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ON GENERALIZED WHITTAKER FUNCTIONS ON SIEGEL'S UPPER HALF SPACE OF DEGREE 2

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Dedicated to Professor Tomio Kubota on his sixtieth birthday

In [5], H. Maass showed that the dimension of a space of generalized Whittaker functions satisfying certain system of differential equations on Siegel's upper half space H_2 of degree 2 is three. First of all, we shall investigate the structure of a space of generalized Whittaker functions which are eigen functions for the algebra of invariant differential operators on H_2. The theory of generalized Whittaker functions is discussed in Yamashita [12], [13], [14], [15] with full generality. But, we will get an outlook of the space of generalized Whittaker functions by using elementary calculus instead of representation theory of Lie groups. Generalized Whittaker functions, naturally appear in the theory of indefinite theta function, and so we shall next show commutation relations between the invariant differential operators on H_2 and those on the product $H_1 \times H_1$ of two copies of the upper half plane H_1 operated on a theta function. The relations are analogies of commutation relations for Hecke operators in [1], [16], [17] and are proved in some cases with the Laplacian in [8], [2]. We essentially use the result in Nakajima [10] where the generators of the center of the universal enveloping algebra of $\mathfrak{sp}(2, \mathbb{R})$ are explicitly given. By commutation relations we can construct an automorphic form F on H_2 corresponding to an L-function with Grössencharacter of a certain biquadratic field. Generalized Whittaker functions investigated in the present paper appear in the Fourier expansion of F with respect to translations in H_2 and so we can define the "constant part" of the Fourier coefficient as the ratio of the Fourier coefficient to a generalized Whittaker function. The constant part of a certain Fourier coefficient of an automorphic form analogous to F is given in 2. (See in particular (2.10).)

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§ 1.

We denote, as usual, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ the ring of rational integers, the rational number field, the real number field, and the complex number field. We denote the algebra of invariant differential operators on H_2 by $\mathscr{D}$ and the center of the universal enveloping algebra of $\mathfrak{sp}(2, \mathbb{R})$ by $\mathscr{C}$. S. Nakajima calculated generators of $\mathscr{D}$, $\mathscr{C}$ in [9], [10]. The generators of $\mathscr{D}$ are

$$\Delta_1 = \sum_{i,j=1}^{3} y_i y_j \partial_i \bar{\partial}_j - d(\partial_1\bar{\partial}_3 + \bar{\partial}_1\partial_3 - (1/2)\partial_2\bar{\partial}_2)$$

and

$$\begin{aligned}\Delta_2 = {} & d^2(\partial_1\partial_3 - (1/4)\partial_2^2)(\bar{\partial}_1\bar{\partial}_3 - (1/4)\bar{\partial}_2^2) \\ & + \sqrt{-1}(1/4)d\Big(\sum_{i=1}^{3} y_i\partial_i\Big)(\bar{\partial}_1\bar{\partial}_3 - (1/4)\bar{\partial}_2^2) \\ & + \sqrt{-1}(1/4)d\Big(\sum_{i=1}^{3} y_i\bar{\partial}_i\Big)(\partial_1\partial_3 - (1/4)\partial_2^2) \\ & + (1/16)d(\partial_1\bar{\partial}_3 + \bar{\partial}_1\partial_3 - (1/2)\partial_2\bar{\partial}_2)\end{aligned}$$

where we put

$$z_i = x_i + \sqrt{-1}y_i \qquad (1 \leq i \leq 3)\,,$$

$$\partial_i = \frac{\partial}{\partial z_i} = (1/2)\Big(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\Big) \qquad (1 \leq i \leq 3)\,,$$

$$\bar{\partial}_i = \frac{\partial}{\partial \bar{z}_i} = (1/2)\Big(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\Big) \qquad (1 \leq i \leq 3)\,,$$

and $d = y_1 y_3 - y_2^2$ for $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in H_2$. For two complex numbers d_1, d_2, we consider the space $\mathscr{W}$ formed by functions $f(Z) = g(Y)e^{2\pi i \operatorname{tr} X}$ of $Z = X + iY \in H_2$, satisfying $\Delta_1 f = d_1 f$ and $\Delta_2 f = d_2 f$ with some functions g of Y, which we call generalized Whittaker functions. If $g(Y)e^{2\pi i \operatorname{tr} X}$ belongs to $\mathscr{W}$, g has to satisfy

$$\begin{aligned}(1.1)\quad & \frac{1}{8}\Big(4y_3y_2\frac{\partial^2}{\partial y_3 \partial y_2} + 4y_2^2\frac{\partial^2}{\partial y_3\partial y_1} + 2y_3^2\frac{\partial^2}{\partial y_3^2} + 4y_2y_1\frac{\partial^2}{\partial y_2\partial y_1} \\ & + y_3y_1\frac{\partial^2}{\partial y_2^2} + y_2^2\frac{\partial^2}{\partial y_2^2} + 2y_1^2\frac{\partial^2}{\partial y_1^2} - 8\pi^2y_3^2 - 16\pi^2y_2^2 - 8\pi^2y_1^2\Big)g = d_1 g\end{aligned}$$

and

$$
\begin{aligned}
(1.2)\qquad & -\frac{d}{256}\Big((2(16d\pi^2+1)\frac{\partial^2}{\partial y_2^2}-32(8d\pi^2-1)\pi^2 \\
&+d\Big(8\frac{\partial^4}{\partial y_1\partial y_2^2\partial y_3}-16\frac{\partial^4}{\partial y_1^2\partial y_3^2}+64\pi^2\frac{\partial^2}{\partial y_1^2}-\frac{\partial^4}{\partial y_2^4}+64\pi^2\frac{\partial^2}{\partial y_3^2}\Big) \\
&+16y_2\frac{\partial^3}{\partial y_1\partial y_2\partial y_3}-4y_1\frac{\partial^3}{\partial y_1\partial y_2^2}+16y_3\frac{\partial^3}{\partial y_1\partial y_3^2}-8\frac{\partial^2}{\partial y_1\partial y_3} \\
&+16y_1\frac{\partial^3}{\partial y_1^2\partial y_3}-64\pi^2y_3\frac{\partial}{\partial y_1}-4y_2\frac{\partial^3}{\partial y_2^3}-4y_3\frac{\partial^3}{\partial y_2^2\partial y_3} \\
&+64\pi^2y_2\frac{\partial}{\partial y_2}-64\pi^2y_1\frac{\partial}{\partial y_1}\Big)g=d_2g
\end{aligned}
$$

with $d=y_1y_3-y_2^2$. Put

$$
(1.3)\qquad Y=\begin{pmatrix} y_1 & y_2\\ y_2 & y_3\end{pmatrix}=\begin{pmatrix} t_1 & 0\\ 0 & t_2\end{pmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix},\qquad g(Y)=h(t_1,t_2,\theta)\,.
$$

Then h is periodic with respect to θ, so that h has a Fourier expansion

$$
(1.4)\qquad h(t_1,t_2,\theta)=\sum_{n\in\boldsymbol{Z}}B_n(t_1,t_2)e^{2ni\theta}
$$

where $B_n(t_1,t_2)$ is a solution of differential equations

$$
\begin{aligned}
(1.5)\qquad & -4^{-1}(t_1-t_2)^{-2}\Big(4(t_1^2+t_2^2)(t_1-t_2)^2\pi^2+2n^2t_1t_2 \\
&-(t_1-t_2)^2\Big(t_1^2\frac{\partial^2}{\partial t_1^2}+t_2^2\frac{\partial^2}{\partial t_2^2}\Big)-t_1t_2(t_1-t_2)\Big(\frac{\partial}{\partial t_1}-\frac{\partial}{\partial t_2}\Big)\Big)B_n=d_1B_n
\end{aligned}
$$

and

$$
\begin{aligned}
(1.6)\qquad & \frac{t_1t_2}{64(t_1-t_2)^4}\Big(2(16(t_1-t_2)^2\pi^2t_1t_2+5t_1^2-18t_1t_2+5t_2^2)n^2 \\
&+8(8\pi^2t_1t_2-1)(t_1-t_2)^4\pi^2+4n^4t_1t_2 \\
&-(t_1-t_2)(4(2t_1-t_2)n^2t_2-16(t_1-t_2)^2\pi^2t_1^2-3t_1^2+8t_1t_2-3t_2^2)\frac{\partial}{\partial t_2} \\
&-(t_1-t_2)^2t_2(16(t_1-t_2)^2\pi^2t_1+3t_1-2t_2)\frac{\partial^2}{\partial t_2^2} \\
&-(t_1-t_2)(16(t_1-t_2)^2\pi^2t_2^2+4(t_1-2t_2)n^2t_1+3t_1^2-8t_1t_2+3t_2^2)\frac{\partial}{\partial t_1} \\
&-t_1(t_1-t_2)^2(16(t_1-t_2)^2\pi^2t_2^2-2t_1+3t_2)\frac{\partial^2}{\partial t_1^2} \\
&+4t_2^2(t_1-t_2)^3\frac{\partial^3}{\partial t_1\partial t_2^2}-4t_1^2(t_1-t_2)^3\frac{\partial^3}{\partial t_1^2\partial t_2} \\
&+2(4n^2+1)t_1t_2(t_1-t_2)^2\frac{\partial^2}{\partial t_1\partial t_2}+4t_1t_2(t_1-t_2)^4\frac{\partial^4}{\partial t_1^2\partial t_2^2}\Big)B_n=d_2B_n\,.
\end{aligned}
$$

Since the differential equations do not change when $n \to -n$, we assume $n \geq 0$ hereafter. Now we introduce variables $x = t_1 - t_2$, $y = t_1 + t_2$ and put

$$B_n(t_1, t_2) = C_n(x, y) \tag{1.7}$$

where $y > 0$, $-y < x < y$. Then the differential equations for B_n yield the following condition for C_n:

$$\begin{aligned} -\frac{1}{8x^2}\Big(&4\pi^2x^2(x^2+y^2) + (y^2-x^2)n^2 - x^2(x^2+y^2)\frac{\partial^2}{\partial y^2} \\ &- x^2(x^2+y^2)\frac{\partial^2}{\partial x^2} - x(y^2-x^2)\frac{\partial}{\partial x} - 4x^3y\frac{\partial^2}{\partial x\partial y}\Big)C_n = d_1C_n\,, \end{aligned} \tag{1.8.1}$$

$$\begin{aligned} \frac{(y^2-x^2)}{256x^4}\Big(&(n^4+16\pi^4x^4)(y^2-x^2) + 2(4\pi^2x^2y^2 - 4\pi^2x^4 - 2y^2 + 7x^2)n^2 \\ &- 8\pi^2x^4 + 4x^3y\Big(x\frac{\partial^3}{\partial y\partial x^2} + \frac{\partial^2}{\partial x\partial y} - x\frac{\partial^3}{\partial y^3}\Big) \\ &+ x^4(y^2-x^2)\Big(\frac{\partial^4}{\partial y^4} - 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial x^4}\Big) \\ &- 2x^3(y^2+x^2)\Big(\frac{\partial^3}{\partial x\partial y^2} - \frac{\partial^3}{\partial x^3}\Big) \\ &- x^2(2n^2y^2 - 2n^2x^2 + 8\pi^2y^2x^2 - 8\pi^2x^4 + y^2 - 3x^2)\frac{\partial^2}{\partial x^2} \\ &+ x(2n^2y^2 - 6n^2x^2 - 8\pi^2x^2y^2 - 8\pi^2x^4 + y^2 - 7x^2)\frac{\partial}{\partial x} \\ &+ 2x^2(n^2y^2 - n^2x^2 - 4\pi^2x^2y^2 + 4\pi^2x^4 + x^2)\frac{\partial^2}{\partial y^2} \\ &- 4x^2y(n^2 + 4\pi^2x^2)\frac{\partial}{\partial y}\Big)C_n = d_2C_n\,. \end{aligned} \tag{1.8.2}$$

Put $C_n(x, y) = (x^2 - y^2)\sum_{k=0}^{\infty} a_k(y)x^k$ where $a_k(y) = 0$ if $k \leq m$ with an integer $m \geq 0$ such that $a_m(y) \neq 0$. Then the equation (1.8.1) implies

$$\begin{aligned} \frac{1}{8}(&(k^2 + 2k + n^2 - 4\pi^2y^2 - 2)a_k + ((k+2)^2 - n^2)y^2a_{k+2} - 4\pi^2a_{k-2} \\ &+ a''_{k-2} + y^2a''_k + (4k+4)ya'_k) = d_1a_k\,, \end{aligned} \tag{1.9.1}$$

especially $(m^2 - n^2)y^2a_m = 0$, $((m+1)^2 - n^2)y^2a_{m+1} = 0$ which results in $m = n$, $a_{m+1} = 0$ and therefore $a_k = 0$ unless $k \equiv m \bmod 2$. The equation (1.8.2) implies

$$
\begin{aligned}
(1.9.2) \qquad & -\frac{1}{256}(2(k+n+2)(k-n+2)y^3(ya''_{k+2}+2a'_{k+2}) \\
& -(k^4-2k^2n^2+16k^2\pi^2y^2-6k^2+16k\pi^2y^2 \\
& -16k+n^4-16n^2\pi^2y^2-10n^2+16\pi^4y^4+40\pi^2y^2+12)a_k \\
& -2(2k^2+2k-2n^2-4\pi^2y^2-1)y^2a''_k \\
& +2(k^2+2k-n^2+4\pi^2y^2+7)(k+n+2)(k-n+2)y^2a_{k+2} \\
& +8(k^2-2k-n^2+4\pi^2y^2-3)\pi^2a_{k-2} \\
& +2(k^2-2k-n^2-8\pi^2y^2+7)a''_{k-2} \\
& -4(k^2-8k-n^2-4\pi^2y^2-8)ya'_k \\
& -(k+n+4)(k+n+2)(k-n+4)(k-n+2)y^4a_{k+4} \\
& -a^{(4)}_{k-4}+8\pi^2a''_{k-4}+2y^2a^{(4)}_{k-2}+4ya^{(3)}_{k-2}-16\pi^2ya'_{k-2} \\
& -y^4a^{(4)}_k-4y^3a^{(3)}_k-16\pi^4a_{k-4})=d_2a_k
\end{aligned}
$$

for all integers k. Since we especially have

$$
\begin{aligned}
(1.10) \qquad a_{n+2} &= \frac{1}{4(n+1)y^2}(2(4d_1-n^2-n+2\pi^2y^2+1)a_n \\
&\qquad\qquad -4(n+1)ya'_n-y^2a''_n), \\
a_{n+4} &= \frac{1}{8(n+2)y^2}(2(4d_1-n^2-3n+2\pi^2y^2-3)a_{n+2} \\
&\qquad\qquad -4(n+3)ya'_{n+2}-y^2a''_{n+2}-a''_n+4\pi^2a_n)
\end{aligned}
$$

and

$$
\begin{aligned}
(1.11) \qquad & 4(64d_2+4n^2-4n\pi^2y^2+4n-4\pi^4y^4-10\pi^2y^2-3)a_n \\
& +8y^2(2n+4\pi^2y^2+7)(n+1)a_{n+2}+16(2n+\pi^2y^2+2)ya_n \\
& -2(2n-4\pi^2y^2-1)y^2a''_n-32(n+2)(n+1)a_{n+4}y^4 \\
& +8y^3(n+1)(ya''_{n+2}+2a'_{n+2})-y^4a^{(4)}_n-4y^3a^{(3)}_n=0,
\end{aligned}
$$

we obtain from (1.9.1), (1.9.2) an ordinary differential equation

$$
\begin{aligned}
(1.12) \qquad & (8(n^2+n+1)d_1-16d_1^2+64d_2-n^4-2n^3+n^2+2n+8\pi^2y^2)a_n \\
& +4(4(n+1)d_1-n^3-3n^2-2n+4\pi^2y^2)ya'_n \\
& +2(4d_1-3n^2-9n+2\pi^2y^2-6)y^2a''_n \\
& -4(n+2)y^3a^{(3)}_n-y^4a^{(4)}_n=0,
\end{aligned}
$$

for a_n. It is more convenient to introduce parameters λ_1, λ_2 defined by

$$
(1.13) \qquad d_1=\frac{\lambda_1+\lambda_2-2}{8}, \qquad d_2=\frac{(\lambda_1-\lambda_2)^2}{256}-\frac{\lambda_1+\lambda_2}{32}+\frac{3}{64}
$$

to describe the solutions of (1.12). With these λ_1, λ_2 (1.12) becomes

$$
\begin{aligned}
(1.14)\qquad & -(\lambda_1\lambda_2 - n(n+1)\lambda_1 - n(n+1)\lambda_2 + n^4 + 2n^3 + n^2 - 8\pi^2y^2)a_n \\
& + 2((\lambda_1+\lambda_2)(n+1) - 2n^3 - 6n^2 - 6n + 8\pi^2y^2 - 2)ya_n' \\
& + (\lambda_1 + \lambda_2 - 6n^2 - 18n + 4\pi^2y^2 - 14)y^2a_n'' \\
& - 4(n+2)y^3a_n^{(3)} - y^4a_n^{(4)} = 0\,.
\end{aligned}
$$

For $\nu \in \mathbb{C}$, $m \in \mathbb{Z}$, denote the associated Legendre function of the first kind by $P_\nu^m(z)$ and that of the second kind by $Q_\nu^m(z)$ as usual. Then we have

$$
\begin{aligned}
(1.15)\qquad P_\nu^m(z) &= \frac{\Gamma(\nu+m+1)}{\pi\Gamma(\nu+1)}\int_0^\pi (z + (z^2-1)^{1/2}\cos t)^\nu \cos mt dt\,, \\
Q_\nu^m(z) &= (-1)^m \frac{\Gamma(\nu+m+1)}{2^{\nu+1}\Gamma(\nu+1)}(z^2-1)^{m/2}\int_{-1}^1 \frac{(1-t^2)^\nu}{(z-t)^{\nu+m+1}}dt
\end{aligned}
$$

for z not on the real axis between 1 and $-\infty$, assuming $w^\mu = e^{\mu \log w}$, $\log w = \log|w| + i\arg w$, $-\pi < \arg w < \pi$ for w, $\mu \in \mathbb{C}$. We put

$$
\begin{aligned}
(1.16)\qquad P_\nu^m(x) &= \lim_{y\to+0} e^{3m\pi i/2}P_\nu^m(x+iy) = \lim_{y\to+0} e^{m\pi i/2}P_\nu^m(x-iy)\,, \\
Q_\nu^m(x) &= \frac{1}{2}\lim_{y\to+0}(e^{-m\pi i/2}Q_\nu^m(x+iy) - e^{m\pi i/2}Q_\nu^m(x-iy))
\end{aligned}
$$

for $-1 < x < 1$. $P_\nu^m(z)$ and $Q_\nu^m(z)$ are independent solutions of Legendre's differential equation

$$
(1.17)\qquad \frac{d}{dz}\left((1-z^2)\frac{d}{dz}\right)u + \nu(\nu+1)u - \frac{m^2}{1-z^2}u = 0\,.
$$

Put

$$
\begin{aligned}
(1.18)\qquad c_{11} &= -Q_{\nu_2}^0(0) = \sqrt{\pi}\,2^{-1}\sin\left(\frac{\nu_2}{2}\pi\right)\Gamma\left(\frac{\nu_2+1}{2}\right)\Big/\Gamma\left(\frac{\nu_2}{2}+1\right), \\
c_{12} &= P_{\nu_2}^0(0) = \sqrt{\pi}\Big/\left(\Gamma\left(\frac{\nu_2}{2}+1\right)\Gamma\left(\frac{-\nu_2+1}{2}\right)\right), \\
c_{21} &= -\frac{d}{dz}Q_{\nu_2}^0(0) = -\sqrt{\pi}\cos\left(\frac{\nu_2}{2}\pi\right)\Gamma\left(\frac{\nu_2}{2}+1\right)\Big/\Gamma\left(\frac{\nu_2+1}{2}\right), \\
c_{22} &= \frac{d}{dz}P_{\nu_2}^0(0) = 2\sin\left(\frac{\nu_2}{2}\pi\right)\Gamma\left(\frac{\nu_2}{2}+1\right)\Big/\left(\sqrt{\pi}\,\Gamma\left(\frac{\nu_2+1}{2}\right)\right).
\end{aligned}
$$

Then $c_{12} \neq 0$, $c_{22} \neq 0$ for $-1 < \Re\nu_2 < 0$. Put

$$
\begin{aligned}
(1.19)\qquad R_{\nu_2}^0(z) &= c_{11}P_{\nu_2}^0(z) + c_{12}Q_{\nu_2}^0(z)\,, \\
S_{\nu_2}^0(z) &= c_{21}P_{\nu_2}^0(z) + c_{22}Q_{\nu_2}^0(z)\,.
\end{aligned}
$$

Then we obtain

PROPOSITION 1. *Put* $\nu_1 = \dfrac{-1+\sqrt{1+4\lambda_1}}{2}$, $\nu_2 = \dfrac{-1+\sqrt{1+4\lambda_2}}{2}$, *and assume that* $-1 < \Re\,\nu_1 < 0$, $-1 < \Re\,\nu_2 < 0$ *and that* λ_1, λ_2 *are not integers. Then there exist polynomials* h_1, h_2 *in* y^{-1} *of degree* $n-1$, n *such that*

$$A_n(y) = \int_0^\infty \left(\int_1^\infty R^0_{\nu_1}(z_1)P^n_{\nu_2}(z_2)(-2\pi y z_2)^{-n}(z_2^2-1)^{n/2}e^{-2\pi z_1 z_2 y}dz_2\right)dz_1 - h_1\,,$$

$$B_n(y) = \int_0^\infty \left(\int_1^\infty S^0_{\nu_1}(z_1)P^n_{\nu_2}(z_2)(-2\pi y z_2)^{-n}(z_2^2-1)^{n/2}e^{-2\pi z_1 z_2 y}dz_2\right)dz_1 - h_2\,,$$

$$C_n(y) = \int_1^\infty \int_1^\infty P^n_{\nu_1}(z_1)P^n_{\nu_2}(z_2)(z_1^2-1)^{n/2}(z_2^2-1)^{n/2}e^{-2\pi z_1 z_2 y}dz_1 dz_2$$

are linearly independent solutions of the equation (1.14). *Further assume that* λ_1, λ_2 *are real. Then there exists a polynomial* h_3 *in* y^{-1} *of degree* n *such that together with above three functions*

$$\begin{aligned} D_n(y) = &\int_0^1 \left(\int_{-1}^1 P^n_{\nu_1}(z_1)P^n_{\nu_2}(-z_2)(z_1^2-1)^{n/2}(z_2^2-1)^{n/2}e^{-2\pi z_1 z_2 y}dz_2\right)dz_1 \\ &+ \sqrt{-1}^{\,-n}\frac{\Gamma(\nu_1+n+1)}{\Gamma(\nu_1-n+1)}\int_0^1 \left(\int_1^\infty P^0_{\nu_1}(z_1)P^n_{\nu_2}(-z_2)(-2\pi y z_2)^{-n}\right. \\ &\qquad \left.\times (z_2^2-1)^{n/2}e^{-2\pi z_1 z_2 y}dz_2\right)dz_1 + h_3 \end{aligned}$$

generates all the solutions of the equation (1.14).

Note that polynomials h_1, h_2, h_3 can be given explicitly.

Proof. Put

$$\begin{aligned} (1.20)\qquad L = &-(\lambda_1\lambda_2 - n(n+1)\lambda_1 - n(n+1)\lambda_2 + n^4 + 2n^3 + n^2 - 8\pi^2 y^2) \\ &+ 2((\lambda_1+\lambda_2)(n+1) - 2n^3 - 6n^2 - 6n + 8\pi^2 y^2 - 2)y\frac{\partial}{\partial y} \\ &+ (\lambda_1 + \lambda_2 - 6n^2 - 18n + 4\pi^2 y^2 - 14)y^2\frac{\partial^2}{\partial y^2} \\ &- 4(n+2)y^3\frac{\partial^3}{\partial y^3} - y^4\frac{\partial^2}{\partial y^2}\,, \end{aligned}$$

$$\begin{aligned} (1.21)\qquad L_1^m &= \frac{d}{dz_1}\left((z_1^2-1)\frac{d}{dz_1}\right) - \frac{m^2}{z_1^2-1}\,, \\ L_2^m &= \frac{d}{dz_2}\left((z_2^2-1)\frac{d}{dz_2}\right) - \frac{m^2}{z_2^2-1}\,, \end{aligned}$$

$$\begin{aligned} (1.22)\qquad L' = &-(L_1^0 L_2^n - n(n+1)L_1^0 - n(n+1)L_2^n + n^4 + 2n^3 + n^2 - 8\pi^2 y^2) \\ &+ 2((L_1^0 + L_2^n)(n+1) - 2n^3 - 6n^2 - 6n + 8\pi^2 y^2 - 2)y\frac{\partial}{\partial y} \end{aligned}$$

$$+ (L_1^0 + L_2^n - 6n^2 - 18n + 4\pi^2 y^2 - 14) y^2 \frac{\partial^2}{\partial y^2}$$
$$- 4(n+2) y^3 \frac{\partial^3}{\partial y^3} - y^4 \frac{\partial^4}{\partial y^4} .$$

Then we have

$$\begin{aligned}
(1.23) \quad & L \int_0^\infty \left(\int_1^\infty R_{\nu_1}^0(z_1) P_{\nu_2}^n(z_2)(-2\pi y z_2)^{-n}(z_2^2 - 1)^{n/2} e^{-2\pi z_1 z_2 y} dz_2 \right) dz_1 \\
&= \int_0^\infty \int_1^\infty \Big(-L_1^0 R_{\nu_1}^0(z_1) L_2^n P_{\nu_2}^n(z_2) + n(n+1)(L_1^0 R_{\nu_1}^0(z_1) P_{\nu_2}^n(z_2) \\
&\quad + R_{\nu_1}^0(z_1) L_2^n P_{\nu_2}^n(z_2)) - (n^4 + 2n^3 + n^2 - 8\pi^2 y^2) R_{\nu_1}^0(z_1) P_{\nu_2}^n(z_2) \\
&\quad + 2((n+1)(L_1^0 R_{\nu_1}^0(z_1) P_{\nu_2}^n(z_2) + R_{\nu_1}^0(z_1) L_2^n P_{\nu_2}^n(z_2)) \\
&\quad + R_{\nu_1}^0(z_1) P_{\nu_2}^n(z_2)(-2n^3 - 6n^2 - 6n + 8\pi^2 y^2 - 2)) y \frac{\partial}{\partial y} \\
&\quad + (L_1^0 R_{\nu_1}^0(z_1) P_{\nu_2}^n(z_2) + R_{\nu_1}^0(z_1) L_2^n P_{\nu_2}^n(z_2) \\
&\quad + R_{\nu_1}^0(z_1) P_{\nu_2}^n(z_2)(-6n^2 - 18n + 4\pi^2 y^2 - 14)) y^2 \frac{\partial^2}{\partial y^2} - R_{\nu_1}^0(z_1) P_{\nu_2}^n(z_2) \\
&\quad \times \left(4(n+2) y^3 \frac{\partial^3}{\partial y^3} + y^4 \frac{\partial^4}{\partial y^4} \right) \Big) (-2\pi y z_2)^{-n}(z_2^2 - 1)^{n/2} e^{-2\pi z_1 z_2 y} dz_1 dz_2 \\
&= \int_0^\infty \int_1^\infty R_{\nu_1}^0(z_1) P_{\nu_2}^n(z_2) L'((-2\pi y z_2)^{-n}(z_2^2 - 1)^{n/2} e^{-2\pi z_1 z_2 y}) dz_1 dz_2 \\
&\quad + n(n+1) \frac{\partial}{\partial z_1} R_{\nu_1}^0(0) \int_1^\infty P_{\nu_2}^n(z_2)(-2\pi y z_2)^{-n}(z_2^2 - 1)^{n/2} z_2^{-2} dz_2
\end{aligned}$$

by partial integration. We note that all integrals should be considered as limits of integrals of finite interval. Since we easily see that

$$\begin{aligned}
(1.24) \quad & L'((-2\pi y z_2)^{-n}(z_2^2 - 1)^{n/2} e^{-2\pi z_1 z_2 y}) = 0 , \\
& L y^{-m} = \frac{1}{64}(\lambda_1 - (n-m)^2 - (n-m))(\lambda_2 - (n-m)^2 - (n-m)) y^{-m} \\
& \qquad - \frac{1}{16}\pi^2 (n-1)(n-2) y^{-m+2} ,
\end{aligned}$$

there exists a polynomial h_1 in y^{-1} of degree n such that A_n is a solution of (1, 14). In the same way we can show existence of h_2, h_3, h_4 such that B_n, C_n, D_n are solutions of (1.14). It is easy to see that yA_n, yB_n are bounded when $y \to \infty$ and that C_n and D_n are rapidly decreasing and increasing respectively when $y \to \infty$. Since we easily see that A_n, B_n have different asymptotic expansions by considering

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$$(1.25)\qquad \int_0^\varepsilon\left(\int_1^\infty R_{\nu_1}^0(z_1)P_{\nu_2}^n(z_2)(-2\pi y z_2)^{-n}(z_2^2-1)^{n/2}e^{-2\pi z_1 z_2}dz_2\right)dz_1\,,$$
$$\int_0^\varepsilon\left(\int_1^\infty S_{\nu_1}^0(z_1)P_{\nu_2}^n(z_2)(-2\pi y z_2)^{-n}(z_2^2-1)^{n/2}e^{-2\pi z_1 z_2}dz_2\right)dz_1$$

for small ε, functions A_n, B_n, C_n, D_n are linearly independent.

By the recurrence relation (1.9.1) and the partial integration using (1.17), we especially obtain

PROPOSITION 2. *The functions of x, y*

$$C_n(x, y) = (x^2 - y^2)\int_1^\infty\int_1^\infty P_{\nu_1}^n(z_1)P_{\nu_2}^n(z_2)$$
$$\times J_n(2\pi i(z_1^2-1)^{1/2}(z_2^2-1)^{1/2}x)e^{-2\pi z_1 z_2 y}dz_1dz_2$$

are solutions of the equations (1.8.1), (1.8.2) *where J_n denotes the Bessel function of the first kind.*

THEOREM 1. *Put* $\nu_1 = \dfrac{-1+\sqrt{1+4\lambda_1}}{2}$, $\nu_2 = \dfrac{-1+\sqrt{1+4\lambda_2}}{2}$, *and assume that* $-1 < \Re\mathrm{e}\,\nu_1 < 0$, $-1 < \Re\mathrm{e}\,\nu_2 < 0$ *and that* $\lambda_1, \lambda_2 \in \mathbb{R}$ *are not integers.*

Let $f(X+iY) = g(Y)e^{2\pi i\,\mathrm{tr}\,X}$ be a generalized Whittaker functions i.e., $f(X+iY)$ satisfies $\Delta_1 f = d_1 f$, $\Delta_2 f = d_2 f$ with d_1, d_2 in (1.13). *Assume that $g(Y)$ is a real analytic function of $t_1 - t_2$ with*

$$Y = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

and that for all positive integers m, a, b, c, the convergence

$$(\mathrm{tr}\ Y)^m \frac{\partial^{a+b+c}}{\partial y_1^a \partial_2^b \partial y_3^c} g(Y) \to 0$$

holds uniformly on compact sets of $y_1 - y_3$ and y_2 when $\mathrm{tr}\ Y \to 0$.

Then $g(Y)$ is expanded as follows:

$$g(Y) = \sum_{n\in Z} b_n C_n(t_1 + t_2, t_1 - t_2)e^{2ni\theta}$$

where $C_n(x, y)$ is the function defined in Proposition 2 and $b_n \in \mathbb{C}$.

Proof. Put $x = t_1 - t_2$, $y = t_1 + t_2$. Then

$$(1.26)\qquad \frac{\partial}{\partial x^n}\bigg|_{x=0}\int_0^\pi g(Y)e^{-2in\theta}d\theta = \int_0^\pi\left(\frac{\cos 2\theta}{2}\left(\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_3}\right)\right.$$
$$\left. - \frac{\sin 2\theta}{2}\frac{\partial}{\partial y_2}\right)^n\bigg|_{y_1 - y_3 = 0,\, y_2 = 0} g(Y)e^{-2in\theta}d\theta$$

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holds. Since the left hand side of (1.26) is a solution of (1.14), it should be one of the functions in Proposition 1. The right hand side of (1.26) decreases rapidly by the assumption and therefore it equals to $C_n(y)$ in Proposition 1.

§ 2.

Here we introduce a certain theta function and show the commutation relations of invariant differential operators and explain their connection with the generalized Whittaker functions. We deal with L-function of biquadratic fields with Grössencharacter at the end.

For $(g_2, g_2) \in SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$, put

$$\rho(g_1, g_2) = \left(\begin{array}{cc|cc} 0 & 1 & & \\ -1 & 0 & & \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array}\right) \left(\begin{array}{c|c} g_1^{-1} & \\ \hline & g_1^{-1} \end{array}\right) \left(\begin{array}{c|c} a_2 E & c_2 E \\ \hline b_2 E & d_2 E \end{array}\right) \left(\begin{array}{cc|cc} 0 & -1 & & \\ 1 & 0 & & \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array}\right) \tag{2.1}$$

with $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$. Then $\rho(g_1, g_2)$ is an element in the orthogonal group for $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$. For an odd squarefree integer N, we define a lattice $\mathscr{L}$ by

$$\mathscr{L} = \left\{ \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \\ m_{41} & m_{42} \end{pmatrix} \middle| \, m_{ij} \in \mathbb{Z} \text{ for } (i, j) \neq (1, 2),\ N m_{12} \in \mathbb{Z} \right\}. \tag{2.2}$$

For $Z = X + iY \in H_2$ and a Dirichlet character χ modulo N, put

$$\begin{aligned} \Theta_\chi(Z, (g_1, g_2)) = |Y| \sum_{m = \begin{pmatrix} * & m_{12} \\ * & * \\ * & * \\ * & * \end{pmatrix} \in \mathscr{L}} & \chi(N m_{12}) \\ & \times \exp\left(\pi i N \operatorname{tr}\left(X\left(\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}[\rho(g_1, g_2) m]\right)\right)\right) \\ & \times \exp\left(-\pi N \operatorname{tr}\left(Y\left(\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}[\rho(g_1, g_2) m]\right)\right)\right) \end{aligned} \tag{2.3}$$

where $S[T]$ means ${}^t TST$. The function in (2.3) is usually called theta function. Put $g_{x+iy} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}$ for $x + iy \in H_1$, the upper half plane, and put

$$\theta(Z, z_1, z_2) = \Theta_\chi(Z, (g_{z_1}, g_{z_2})). \tag{2.4}$$

Then using Nakajima [10] where the generators of the center $\mathscr{C}$ of the universal enveloping algebra of $\mathfrak{sp}(2, \mathbb{R})$ are given, we have

THEOREM 2.[1] *For* $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2 \in H_1$,

$$\Delta_1\theta(Z, z_1, z_2) = \frac{1}{8}\Big(y_1^2\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}\Big) + y_2^2\Big(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2^2}\Big) - 2\Big)\theta(Z, z_1, z_2),$$

$$\Delta_2\theta(Z, z_1, z_2) = \Big(\frac{1}{256}\Big(y_1^2\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}\Big) - y_2^2\Big(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2^2}\Big)\Big)^2$$
$$- \frac{1}{32}\Big(y_1^2\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}\Big) + y_2^2\Big(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2^2}\Big)\Big) + \frac{3}{64}\Big)\theta(Z, z_1, z_2).$$

Note that Δ_1, Δ_2 defined in § 1 are differential operators with respect to Z not to z_1, z_2.

Proof. Put $V = M_{2,2}(\mathbb{R})$ and define the Weil representation r of $G = Sp(2, \mathbb{R})$ on $\mathscr{S}(V \times V)$ by

$$r\Big(\begin{pmatrix} E & X \\ 0 & E \end{pmatrix}\Big)f\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \exp(2\pi i \operatorname{tr}(X^t X_1 X_2)) f\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \tag{2.5}$$

$$r\Big(\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}\Big)f\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \det(A) f\begin{pmatrix} X_1A \\ X_2A \end{pmatrix},$$

$$r\Big(\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}\Big)f\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \int_V\int_V f\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \exp(2\pi i \operatorname{tr}({}^tY_1X_2 + {}^tY_2X_1))dY_1dY_2$$

for $f \in \mathscr{S}(V \times V)$ where $X_1, X_2 \in V$ and $d\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = dx_{11}dx_{12}dx_{21}dx_{22}$. The representation r of G derives the representation $\tilde{r}$ of $\mathscr{C}$. Let Δ_1, Δ_2 be generators of $\mathscr{C}$ in [10]. Then we can express $\tilde{r}(\Delta_1)$, $\tilde{r}(\Delta_2)$ by differential operators $\dfrac{\partial}{\partial x_{11}}, \cdots, \dfrac{\partial}{\partial x_{42}}$ for

$$f\begin{pmatrix} x_{12} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \\ x_{41} & x_{42} \end{pmatrix} \in \mathscr{S}(V \times V).$$

$\rho(g_1, g_2)$ acts on $\mathscr{S}(V \times V)$ by

$$(\rho(g_1, g_2)f)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = f\Big(\rho(g_1, g_2)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}\Big).$$

ρ derives the representation $\tilde{\rho}$ of the direct product of two copies of the center of the universal enveloping algebra c of $\mathfrak{sl}(2, \mathbb{R})$ which is generated by the Casimir operator λ. $\tilde{\rho}(\lambda, 1)$, $\tilde{\rho}(1, \lambda)$ also can be expressed by $\dfrac{\partial}{\partial x_{11}}, \cdots, \dfrac{\partial}{\partial x_{42}}$.

[1] A computer machine is used to prove this theorem.

Comparing these expressions for $\tilde{r}(\Lambda_1)$, $\tilde{r}(\Lambda_2)$, $\tilde{\rho}(\lambda, 1)$, $\tilde{\rho}(1, \lambda)$ by a computer machine, we obtain desired results.

Let φ_1, φ_2 be Maass wave cusp forms with the character χ satisfying $y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi_1(z) = \lambda_1\varphi_1(z)$, $y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi_2(z) = \lambda_2\varphi_2(z)$. Put $d_0 z = y^{-2}dxdy$ for $z = x + iy$, and define

$$F_{\varphi_1,\varphi_2}(Z) = \int_{\Gamma\backslash H_1}\int_{\Gamma\backslash H_1} \theta(Z, z_1, z_2)\varphi_1(z_1)\varphi_2(z_2)d_0z_1d_0z_2 \tag{2.6}$$

with $\Gamma = \Gamma_0(N)$. Then we have

$$\Delta_1 F_{\varphi_1,\varphi_2}(Z) = d_1 F_{\varphi_1,\varphi_2}(Z), \qquad \Delta_2 F_{\varphi_1,\varphi_2}(Z) = d_2 F_{\varphi_1,\varphi_2}(Z), \tag{2.7}$$

with d_1, d_2 defined by the equalities (1.13) for λ_1, λ_2, and

$$F_{\varphi_1,\varphi_2}(\sigma Z) = \chi(d) F_{\varphi_1,\varphi_2}(Z) \tag{2.8}$$

holds for $\sigma = \left(\begin{array}{cc|cc} * & * & * & * \\ * & * & * & * \\ \hline * & * & a & b \\ * & * & c & d \end{array}\right) \in \Gamma_2$ where

$$\Gamma_2 = \left\{ \sigma = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \middle| \begin{array}{l} \sigma \in Sp(2, \mathbb{Q}), \\ a_{21},\ a_{31},\ a_{32},\ a_{41},\ a_{42} \in N\mathbb{Z}, \\ Na_{13} \in \mathbb{Z},\ \text{other } a_{ij} \in \mathbb{Z} \end{array} \right\}.$$

There exists a lattice $\mathcal{T}$ in $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c \in \mathbb{Q}\right\}$ containing $\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$ such that $F_{\varphi_1,\varphi_2}(Z)$ is expanded as follows:

$$F_{\varphi_1,\varphi_2}(Z) = \sum_{T\in\mathcal{T}} A(T, Y)e^{2\pi i\, \mathrm{tr}\,(TX)}. \tag{2.9}$$

Hence

$$W(Y) = A\left(\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \sqrt{N}^{-1} & 0 \\ 0 & 1 \end{pmatrix} Y \begin{pmatrix} \sqrt{N}^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)$$

is a generalized Whittaker function. A direct calculation shows that $W\left(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right)$ is equal to $\sum_{n\in Z} b_n C_n(t_1 - t_2, t_1 + t_2)e^{ni\theta}$ and b_0 is equal to

$$\sum_{j=1}^{h} \chi(a_j)\varphi_1\left(\frac{-a_j}{c_jN + \sqrt{-N}}\right)\varphi_2\left(\frac{-a_j}{c_jN + \sqrt{-N}}\right) \tag{2.10}$$

up to a constant multiple, where $\mathfrak{U}_j = a_j\mathbb{Z} + (c_jN + \sqrt{-N})\mathbb{Z}$ runs over the full set of representatives of the ideal class group of $\mathbb{Q}(\sqrt{-N})$ with class number h and $C_n(t_1, t_2)$ is same as defined in Proposition 2.

Note that the corresponding L-function (the spinor L-function) of $F_{\varphi_1,\varphi_2}(Z)$ is the product $L(s, \varphi_1)L(s, \varphi_2)$ of L-functions of φ_1 and φ_2. Let $K_i = \mathbb{Q}(\sqrt{d_i})$ be a real quadratic field with discriminant d_i for $i = 1, 2, 3$. Let $d_3 = d_1d_2$ and assume the class number of $K = K_3$ is one. Let $\mathfrak{o}$ be the ring of integers in K and E_+ the group of all totally positive units in $\mathfrak{o}$. Put

$$(2.11) \qquad g_1(z, \xi_m) = \sum_{\substack{\mu \in \mathfrak{o}/E_+ \\ \mu \neq 0}} \xi_m(\mu) y^{1/2} K_{im\kappa}(2\pi |N_{K/\mathbb{Q}}\mu| y) \cos(2\pi x N_{K/\mathbb{Q}}\mu),$$

$$g_2(z, \xi_m) = \sum_{\substack{\mu \in \mathfrak{o}/E_+ \\ \mu \neq 0}} \xi_m(\mu) \psi(\mu) y^{1/2} K_{im\kappa}(2\pi |N_{K/\mathbb{Q}}\mu| y) \cos(2\pi x N_{K/\mathbb{Q}}\mu)$$

where $\xi_m(\mu) = |\mu/\mu'|^{im\kappa}$, $\kappa = 2\pi/\log \varepsilon$, $\psi(\mu) = \left(\frac{d_1}{N_{K/\mathbb{Q}}\mu}\right)$, ε the fundamental unit in K, $K_{im\kappa}$ the modified Bessel function. Then g_1 is a Maass wave form on $\Gamma_0(d_1d_2)$ with character $\left(\frac{d_1d_2}{*}\right)$ and g_2 a Maass wave form on $\Gamma_0(d_1^2d_2)$ with character $\left(\frac{d_1d_2}{*}\right)$. Though in this case $N = d_1^2d_2$ is not squarefree, we can define Θ, θ and F_{φ_1,φ_2} for $\varphi_1 = g_1$, $\varphi_2 = g_2$ in the same way as above. In this case L-function corresponding to F_{φ_1,φ_2} is

$$\sum_{\mathfrak{a}} \xi(N_{F/k}\mathfrak{a}) N_{F/\mathbb{Q}}\mathfrak{a}^{-s}$$

where $\mathfrak{a}$ runs over all integral ideals in $F = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. (2.10) may be a little more complicated in this case.

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