \right) = a^2 \prod_{i=1}^{n-1} x_i - a \sum_{i=1}^{n-1} x_i + a(n-3) + 1.$$

To prove the inductive step, i.e., to show that  $T(n-1)$  implies  $T(n)$  for  $n \geq 3$ , we will use the following algebraic identities that can be verified directly:

$$(2.4) \quad \left( a \prod_{i=1}^{n-1} x_i - 1 \right) (ax_n - 1) = a^2 \prod_{i=1}^n x_i - ax_n + 1 - a \prod_{i=1}^{n-1} x_i,$$

$$(2.5) \quad \begin{aligned} & a \sum_{s=1}^{n-2} \left( \left( \prod_{i=1}^s x_i - 1 \right) (x_{s+1} - 1) \right) = \\ & a \left( \prod_{i=1}^{n-2} x_i - 1 \right) (x_{n-1} - 1) + a \sum_{s=1}^{n-3} \left( \left( \prod_{i=1}^s x_i - 1 \right) (x_{s+1} - 1) \right). \end{aligned}$$

Let us proceed to the proof of the inductive step. We want to show  $T(n)$  assuming  $T(n-1)$ . Let us start by transforming the left side of  $T(n)$  using equations (2.4) and (2.5)

$$(2.6) \quad \begin{aligned} & \left( a \prod_{i=1}^{n-1} x_i - 1 \right) (ax_n - 1) + a \sum_{s=1}^{n-2} \left( \left( \prod_{i=1}^s x_i - 1 \right) (x_{s+1} - 1) \right) = \\ & a^2 \prod_{i=1}^n x_i - ax_n + 1 - a \prod_{i=1}^{n-1} x_i + \\ & + a \left( \prod_{i=1}^{n-2} x_i - 1 \right) (x_{n-1} - 1) + a \sum_{s=1}^{n-3} \left( \left( \prod_{i=1}^s x_i - 1 \right) (x_{s+1} - 1) \right). \end{aligned}$$

Calculating directly, we notice that the following equality holds true

$$(2.7) \quad \begin{aligned} & -a \prod_{i=1}^{n-1} x_i + a \left( \prod_{i=1}^{n-2} x_i - 1 \right) (x_{n-1} - 1) = \\ & -a \prod_{i=1}^{n-1} x_i + a \prod_{i=1}^{n-1} x_i - ax_{n-1} - a \prod_{i=1}^{n-2} x_i + a = a - ax_{n-1} - a \prod_{i=1}^{n-2} x_i. \end{aligned}$$

From equations (2.6) and (2.7), and then using the inductive assumption (2.3), we obtain

$$\begin{aligned} & \left( a \prod_{i=1}^{n-1} x_i - 1 \right) (ax_n - 1) + a \sum_{s=1}^{n-2} \left( \left( \prod_{i=1}^s x_i - 1 \right) (x_{s+1} - 1) \right) \\ & = a^2 \prod_{i=1}^n x_i - ax_n + 1 + a - ax_{n-1} - a \prod_{i=1}^{n-2} x_i + a \sum_{s=1}^{n-3} \left( \left( \prod_{i=1}^s x_i - 1 \right) (x_{s+1} - 1) \right) \stackrel{(2.3)}{=} \\ & a^2 \prod_{i=1}^n x_i - ax_n + 1 + a - ax_{n-1} - a \prod_{i=1}^{n-2} x_i - \left( a \prod_{i=1}^{n-2} x_i - 1 \right) (ax_{n-1} - 1) + \\ & + a^2 \prod_{i=1}^{n-1} x_i - a \sum_{i=1}^{n-1} x_i + a(n-3) + 1 = a^2 \prod_{i=1}^n x_i - a \sum_{i=1}^n x_i + a(n-2) + 1. \end{aligned}$$

Thus, assuming  $T(n-1)$ , we have shown that  $T(n)$  holds, completing the inductive step and concluding the proof of the lemma. ■

**Theorem 2.2** Let  $a, k \in \mathbb{N}$ ,  $b \in \mathbb{N} \cup \{0\}$ . For any integer  $n \geq 2$ , the system of Diophantine equations

$$(2.8) \quad \begin{cases} x_{1,1} + x_{1,2} + \cdots + x_{1,n} &= ax_{2,1} \cdot x_{2,2} \cdot \cdots \cdot x_{2,n} + b, \\ x_{2,1} + x_{2,2} + \cdots + x_{2,n} &= ax_{3,1} \cdot x_{3,2} \cdot \cdots \cdot x_{3,n} + b, \\ &\dots \\ x_{k-1,1} + x_{k-1,2} + \cdots + x_{k-1,n} &= ax_{k,1} \cdot x_{k,2} \cdot \cdots \cdot x_{k,n} + b, \\ x_{k,1} + x_{k,2} + \cdots + x_{k,n} &= ax_{1,1} \cdot x_{1,2} \cdot \cdots \cdot x_{1,n} + b \end{cases}$$

has only finite number of solutions  $x_{i,j}$  which are natural numbers.

**Proof** By adding sides of equations of the system of equations (2.8), we obtain

$$\sum_{i=1}^k \sum_{j=1}^n x_{i,j} = \sum_{i=1}^k a \prod_{j=1}^n x_{i,j} + kb.$$

Hence,

$$\sum_{i=1}^k \left( a^2 \prod_{j=1}^n x_{i,j} - a \sum_{j=1}^n x_{i,j} + a(n-2) + 1 \right) = k(a(n-2) + 1) - kab.$$

By (2.1), we have

$$(2.9) \quad \sum_{i=1}^k \left( \left( a \prod_{j=1}^{n-1} x_{i,j} - 1 \right) (ax_{i,n} - 1) + a \sum_{s=1}^{n-2} \left( \prod_{j=1}^s x_{i,j} - 1 \right) (x_{i,s+1} - 1) \right) = k(a(n-2) + 1) - kab.$$

For given  $a, k, b, n$ , the number of solutions of equation (2.9) in positive integers is bounded above. Hence, the system of equations (2.8) has only a finite number of solutions in positive integers  $x_{i,j}$ . ■

Taking  $k = 1$ , an immediate consequence of Theorem 2.2 is the following result.

**Corollary 2.3** For given  $a \in \mathbb{N}$ ,  $b \in \mathbb{N} \cup \{0\}$  and any integer  $n \geq 2$ , the number of solutions of the equation

$$(2.10) \quad x_1 + x_2 + \cdots + x_n = ax_1 \cdot x_2 \cdot \cdots \cdot x_n + b$$

in positive integers  $x_1 \geq x_2 \geq \cdots \geq x_n \geq 1$  is finite. In particular, in the case  $b = 0$ , the number of solutions  $N_a(n)$  is finite.

**Remark 2.4** Theorem 2.2 is true for all  $a, b \in \mathbb{Q}$ ,  $a \geq 1$ .

**Remark 2.5** In the case of  $b = 0$ , we can provide a different proof of Corollary 2.3.

Let  $z_i = x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n = \frac{1}{x_i} \prod_{j=1}^n x_j \in \mathbb{N}$  for  $i \in \{1, 2, \dots, n\}$ . Notice that from the inequality  $x_1 \geq x_2 \geq \cdots \geq x_n \geq 1$ , we get the inequality  $1 \leq z_1 \leq z_2 \leq \cdots \leq z_n$ . Then, equation (2.10) takes the form

$$(2.11) \quad \frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_n} = a \geq 1.$$

Equation (2.11) has finitely many solutions in positive integers, as we can find upper bounds on  $z_i$ . The bounds we will find are not optimal, but they are sufficient for our purposes. If  $n \geq 2$ , then  $ax_1x_2 \cdots x_n = x_1 + x_2 + \cdots + x_n \geq x_1 + x_2 \geq x_1 + 1 > x_1$ , and hence  $ax_2 \cdots x_n \geq 2$ . From here, we can deduce

$$(n-1)x_2 \geq x_2 + \cdots + x_n = x_1(ax_2 \cdots x_n - 1) \geq x_1.$$

Therefore,  $nx_2 > x_1$  and  $nz_1 > z_2$ . We also have for  $k \in \{2, 3, \dots, n-1\}$ , that

$$nz_1z_2 \cdots z_k \geq z_1z_2 \geq \prod_{i=1}^n x_i \geq z_{k+1}.$$

Thus, for all  $k \in \{1, 2, \dots, n-1\}$ , we have  $z_{k+1} \leq nz_1 \cdot z_2 \cdots z_k$ . Now we can proceed with the inductive proof of the upper bound:  $z_i \leq a^{-1}n^{2^{i-1}}$ , where  $i \in \{1, 2, \dots, n\}$ . Base step, as the  $z_i$  are increasing, we can use equation (2.11) to obtain an inequality:

$$\frac{n}{z_1} \geq \frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_n} = a \geq 1, \text{ hence } z_1 \leq a^{-1}n.$$

If we now make the assumption that  $z_i \leq a^{-1}n^{2^{i-1}}$  for all  $i \in \{1, 2, \dots, k\}$ , where  $k < n$ , then  $z_{k+1} \leq nz_1z_2 \cdots z_k \leq n \frac{n^{2^0+2^1+\dots+2^{k-1}}}{a} = \frac{n^{2^k}}{a}$ ; this establishes the inductive step.

The proof of Theorem 2.2 can be modified in the specific case of  $a, n$  to create an efficient algorithm for finding solutions to equation (2.10).

Kurlandchik and Nowicki [6, Theorem 3] had earlier shown that  $N_1(n)$  is finite for any  $n \geq 2$ .

Schinzel's question can be generalized. For given  $a \in \mathbb{N}$ , does the number  $N_a(n)$  tend to infinity with  $n$ ? We can show with the elementary method the following theorems.

**Theorem 2.6** *If  $a, n \in \mathbb{N}$ , then  $\limsup_{n \rightarrow \infty} N_a(n) = \infty$ .*

**Proof** We shall consider two cases. Let  $a \in \{1, 2\}$ . If  $t \in \{0, 1, \dots, \lfloor \frac{s}{2} \rfloor\}$ , where  $s$  is a nonnegative integer, then

$$\begin{aligned} & \frac{1}{a}((a+1)^{s-t} + 1) + \frac{1}{a}((a+1)^t + 1) + \underbrace{1 + 1 + \cdots + 1}_{\frac{1}{a}((a+1)^s - 1) \text{ times}} = \\ & a \cdot \frac{1}{a}((a+1)^{s-t} + 1) \cdot \frac{1}{a}((a+1)^t + 1) \cdot \underbrace{1 \cdot 1 \cdots 1}_{\frac{1}{a}((a+1)^s - 1) \text{ times}}. \end{aligned}$$

We have  $s-t \geq t$  and  $\frac{1}{a}((a+1)^i + 1) \in \mathbb{N}$ , where  $i$  is a nonnegative integer. Hence,  $N(\frac{1}{a}((a+1)^s + 2a - 1)) \geq \lfloor \frac{s}{2} \rfloor + 1$ . Therefore,  $\limsup_{n \rightarrow \infty} N_a(n) = \infty$ .

Let  $a \geq 3$ . If  $t \in \{1, \dots, \lfloor \frac{s+1}{2} \rfloor\}$ , where  $s \in \mathbb{N}$ , then

$$\begin{aligned} & \frac{1}{a}((a-1)^{2s-2t+1} + 1) + \frac{1}{a}((a-1)^{2t-1} + 1) + \underbrace{1+1+\dots+1}_{\frac{1}{a}((a-1)^{2s-1}-1) \text{ times}} = \\ & a \cdot \frac{1}{a}((a-1)^{2s-2t+1} + 1) \cdot \frac{1}{a}((a-1)^{2t-1} + 1) \cdot \underbrace{1 \cdot 1 \cdot \dots \cdot 1}_{\frac{1}{a}((a-1)^{2s-1}-1) \text{ times}}. \end{aligned}$$

We have  $2s - 2t + 1 \geq 2t - 1$  and  $\frac{1}{a}((a-1)^{2i-1} + 1), \frac{1}{a}((a-1)^{2i} - 1) \in \mathbb{N}$ , where  $i \in \mathbb{N}$ . Hence,  $N(\frac{1}{a}((a-1)^{2s} + 2a - 1)) \geq \lfloor \frac{s+1}{2} \rfloor$ .

Therefore,  $\limsup_{n \rightarrow \infty} N_a(n) = \infty$ . ■

**Remark 2.7** Let  $a \geq 3$ . Depending on the choice of  $a \leq n$ , equation (1.2) may not have solutions. The simplest example is  $a = 3$  and  $n = 4$ . In this case, equation (1.2) is equivalent to

$$(3x_1x_2x_3 - 1)(3x_4 - 1) + 3(x_1x_2 - 1)(x_3 - 1) + 3(x_1 - 1)(x_2 - 1) = 7,$$

but the corresponding equation has no integer solutions  $x_1 \geq x_2 \geq x_3 \geq x_4 \geq 1$ . This gives  $N_3(4) = 0$ .

**Remark 2.8** Due to the solutions  $(\underbrace{2, 2, \dots, 1}_{4a-2 \text{ times}}, \underbrace{m, 1, \dots, 1}_{ma-m+1 \text{ times}})$ , where  $m \in \mathbb{N}$  and certain technical computations based on the method from Remark 2.5, we can prove that:

- (1)  $N_a(a) = N_a(2a - 1) = N_a(3a - 2) = N_a(4a - 3) = 1$ , where  $a \geq 2$ ,
- (2)  $N_2(6) = 2$ ,  $N_a(4a - 2) = 1$ , where  $a \geq 3$ ,
- (3)  $N_a(n) = 0$  if  $n \in ((a, 2a - 1) \cup (2a - 1, 3a - 2) \cup (3a - 2, 4a - 3)) \cap \mathbb{N}$ ,
- (4)  $N_a(ma - m + 1) \geq 1$ , where  $m \in \mathbb{N}$ .

Points (1)–(3) partially explain the basic structure of the right side of Table 1.

It has been proven in [15] that in the case of  $a = 1$ , the following theorem holds.

**Theorem 2.9** If  $n \in \mathbb{N}$ ,  $n \geq 2$ , then

$$(2.12) \quad N_1(n) \geq \left\lfloor \frac{d(n-1)+1}{2} \right\rfloor + \left\lfloor \frac{d(2n-1)+1}{2} \right\rfloor - 1,$$

where  $d(j)$  is the number of positive divisors of  $j$ . Moreover,

$$\begin{aligned} (2.13) \quad & N_1(n) \geq \left\lfloor \frac{d(n-1)+1}{2} \right\rfloor + \left\lfloor \frac{d(2n-1)+1}{2} \right\rfloor - 1 \\ & + \left\lfloor \frac{d_2(3n+1)+1}{2} \right\rfloor + \left\lfloor \frac{d_3(4n+1)+1}{2} \right\rfloor + \left\lfloor \frac{d_3(4n+5)+1}{2} \right\rfloor \\ & - \delta(2|n+1) - \delta(3|n+1) - \delta(3|n+2) \\ & - \delta(5|n+2, n \geq 8) - \delta(7|n+3, n \geq 11) - \delta(11|n+4, n \geq 29), \end{aligned}$$

where  $d_i(m)$  is the number of positive divisors of  $m$  which lie in the arithmetic progression  $i \pmod{i+1}$ . The function  $\delta$  is the Dirac delta function.

Table 1: The table shows values of  $N_a(n)$  for small natural numbers  $a \leq n \leq 10$ . The bold numbers are  $N_a(n)$ , such that  $n \geq 4a - 1$ .

$n \backslash a$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	1														
3	1	1	1													
4	1	1	0	1												
5	3	1	1	0	1											
6	1	2	0	0	0	1										
7	2	1	1	1	0	0	1									
8	2	1	0	0	0	0	0	1								
9	2	2	1	0	1	0	0	0	1							
10	2	1	1	1	0	0	0	0	0	1						
11	3	1	1	0	0	1	0	0	0	0	1					
12	2	2	0	0	0	0	0	0	0	0	0	1				
13	4	2	1	1	1	0	1	0	0	0	0	0	1			
14	2	2	0	1	0	0	0	0	0	0	0	0	0	1		
15	2	2	2	0	0	0	0	1	0	0	0	0	0	0	1	
16	2	1	0	1	0	1	0	0	0	0	0	0	0	0	0	1

**Remark 2.10** In the case  $a = 2$ , equation (1.2) has at least one *typical* solution in the form  $(n-1, \underbrace{1, 1, \dots, 1}_{n-1 \text{ times}})$ . Therefore,  $N_2(n) \geq 1$  for all integers  $n \geq 2$ .

### 3 Main results

We give a lower bound on the number of solutions  $N_a(n)$  of equation (1.2).

**Theorem 3.1** If  $a, n \in \mathbb{N}$ ,  $n \geq 2$ , then

$$(3.1) \quad N_a(n) \geq \left\lfloor \frac{d_{a-1}(a(n-2)+1)+1}{2} \right\rfloor + \left\lfloor \frac{d_{2a-1}(2a(n-1)+1)+1}{2} \right\rfloor - \delta(2a-1|n),$$

where  $d_i(m)$  is the number of positive divisors of  $m$  which lie in the arithmetic progression  $i \pmod{i+1}$ . The function  $\delta$  is the Dirac delta function.

**Proof** In the set  $\mathbb{N}^n$ , we have the following pairwise disjoint families of pairwise different  $(x_1, x_2, \dots, x_n)$  solutions of equation (1.2). Note that in each case  $x_i$  is an integer and  $x_1 \geq x_2 \geq \dots \geq x_n \geq 1$ . We define

$$A_1(n) = \left\{ \left( \frac{n-2+\frac{d+1}{a}}{d}, \frac{d+1}{a}, \underbrace{1, 1, \dots, 1}_{n-2 \text{ times}} \right) : \right. \\ \left. a(n-2)+1 \equiv 0 \pmod{d}, d \equiv -1 \pmod{a}, \right. \\ \left. 1 \leq d \leq \sqrt{a(n-2)+1}, d \in \mathbb{N} \right\}.$$

We also define

$$A_2(n) = \left\{ \left( \frac{n-1+\frac{d+1}{2a}}{d}, \frac{d+1}{2a}, \underbrace{2, 1, 1, \dots, 1}_{n-3 \text{ times}} \right) : \right. \\ \left. 2a(n-1) + 1 \equiv 0 \pmod{d}, d \equiv -1 \pmod{2a}, \right. \\ \left. 4a-1 \leq d \leq \sqrt{2a(n-1)+1}, d \in \mathbb{N} \right\}, \text{ when } n \geq 3.$$

We have  $A_2(2) = \emptyset$ . Moreover,

$$|A_1(n)| = |\{d : a(n-2) + 1 \equiv 0 \pmod{d}, d \equiv -1 \pmod{a}, \\ 1 \leq d \leq \sqrt{a(n-2)+1}, d \in \mathbb{N}\}| = \left\lfloor \frac{d_{a-1}(a(n-2)+1)+1}{2} \right\rfloor.$$

In the case of the set  $A_2(n)$ , we have  $d \neq 2a-1$ ; thus,

$$|A_2(n)| = |\{d : 2a(n-1) + 1 \equiv 0 \pmod{d}, d \equiv -1 \pmod{2a}, \\ 4a-1 \leq d \leq \sqrt{2a(n-1)+1}, d \in \mathbb{N}\}| = \\ = |\{d : 2a(n-1) + 1 \equiv 0 \pmod{d}, d \equiv -1 \pmod{2a}, \\ 1 \leq d \leq \sqrt{2a(n-1)+1}, d \in \mathbb{N}\}| \\ - |\{d : 2a(n-1) + 1 \equiv 0 \pmod{d}, d = 2a-1\}| = \\ \left\lfloor \frac{d_{2a-1}(2a(n-1)+1)+1}{2} \right\rfloor - \delta(2a-1|n).$$

The sets  $A_1(n)$ ,  $A_2(n)$  are disjoint. Hence,  $N_a(n) \geq |A_1(n)| + |A_2(n)|$ . Thus, we get immediately (3.1). ■

**Corollary 3.2** If  $n \in \mathbb{N}$ ,  $n \geq 2$ , then

$$(3.2) \quad N_2(n) \geq \left\lfloor \frac{d(2n-3)+1}{2} \right\rfloor + \left\lfloor \frac{d_3(4n-3)+1}{2} \right\rfloor - \delta(3|n).$$

The following corollary is almost immediate.

**Corollary 3.3** If  $n \in \mathbb{N}$ ,  $n \geq 2$ , then

$$(3.3) \quad N_2(n) \geq \frac{1}{2}d(2n-3).$$

**Proof** Formula (3.3) follows at once from Corollary 3.2 and inequalities

$$\left\lfloor \frac{d_3(4n-3)+1}{2} \right\rfloor \geq \delta(3|n), \left\lfloor \frac{x+1}{2} \right\rfloor \geq \frac{1}{2}x, \text{ where } x \in \mathbb{Z}. \quad \blacksquare$$

For the convenience of the reader, values of  $N_2(n)$  for small values of  $n$  are presented in Table 2.

Table 2: The table lists the numbers  $N_2(n)$  for  $2 \leq n \leq 51$ .

$n$	$N_2(n)$	$n$	$N_2(n)$	$n$	$N_2(n)$	$n$	$N_2(n)$	$n$	$N_2(n)$	$n$	$N_2(n)$	$n$	$N_2(n)$	$n$	$N_2(n)$	$n$	$N_2(n)$	$n$	$N_2(n)$
2	1	7	1	12	2	17	1	22	1	27	3	32	1	37	1	42	4	47	2
3	1	8	1	13	2	18	2	23	1	28	2	33	3	38	1	43	2	48	4
4	1	9	2	14	2	19	2	24	3	29	2	34	3	39	3	44	2	49	2
5	1	10	1	15	2	20	2	25	1	30	2	35	3	40	2	45	2	50	1
6	2	11	1	16	1	21	2	26	2	31	2	36	2	41	2	46	1	51	3



**Corollary 3.4** Let  $n \in \mathbb{N}$ ,  $n \geq 3$ . If the equation

$$(3.4) \quad x_1 + x_2 + \cdots + x_n = 2x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

has exactly one solution  $(n-1, \underbrace{1, 1, \ldots, 1}_{n-1 \text{ times}})$  in the natural numbers  $x_1 \geq x_2 \geq \cdots \geq x_n \geq 1$ , then  $2n-3$  is a prime number.

**Proof** If  $N_2(n) = 1$ , then by Corollary 3.3 we get  $2 \geq d(2n-3)$ . Since  $2n-3 \geq 3$ , it follows that  $2n-3$  is a prime number. ■

**Remark 3.5** If  $N_1(n) = 1$ , then  $n-1$  must be a Sophie Germain prime number (see [8]).

## 4 The set of exceptional values

Let  $E_{\leq k}^2 = \{n : N_2(n) \leq k, n \geq 2\}$ , where  $k \in \mathbb{N}$ . In particular,  $E_{\leq 1}^2 = \{n : N_2(n) = 1, n \geq 2\}$ .

**Theorem 4.1** The set  $E_{\leq k}^2$  has natural density 0, i.e., the ratio

$$\frac{1}{x} |E_{\leq k}^2 \cap [1, x]|$$

tends to 0 as  $x \rightarrow \infty$ .

**Proof** Let  $\Omega(m)$  count the total number of prime factors of  $m$ . We have  $\Omega(m) \leq d(m) - 1$  for every natural  $m$ . Let  $\pi_i(x) = |\{m : \Omega(m) = i, 1 \leq m \leq x\}|$ , i.e., the number of  $1 \leq m \leq x$  with  $i$  prime factors (not necessarily distinct). By Corollary 3.3, we have  $N_2(n) \geq \frac{1}{2}d(2n-3)$ . Thus, if  $n \in E_{\leq k}^2$ , then  $d(2n-3) \leq 2k$  and consequently  $\Omega(2n-3) \leq 2k-1$ . Therefore,

$$|E_{\leq k}^2 \cap [1, x]| \leq \sum_{i=0}^{2k-1} \pi_i(2x-3),$$

where  $x \geq 2$ . Using the sieve of Eratosthenes, one can show that (see [2, p. 75])

$$\pi_i(x) \leq \frac{1}{i!} x^{\frac{(A \log \log x + B)^i}{\log x}}$$

for some constants  $A, B > 0$ . There follows that

$$0 \leq \frac{1}{x} |E_{\leq k}^2 \cap [1, x]| \leq \frac{2x-3}{x} \sum_{i=0}^{2k-1} \frac{1}{i!} \frac{(A \log \log (2x-3) + B)^i}{\log (2x-3)}.$$

For a fixed  $k$ , the right-hand side tends to 0, as  $x \rightarrow \infty$ . Thus,

$$\lim_{x \rightarrow \infty} \frac{1}{x} |E_{\leq k}^2 \cap [1, x]| = 0.$$

This completes the proof. ■

The above theorem implies that the set  $E_k^2 = \{n : N_2(n) = k, n \geq 2\}$  has zero natural density for any fixed  $k \geq 1$ . This observation might suggest that the set  $E_k^2 = \{n : N_2(n) = k, n \geq 2\}$  is finite for any fixed  $k \geq 1$  and that the number  $N_2(n) \rightarrow \infty$  as  $n \rightarrow \infty$ . In the next theorem, we study the average behavior of  $N_2(n)$ .

**Theorem 4.2** *If  $\varepsilon > 0$ , then for sufficiently large  $x$ , we have*

$$\sum_{1 < n \leq x} N_2(n) \geq \frac{1-\varepsilon}{8} x \log x.$$

**Proof** By [9, 14], there exists constant  $c > 0$  such that

$$\left| \sum_{\substack{1 \leq n \leq x, \\ n \equiv 1 \pmod{2}}} d(n) - \frac{x}{4} \log x \right| \leq cx,$$

for sufficiently large  $x > x_0$ . It follows that

$$\sum_{\substack{1 \leq n \leq x, \\ n \equiv 1 \pmod{2}}} d(n) \geq \frac{x}{4} \log(x) - cx$$

for  $x > x_0$ . By Corollary 3.3, for  $n \geq 2$ , we have  $N_2(n) \geq \frac{1}{2}d(2n-3)$ . Therefore,

$$\begin{aligned} \frac{1}{x} \sum_{1 < n \leq x} N_2(n) &\geq \frac{1}{x} \sum_{1 < n \leq x} \frac{1}{2} d(2n-3) = \frac{1}{2x} \sum_{\substack{1 \leq m \leq 2x-3, \\ m \equiv 1 \pmod{2}}} d(m) \\ &\geq \frac{1}{8} \log(2x-3) - c \frac{2x-3}{2x} \end{aligned}$$

for  $2x-3 > x_0$ . Let  $\varepsilon > 0$ , then for sufficiently large  $x$ , we have

$$\frac{1}{x} \sum_{1 < n \leq x} N_2(n) \geq (1-\varepsilon) \frac{1}{8} \log x. \quad \blacksquare$$

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