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Witnessing flows in arithmetic

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Abstract

One of the elegant achievements in the history of proof theory is the characterization of the provably total recursive functions of an arithmetical theory by its proof-theoretic ordinal as a way to measure the time complexity of the functions. Unfortunately, the machinery is not sufficiently fine-grained to be applicable on the weak theories, on the one hand and to capture the bounded functions with bounded definitions of strong theories, on the other. In this paper, we develop such a machinery to address the bounded theorems of both strong and weak theories of arithmetic. In the first part, we provide a refined version of ordinal analysis to capture the feasibly definable and bounded functions that are provably total in $PA + \bigcup_{\beta \prec \alpha} TI(\prec_\beta)$, the extension of Peano arithmetic by transfinite induction up to the ordinals below α. Roughly speaking, we identify the functions as the ones that are computable by a sequence of PV-provable polynomial time modifications on an initial polynomial time value, where the computational steps are indexed by the ordinals below α, decreasing by the modifications. In the second part, and choosing $l \leq k$, we use similar technique to capture the functions with bounded definitions in the theory T_2^k (resp. S_2^k) as the functions computable by exponentially (resp. polynomially) long sequence of PV_{k-l+1}-provable reductions between l-turn games starting with an explicit PV_{k-l+1}-provable winning strategy for the first game.

Keywords: Total search problems; ordinal analysis; bounded arithmetic; local search programs

1. Introduction

One of the elegant achievements in the history of proof theory is the *witnessing* techniques connecting the provability of a formula of a certain form to the existence of a computational entity (algorithm (Troelstra 1998), function (Fairtlough and Wainer 1998), term in a type theory (Avigad and Feferman 1998), etc.) that *witnesses* the truth of the formula. These connections identify the power of the theories, and they are useful to establish the unprovability of a formula by showing the nonexistence of the corresponding witness. As an example, consider the *ordinal analysis* as one of the well-known witnessing techniques that among many other things provides a characterization for the provably total recursive functions of some mathematical theories (Buss 1994; Fairtlough and Wainer 1998; Kreisel 1952). (For a comprehensive high-level explanation, see Rathjen 1999.) It connects the provability of the totality of a Σ_1^0-definable function to its time complexity, measured by the proof-theoretic ordinal of the theory. The characterization then leads to some independence results for the formulas in the form $A = \forall x \exists y B(x, y)$, where $B \in \Sigma_1^0$ is a definition of a function with a faster growth rate and hence higher time complexity than what the theory can actually reach (Fairtlough and Wainer 1998).

There are, however, some settings in which the witnessing techniques and especially the one based on ordinal analysis break down. Sometimes, we are only interested in the formulas with no

existential quantifiers to witness (e.g., $A = \forall x B(x)$, where B is a quantifier-free formula). Other times, the theory is so weak that even the basics of the witnessing machinery goes beyond the power of the theory. Even working with powerful theories, there can be some problematic situations. For instance, one may be interested in bounded formulas (e.g., $\forall x \exists y \leq t(x) B(x, y)$, where all quantifiers in B are also bounded) provable in Peano arithmetic, denoted by PA. Here, what the usual witnessing methods provide is rather weak or even useless. For instance, using ordinal analysis for PA, the best thing we can learn in the bounded setting is the existence of an algorithm to compute y using a huge amount of time measured by ε_0, the ordinal of the theory. This is much weaker than what we started with, that is, the provability of the totality of the function with a *bounded* definition. The reason roughly is that the algorithm leads to the existence of the definition $\exists w C(x, w, y)$ for the function, where w encodes the computation and PA proves $\forall x \exists y w C(x, w, y)$. However, the computation w can be huge and hence unbounded by the terms in the language and in this sense proving the totality of a *bounded* function with a *bounded* definition is stronger than the existence of such an algorithm.

To solve this type of issues and to address both weak theories and low complexity formulas, many new witnessing techniques were designed, from witnessing the universal provable formulas by short propositional proofs (Buss 1997; Cook 1975; Krajíček and Pudlák 1990; Paris and Wilkie 1981) to witnessing provable bounded formulas in first-order bounded theories of arithmetic in special cases (Buss 1986; Buss and Krajíček 1994; Krajíček et al. 2007) and then in general cases (Beckmann and Buss 2010; Skelley and Thapen 2011; Thapen 2011), using game reductions and different versions of local search problems. A similar technique is also developed for second-order bounded theories of arithmetic (Beckmann and Buss 2017, 2014; Buss et al. 1993; Kołodziejczyk et al. 2011) and even for Peano arithmetic (Beckmann 2009). In this paper, we will continue this line of research by providing a general witnessing machinery to witness the low-complexity theorems both in strong and weak theories of arithmetic using a computational entity that we call a flow. Flows are meant to formalize the idea of *flowing* information and they formally are uniform suitably long sequences of PV-provable implications between formulas in a suitable class, where PV is Cook's theory for polynomial time functions. We will work with two different types of flows in this paper, *ordinal flows* and *k-flows*.

Ordinal flows

An ordinal flow is a transfinite uniform sequence of PV-provable implications between universal formulas. We use ordinal flows to witness low-complexity theorems of the theory $\text{PA} + \bigcup_{\beta \prec \alpha} \text{TI}(\prec_\beta)$, where α is an ordinal with a certain polynomial time representation and $\text{TI}(\prec_\beta)$ means the transfinite induction up to the ordinal β. More precisely, we witness the provability of an implication between two universal formulas in $\text{PA} + \bigcup_{\beta \prec \alpha} \text{TI}(\prec_\beta)$ by a uniform sequence of PV-provable implications of length $\beta \prec \alpha$. Using Herbrand's theorem for PV, we push the witnessing further to witness the $\text{PA} + \bigcup_{\beta \prec \alpha} \text{TI}(\prec_\beta)$-provable formulas in the form $A = \forall \bar{x} \exists \bar{y} B(\bar{x}, \bar{y})$, where B is a polynomial time computable predicate by an algorithm to compute $\bar{y}$ by a sequence of PV-provable polynomial time modifications on an initial polynomial time value, where the computational steps are indexed by the ordinals below α, decreasing by the modifications. Our result generalizes the main theorem of Beckmann (2009) that developed a similar characterization for PA. However, as we will explain below, even for that special case, we use a simpler and easier to generalize methodology.

To compare our result to the existing literature on ordinal analysis, it is important to focus on the role of the polynomial time computable functions and the theory PV in our contribution. First, note that changing the polynomial time functions and PV in our characterization to the elementary or primitive recursive functions and ERA or PRA, respectively, makes the characterization an easy consequence of the known facts in the ordinal analysis literature. For instance,

one can use the powerful witnessing theorems in Friedman and Sheard (1995), Avigad (2002) or the interesting algebraic presentation of the ordinals in Beklemishev (2004). What is not trivial, though, is providing a low-complexity version suitable to witness the low-complexity theorems of arithmetic. To reach such a version, we have two options. The first, as followed in the above-mentioned paper (Beckmann 2009), rewrites the continuous cut elimination technique (Buchholz 1991, 1997), replacing all primitive recursive functions by more careful polynomial time computable operations (Beckmann et al. 2003). The second as an indirect approach uses the known results in ordinal analysis as a *black-box* and rewitness them in a feasible manner to circumvent redoing the tedious ordinal analysis argument. This option is what we follow in the present paper. More precisely, we first use the refined ordinal analysis in Friedman and Sheard (1995) to show that a Π_2^0-formula is provable in the theory $PA + \bigcup_{\beta \prec \alpha} TI(\prec_\beta)$ iff it is provable in an extension of PRA with a weak form of transfinite induction. Then, using a suitable polynomial time representation for the ordinals below α, we will transform a proof in the weaker theory to a sequence of PV-provable polynomial time modifications described above. Our technique of using ordinally long sequence of easy modifications is similar to what used in Avigad (2002), although its machinery has a more model-theoretic character and also implements the ordinal analysis from the scratch. Roughly speaking, Avigad (2002) provides a similar witnessing theorem using elementary functions rather than polynomial time functions in its ordinal flows. However, to have a verifiablity criterion, it insists on having the whole witnessing process provable inside the meta-theory PRA. The witnessing machinery of Avigad (2002) cannot be directly used to prove the low-complexity version we are interested in here. The reason is its use of PRA-formalized Herbrand's theorem for first-order logic that uses cut elimination, and it is extremely costly to be directly formalizable in PV. To solve the issue, as Avigad (2002) also suggests, one must witness the Herbrand's theorem part by a sequence of PV-verifiable modifications or equivalently witness the first-order logic by such modifications, directly. This is one of the things we do in the present paper. Therefore, although our work is inspired by Beckmann (2009) and the witnessing theorems in bounded arithmetic and hence its technique was developed independent from Avigad (2002), one can interpret our contribution as a generalization of Avigad (2002) making its machinery applicable even in the low-complexity settings.

k-flows

A (polynomial) k-flow is a uniform (polynomially) exponentially long sequence of PV-provable implications between $\hat{\Pi}_k^b$-formulas. Recall that $\hat{\Pi}_k^b$- ($\hat{\Sigma}_k^b$-formulas) are roughly the formulas with k-many bounded quantifier blocks starting with a universal (existential) block and followed by a quantifier-free formula over the language $\mathcal{L}_{PV}$ that has a term for any polynomial time computable function. We will witness the provability of an implication between $\hat{\Pi}_k^b$-formulas in T_2^k (resp. S_2^k) by a k-flow (resp. polynomial k-flow). To push the witnessing further, we can use Herbrand's theorem again for the universal theory PV. However, this time the formulas are in $\hat{\Pi}_k^b$ and hence we have k-many layers of quantifier to peel off. To control the number of layers, we intend to remove, and we will follow a relative approach. We fix a number $l \le k$ and only peel off the outmost l many quantifier blocks. More precisely, we first move the PV-provable implications from PV to PV_{k-l+1}, a universal theory for the functions in the $(k-l+1)$-th level of the polynomial hierarchy. This way we can pretend that all the formulas in $\hat{\Sigma}_{k-l}^b \cup \hat{\Pi}_{k-l}^b$ are quantifier-free. Therefore, only l many quantifier blocks are left to witness. Using Herbrand's theorem for the theory PV_{k-l+1} and reading any quantifier-free formula in the language of PV_{k-l+1} as an l-turn game (Skelley and Thapen 2011), we can then witness any PV-provable implication by an explicit PV_{k-l+1}-verifiable reduction between l-turn games. These reductions are somewhat *nondeterministic* mapping their input values to *some* possible instances, where one of the options may work, (see the second part in Theorem 2 to see what we mean by nondeterminism in this

context). Finally, using these reductions, we show that a formula in the form $\forall \bar{x} \exists y \leq r(\bar{x}) B(\bar{x}, y)$, where $B \in \hat{\Sigma}_{k-l}^b \cup \hat{\Pi}_{k-l}^b$ is provable in T_2^k (resp. S_2^k) iff there is a uniform (polynomially) exponentially long sequence of PV_{k-l+1}-verifiable reductions between l-turn games, starting from an explicit PV_{k-l+1}-verifiable winning strategy for the first game. We will only spell out the details for $l = 1, 2$. For $l = 1$, we show that our witnessing theorem reproves some of the well-known witnessing theorems for S_2^k and T_2^k including the usual witnessing of $\hat{\Sigma}_k^b$-definable functions of S_2^k by $\Box_k^p$-functions (Buss 1986) and $\hat{\Sigma}_1^b$-definable multifunctions of T_2^1 by polynomial local search problems Buss and Krajíček (1994). For $l = 2$, we provide new witnessing theorems. For T_2^k, there are other witnessing methods providing similar characterizations as ours based on better (i.e., deterministic) game reductions (Skelley and Thapen 2011; Thapen 2011). The theory of flows can also prove these stronger characterizations. However, it needs to work with more involved notions of a k-flow than what we have here. We leave such investigations to another paper. For S_2^k, however, our result, to the best of our knowledge, is the only characterization in the same style of the original witnessing theorems (Buss 1986) that reduce the provability in S_2^k to a *polynomially* long sequence of feasible modifications. Of course, one can use the conservativity of S_2^k over T_2^{k-1} for $\hat{\Sigma}_k^b$-formulas and then using the witnessing for T_2^{k-1} by the deterministic game reductions (Skelley and Thapen 2011; Thapen 2011) or any other characterization (Beckmann and Buss 2009, 2010), find a witnessing theorem for S_2^k. Using this approach, the characterizations provide an exponentially long sequence of deterministic reductions while we provide a polynomially long sequence of more complex nondeterministic reductions. These two different approaches can be seen as an instance of the usual phenomenon of simulating the huge power of the deterministic exponential time with polynomial time nondeterminism, where the latter, if possible, is more informative than the former.

Finally, to compare our witnessing method to the rich literature on witnessing theorems in bounded arithmetic, let us emphasize two points that we find unique to our characterization. First, unlike the methods used in Buss and Krajíček (1994), Krajíček et al. (2007), Skelley and Thapen (2011), Thapen (2011), Beckmann and Buss (2009, 2010), our machinery is sufficiently general to *directly* witness bounded theories arising from practically *any* type of bounded induction Akbar Tabatabai (2018). For instance, for any $m \geq 2$, consider the language $\mathcal{L}_{PV} \cup \{\#_m\}$, where $x \#_2 y = 2^{|x||y|}$ and $x \#_{i+1} y = 2^{|x| \#_i |y|}$ and define the class $\hat{\Pi}_k^b(\#_m)$ and the theory $PV(\#_m)$ over the new language similar to $\hat{\Pi}_k^b$ and PV over $\mathcal{L}_{PV}$. Now, for any $n \geq 0$, $m \geq n + 2$, and $k \geq 1$, define the theory $R_{m,n}^k$ as the extension of a basic universal theory to handle the function symbols, by the induction axiom

$$A(0) \wedge \forall x(A(x) \rightarrow A(x+1)) \rightarrow \forall x A(|x|_n)$$

where $A \in \hat{\Pi}_k^b(\#_m)$, $|x|_0 = x$, and $|x|_{j+1} = ||x|_j|$. It is easy to imitate our technique in the present paper to witness $R_{m,n}^k$-provable implications between $\hat{\Pi}_k^b(\#_m)$-formulas by a uniform sequence of $PV(\#_m)$-provable implications between $\hat{\Pi}_k^b(\#_m)$-formulas with the length $|t|_n$, for some term t. This can be even more generalized to any type of induction satisfying some basic properties Akbar Tabatabai (2018).

The second point is that the length of our witnessing flows *honestly* reflects the type of the induction we use. For instance, for S_2^k and T_2^k, we use polynomially long and exponentially long k-flows, respectively, and more generally, in $R_{m,n}^k$ where the induction is up to $|x|_n$, the length of the witnessing flow is $|t|_n$, for some term t, see Akbar Tabatabai (2018). This honest correspondence is not typical with the above-mentioned characterizations. For instance, the polynomially long adaptation of the known characterizations for T_2^k (Skelley and Thapen 2011; Thapen 2011), that is, polynomially long sequence of PV-verifiable deterministic reductions between k-turn games, does not witness S_2^k-provable implications. The reason is that any polynomially long iteration of

a deterministic reduction is again a deterministic reduction itself. Therefore, if such a witnessing theorem holds, one can witness the implications in S_2^k between $\hat{\Pi}_k^b$-formulas by a polynomially long sequence of reductions and hence only one reduction. Thus, the $\hat{\Sigma}_k^b$-definable functions of S_2^k must be all polynomial time computable and as all the functions in $\square_k^p$ are $\hat{\Sigma}_k^b$-definable in S_2^k, the polynomial hierarchy must collapse. This simple observation shows that the nondeterminism we use in our reductions is essential to have an honest characterization. Moreover, it shows that our characterization for S_2^k is not a simple consequence of the methodologies used for T_2^k in Skelley and Thapen (2011), Thapen (2011) or even in Beckmann and Buss (2009, 2010).

Here is the structure of the paper. In Section 2, we recall the basic definitions of different languages and arithmetical systems we use in this paper. In Section 3, we introduce our version of polynomial time ordinal representation and we recall the one introduced in Beckmann et al. (2003) for ε_0. In Section 4, we present ordinal flows and the witnessing technique to reduce the provability of the low complexity statements in the theory $\mathrm{PA} + \bigcup_{\beta \prec \alpha} \mathrm{TI}(\prec_\beta)$. Finally, in Section 5, we introduce k-flows to witness the provability of the low complexity statements in the theories S_2^k and T_2^k.

2. Preliminaries

For any first-order language $\mathcal{L}$, by an $\mathcal{L}$-formula, we mean any expression constructible by the connectives $\{\wedge, \vee, \forall, \exists\}$ from the atomic formulas (including $\bot$ and $\top$) and their negations. The formula $\neg A$ is defined via de Morgan laws and $A \to B$ is an abbreviation for $\neg A \vee B$. By an $\mathcal{L}$-term, we simply mean a term in the language $\mathcal{L}$. By $\bar{t}$, we mean a sequence of terms in the language and $\bar{x}$ means a sequence of variables.

To introduce the system PV, let us recall Cobham's machine-independent characterization of polynomial-time computable (*ptime*, for short) functions (Cobham 1965). It states that a function is ptime iff it is constructible from certain basic functions by composition and a weak sort of recursion called the *bounded recursion on notation*. Any such construction provides an algorithm to compute the corresponding ptime function. Let $\mathcal{L}_{\mathrm{PV}}$ be a first-order language with a function symbol for any such algorithm. In Cook (1975), Cook introduced an equational theory over the language $\mathcal{L}_{\mathrm{PV}}$ to reason about ptime functions. The theory essentially consists of the defining axioms for the function symbols together with a sort of induction rule. Later, a conservative first-order extension of PV, denoted by PV_1, was introduced by Krajíček et al. (1991). The theory has the *polynomial induction axiom scheme* denoted by PInd

$$A(0) \wedge \forall x (A(\lfloor \tfrac{x}{2} \rfloor) \to A(x)) \to \forall x A(x),$$

for any quantifier-free formula $A(x)$ and is universally axiomatizable (Krajíček et al. 1991). In this paper, we will only use the theory PV_1 and not PV. Therefore, by abuse of notation, we will use the name PV to denote its first-order extension PV_1.

In any language extending $\mathcal{L}_{\mathrm{PV}}$, by a *bounded quantifier*, we mean a quantifier in the form $\forall x (x \leq t \to A(x))$ or $\exists x (x \leq t \wedge A(x))$, abbreviated by $\forall x \leq t\, A(x)$ and $\exists x \leq t\, A(x)$, respectively. For any sequence of variables $\bar{x} = (x_1, \ldots, x_n)$ and terms $\bar{t} = (t_1, \ldots, t_n)$, by $Q\bar{x} \leq \bar{t}\, A(\bar{x})$, we mean $Q x_1 \leq t_1 Q x_2 \leq t_2 \ldots A(x_1, \ldots, x_n)$, for any $Q \in \{\forall, \exists\}$.

By recursion on k, define the classes $\hat{\Sigma}_k^b$ and $\hat{\Pi}_k^b$ of $\mathcal{L}_{\mathrm{PV}}$-formulas in the following way:

- $\hat{\Pi}_0^b = \hat{\Sigma}_0^b$ is the class of all quantifier-free formulas,
- $\hat{\Sigma}_k^b \subseteq \hat{\Sigma}_{k+1}^b$ and $\hat{\Pi}_k^b \subseteq \hat{\Pi}_{k+1}^b$,
- $\hat{\Pi}_k^b$ and $\hat{\Sigma}_k^b$ are closed under conjunction and disjunction,

- If $B(x) \in \hat{\Sigma}_k^b$, then $\exists x \le t \, B(x) \in \hat{\Sigma}_k^b$ and $\forall x \le t \, B(x) \in \hat{\Pi}_{k+1}^b$ and
- If $B(x) \in \hat{\Pi}_k^b$, then $\forall x \le t \, B(x) \in \hat{\Pi}_k^b$ and $\forall x \le t \, B(x) \in \hat{\Sigma}_{k+1}^b$.

Define $\hat{\Sigma}_\infty^b = \hat{\Pi}_\infty^b$ as $\bigcup_{k=0}^\infty \hat{\Sigma}_k^b$ that is the same as $\bigcup_{k=0}^\infty \hat{\Pi}_k^b$. For the sake of simplicity, we suppressed the free variables in our notation in the above definition while they are also allowed to be used in the formulas.

By the axiom scheme $\hat{\Pi}_k^b - \text{PInd}$, we mean

$$A(0) \wedge \forall x(A(\lfloor \frac{x}{2} \rfloor) \to A(x)) \to \forall x A(x),$$

for any $A \in \hat{\Pi}_k^b$ and by $\hat{\Pi}_k^b - \text{Ind}$, we mean

$$A(0) \wedge \forall x(A(x) \to A(x+1)) \to \forall x A(x),$$

for $A \in \hat{\Pi}_k^b$. The schemes $\hat{\Sigma}_k^b - \text{PInd}$ and $\hat{\Sigma}_k^b - \text{Ind}$ are defined similarly. For any $k \ge 1$, define the theories S_2^k and T_2^k as $\text{PV} + \hat{\Pi}_k^b - \text{PInd}$ and $\text{PV} + \hat{\Pi}_k^b - \text{Ind}$, respectively. It is known that S_2^k (resp., T_2^k) proves $\hat{\Sigma}_k^b - \text{PInd}$ (resp., $\hat{\Sigma}_k^b - \text{Ind}$). It is also useful to mention that the following axiom scheme, denoted by $\hat{\Pi}_k^b - \text{LInd}$

$$A(0) \wedge \forall x(A(x) \to A(x+1)) \to \forall x A(|x|),$$

where $A \in \hat{\Pi}_k^b$, is provable in S_2^k. The same also holds for $\hat{\Sigma}_k^b - \text{LInd}$, where we replace $\hat{\Pi}_k^b$ by $\hat{\Sigma}_k^b$ (Buss 1986; Krajíček 1995). The following theorem is true for theories S_2^k and T_2^k Krajíček (1995).

Theorem. *(Parikh) Let T be either S_2^k or T_2^k, for some $k \ge 1$ and $A(\bar{x}, y)$ be an $\mathcal{L}_{\text{PV}}$-formula in $\hat{\Sigma}_\infty^b$. Then, if $T \vdash \forall \bar{x} \exists y A(\bar{x}, y)$, then there exists an $\mathcal{L}_{\text{PV}}$-term $t(\bar{x})$ such that $T \vdash \forall \bar{x} \exists y \le t(\bar{x}) A(\bar{x}, y)$.*

It is possible to define a universal theory for any level in the polynomial hierarchy, similar to what PV_1 does for the polynomial time computable functions. More precisely, for any $k \ge 2$, one can define a universal theory PV_k over an extended language $\mathcal{L}_{\text{PV}_k}$ that has a term for any function in the kth level of the polynomial hierarchy, denoted by $\Box_k^p$ (Krajíček et al. 1991). We do not spell out the details of these theories. The only thing we need to know is that PV_k has an explicit term for the characteristic functions of $\hat{\Sigma}_k^b$-formula and its term construction allows defining functions by bounded recursion on notation (Krajíček et al. 1991; Krajíček 1995). As PV_k is universal, it enjoys Herbrand's theorem (Buss 1998b; Krajíček 1995):

Theorem. *(Herbrand) Let $A(\bar{x}, y)$ and $B(\bar{x}, y, z)$ be two quantifier-free $\mathcal{L}_{\text{PV}_k}$-formulas. Then:*

- *If $\text{PV}_k \vdash \exists y A(\bar{x}, y)$, then there exists an $\mathcal{L}_{\text{PV}_k}$-term $f(\bar{x})$ such that $\text{PV}_k \vdash A(\bar{x}, f(\bar{x}))$.*
- *If $\text{PV}_k \vdash \exists y \forall z B(\bar{x}, y, z)$, then there are $\mathcal{L}_{\text{PV}_k}$-terms $f_0(\bar{x}), f_1(\bar{x}, z_0), f_2(\bar{x}, z_0, z_1), \dots, f_m(\bar{x}, z_0, z_1, \dots, z_{m-1})$ such that $\bigvee_{i=0}^m B(\bar{x}, f_i(\bar{x}, z_0, \dots, z_{i-1}), z_i)$ is provable in PV_k.*

It is possible to generalize this theorem to a *generalized Herbrand's theorem* to cover more alternations of quantifiers. However, in this paper, one can restrict oneself only to these two levels (Buss 1998b).

The system PV_k proves the scheme PInd for any quantifier-free $\mathcal{L}_{\text{PV}_k}$-formula. As any $\mathcal{L}_{\text{PV}_k}$-term can be defined by an $\mathcal{L}_{\text{PV}}$-formula in $\hat{\Sigma}_k^b$, it is possible to represent any quantifier-free $\mathcal{L}_{\text{PV}_k}$-formula by two $\mathcal{L}_{\text{PV}}$-formulas, one in $\hat{\Sigma}_k^b$ and one in $\hat{\Pi}_k^b$. Using this fact, one can interpret PV_k inside the theory S_2^k.

Going beyond bounded theories of arithmetic, in a similar fashion to PV and using the construction of primitive recursive functions by composition and primitive recursion on certain basic functions, it is possible to extend the language $\mathcal{L}_{PV}$ by a *fresh* function symbol for any primitive recursive function. Denote this new language by $\mathcal{L}_{PRA}$ and set the first-order theory PRA over $\mathcal{L}_{PRA}$ as PV extended by the defining axioms for the new functional symbols and the induction axiom $A(0) \wedge \forall x(A(x) \to A(x+1)) \to \forall x A(x)$, for any quantifier-free formula in the new language. This is of course different from the usual definition of PRA as its language is extended by the ptime function symbols in $\mathcal{L}_{PV}$, and the theory itself is extended by the theory PV. Moreover, the formula in the induction axiom of PRA may contain the symbols from $\mathcal{L}_{PV}$. However, as the functions in the Cobham calculus are constructible as primitive recursive functions, it is clear that the separation of the primitive recursive function symbols and ptime function symbols is just a technical point and is totally immaterial. In fact, our presentation of PRA is a conservative extension of the usual PRA and hence has nothing essentially different from the usual PRA.

By Peano arithmetic, denoted by PA, we mean the theory PV extended by full induction axiom scheme $A(0) \wedge \forall x(A(x) \to A(x+1)) \to \forall x A(x)$, for *any* formula $A(x)$. This is also different from the usual definition of PA. However, as all of the function symbols in $\mathcal{L}_{PV}$ are definable in the usual language of PA and their functionality and totality are provable in the usual PA, it is easy to see that our PA is a conservative extension of the usual PA.

By Π_2^0, we mean the class of $\mathcal{L}_{PV}$-formulas in the form $\forall \bar{x} \exists \bar{y} A(\bar{x}, \bar{y})$, where any quantifier in $A(\bar{x}, \bar{y})$ is bounded. For two theories T and S and a class of formulas Φ, by $T \equiv_\Phi S$, we mean $T \vdash A$ iff $S \vdash A$, for any $A \in \Phi$.

Finally, let us recall some basics of the ordinal arithmetic. Apart from addition, multiplication, and exponentiation of the ordinals, it is also possible to define subtraction $\dot{-}$ from *left* such that $\alpha \dot{-} \beta = 0$, if $\alpha \prec \beta$ and $\alpha \dot{-} \beta = \gamma$, if $\beta \preceq \alpha$, where γ is the *unique* ordinal with the property that $\beta + \gamma = \alpha$. Similarly, it is possible to define the division d from *left* such that if $\beta \neq 0$, then $d(\alpha, \beta)$ is the *unique* γ such that $\alpha = \beta \gamma + \delta$, for some $\delta \prec \beta$.

3. Polynomial-Time Ordinal Representations

In this section, we will introduce polynomial time ordinal representations and recall the concrete representation for the ordinal ε_0 provided in Beckmann et al. (2003). Both parts will be of essential use in Section 4.

Definition 1. *Let α be an infinite ordinal closed under addition, multiplication, and the operation $\beta \mapsto \omega^\beta$. We call the tuple*

$$\mathfrak{O} = (\mathcal{O}, \prec, +, \cdot, \dot{-}, d(\cdot, \cdot), o, x \mapsto \omega^x, \mathbf{0}, \mathbf{1}, \omega)$$

a polynomial time representation with a primitive recursive exponentiation (ptime representation, for short) for the ordinal α, if:

- $\mathcal{O}$ *is a unary polynomial time relation on the natural numbers represented as a quantifier-free $\mathcal{L}_{PV}$-formula. Its intended meaning is the set of all the representations of the ordinals below α. We use small Greek letters to denote the elements of $\mathcal{O}$. For instance, by $\forall \beta \, A(\beta)$, we actually mean $\forall x(\mathcal{O}(x) \to A(x))$.*

- $\prec$ *is a binary polynomial time relation on the natural numbers, represented as a quantifier-free $\mathcal{L}_{PV}$-formula. Its intended meaning is the order over the ordinals below α. We define the relation $(\gamma \preceq \beta)$ as $(\gamma \prec \beta) \vee (\gamma = \beta)$.*

- $+, \cdot, \dot{-}$ *and $d(\cdot, \cdot)$ are binary polynomial time functions, represented as $\mathcal{L}_{PV}$-terms. Their intended meaning is the ordinal addition, multiplication, subtraction from left, and division from left, respectively.*

- *o is a unary polynomial time function represented as an $\mathcal{L}_{PV}$-term. Its intended meaning is the function that maps the natural numbers to the representation of their order-types below α. For instance, $o(0)$ is the least element of $\mathcal{O}$ while $o(1)$ is its second least element.*
- *ω^x is a primitive recursive unary function represented as an $\mathcal{L}_{PRA}$-term. Its intended meaning is the function that maps the ordinal $\beta \prec \alpha$ to the ordinal $\omega^\beta \prec \alpha$.*
- *$\mathbf{0}$, $\mathbf{1}$ and ω are three numbers representing the ordinals zero, one and ω, respectively.*
- *The structure $(\mathfrak{O}, \prec)$ is isomorphic to $(\alpha, \prec_\alpha)$, where $\prec_\alpha$ is the order on α.*
- *PV proves that $\prec$ is a total ordering on $\mathcal{O}$ with the minimum $\mathbf{0}$.*
- *PV proves that $\prec$ is discrete over $\mathcal{O}$, that is, for all $\beta, \gamma \in \mathcal{O}$, if $\gamma \prec \beta + \mathbf{1}$, then either $\gamma \prec \beta$ or $\gamma = \beta$.*
- *PV proves the associativity of the addition and multiplication, the left distributivity of multiplication over the addition, the neutrality of $\mathbf{0}$ for the addition, the neutrality of $\mathbf{1}$ for the multiplication and the identity $\mathbf{0}\beta = \beta\mathbf{0} = \mathbf{0}$.*
- *PV proves that the addition and the nonzero multiplication from left respect the order $\prec$, that is, if $\delta \prec \gamma$ then $\beta + \delta \prec \beta + \gamma$ and if we also have $\beta \neq \mathbf{0}$, then $\beta\delta \prec \beta\gamma$.*
- *PV proves that the addition and multiplication from right respects $\preceq$, that is, if $\delta \preceq \gamma$ then $\delta + \beta \preceq \gamma + \beta$ and $\delta\beta \preceq \gamma\beta$.*
- *PV proves the defining axioms of $\dot{-}$, that is, if $\alpha \prec \beta$ then $\alpha \dot{-} \beta = \mathbf{0}$ and if $\alpha \succeq \beta$ then $\alpha = \beta + (\alpha \dot{-} \beta)$.*
- *PV proves the defining axioms of d, that is, if $\beta \neq \mathbf{0}$, then $\beta d(\alpha, \beta) \preceq \alpha$ and $\alpha \dot{-} \beta d(\alpha, \beta) \prec \beta$.*
- *PV proves that o is an order-isomorphism between the natural numbers and the ordinals below ω, mapping 0 and 1 to $\mathbf{0}$ and $\mathbf{1}$, respectively, that is, PV proves $o(0) = \mathbf{0}$, $o(1) = \mathbf{1}$, $\forall x[\mathcal{O}(o(x)) \wedge o(x) \prec \omega]$, $\forall \beta \prec \omega \exists! y\, o(y) = \beta$, and $\forall xy(x < y \leftrightarrow o(x) \prec o(y))$). Where there is no risk of confusion, we will use the numbers and their ordinal reinterpretations, interchangeably. For instance, we use 1 for $\mathbf{1}$.*
- *PRA proves that $\omega^{\mathbf{0}} = 1$ and $\omega^{\mathbf{1}} = \omega$. It also proves that ω^β respects $\preceq$ and maps the addition to the multiplication.*
- *If there is no $\gamma \in \mathcal{O}$ such that $\beta = \gamma + 1$, then ω^β is the supremum of the set $\{\omega^\gamma \mid \gamma \prec \beta\}$, that is, for any $\delta \in \mathcal{O}$, if $\omega^\gamma \preceq \delta$, for any $\gamma \prec \beta$, then $\omega^\beta \preceq \delta$.*
- *PRA proves that for every $\beta \in \mathcal{O}$, there is a unique expansion $\beta = \omega^{\gamma_1} + \ldots + \omega^{\gamma_n}$ such that $\gamma_n \preceq \gamma_{n-1} \preceq \cdots \preceq \gamma_1$.*

Remark 2. Here are some remarks. First, notice that the relations of being a successor and a limit ordinal are both definable by the predicates $\exists \gamma (\beta = \gamma + 1)$ and $\forall \gamma \prec \beta (\gamma + 1 \prec \beta)$, respectively. It is also easy to see that PV can prove the dichotomy that for any $\beta \in \mathcal{O}$, it is either a successor or a limit. Second, using the compatibility of the order with the addition and the multiplication, one can easily prove in PV that if $\beta = \gamma + \delta = \gamma + \eta$, then $\beta \dot{-} \gamma = \delta = \eta$. This observation proves that for any $\gamma \prec \beta$, the interval $(\mathbf{0}, \beta \dot{-} \gamma)$ in $\mathcal{O}$ is in one-to-one correspondence with the interval (γ, β), via the map $\delta \mapsto \gamma + \delta$. Similarly, PV proves that if $\gamma \neq \mathbf{0}$, then $\beta = \gamma\delta = \gamma\eta$ implies $d(\beta, \gamma) = \delta = \eta$. Therefore, $d(\gamma\delta, \gamma) = \delta$, for $\gamma \neq \mathbf{0}$. Third, let us explain the discrepancy between the polynomial time character of the order, addition, multiplication, subtraction, and division and the primitive recursive character of the function $x \mapsto \omega^x$ in our definition. For that purpose, first, pretend that our definition uses the primitive recursive functions and predicates and PRA everywhere when it actually uses polynomial time functions and predicates and PV. Then, one can easily see that this primitive recursive version of our representation is just a mild extension of the primitive recursive (even elementary) ordinal representation employed in Friedman and Sheard (1995). (Their conditions are different, but it is easy to show that our axioms imply theirs.) As we

use a proof-theoretic result of Friedman and Sheard ([1995]), using the primitive recursive version of our definition is completely justified. However, there is another role for our ordinal representation. As it is clear, in this paper, we intend to address the lower complexity formulas and for that purpose, some basic ordinal arithmetic (up to addition and multiplication and hence subtraction and division from left) is required to be implemented in polynomial time. Therefore, we are forced to lower the complexity of some parts of the representation. However, as the use of the exponentiation is only restricted to the result from Friedman and Sheard ([1995]) that we use as a black box here, we decided to lower the complexity up to the point we need and let the exponentiation parts intact. This way we can accept more ptime representations.

Let $\beta \in \mathcal{O}$. By the axiom scheme TI($\prec_\beta$), we mean the transfinite induction up to β, that is

$$\forall \gamma \prec \beta [\forall \delta \prec \gamma A(\delta) \to A(\gamma)] \to \forall \gamma \prec \beta A(\gamma),$$

where A can be any formula in $\mathcal{L}_{PV}$. In Friedman and Sheard ([1995]), a refined method of ordinal analysis is provided showing that the Π_2^0-consequences of the theory $PA + \bigcup_{\beta \in \mathcal{O}} TI(\prec_\beta)$ are actually provable in a smaller theory extending PRA with a weak form of transfinite induction stating that for any $\beta \prec \alpha$, there is no primitive recursive decreasing sequence of ordinals below β. For more, see Friedman and Sheard ([1995]), Rathjen ([1999]).

Theorem. *Let α be an ordinal and $\mathfrak{O}$ be its ptime representation. Then,*

$$PA + \bigcup_{\beta \in \mathcal{O}} TI(\prec_\beta) \equiv_{\Pi_2^0} PRA + \bigcup_{\beta \in \mathcal{O}} PRWO(\prec_\beta),$$

where PRWO($\prec_\beta$) is the scheme $\forall \bar{x} \exists y [f(\bar{x}, y+1) \not\prec f(\bar{x}, y) \lor \neg \mathcal{O}(f(\bar{x}, y)) \lor f(\bar{x}, y) \not\prec \beta]$, for any function symbol f in $\mathcal{L}_{PRA}$.

3.1 A polynomial-time representation for ε_0

In this subsection, we will recall the basics of the ptime notation system for the ordinal ε_0, introduced in Beckmann et al. ([2003]). Define $\mathcal{O}_0$ and $\prec_0$ inductively and simultaneously in the following way: $\mathcal{O}_0$ is the least set of expressions containing the empty string **0** and is closed under the operation $(\alpha_1, \ldots, \alpha_n) \mapsto \omega^{\alpha_1} a_1 + \ldots + \omega^{\alpha_n} a_n$, where $a_i \neq 0$ are natural numbers and $\alpha_n \prec_0 \ldots \prec_0 \alpha_2 \prec_0 \alpha_1$ and set $\omega^{\alpha_1} a_1 + \ldots + \omega^{\alpha_n} a_n \prec_0 \omega^{\beta_1} b_1 + \ldots + \omega^{\beta_m} b_m$, if there exists $i \leq \min\{m, n\}$ such that $\alpha_j = \beta_j$ and $a_j = b_j$, for any $j \leq i$ and one of the following takes place:

- $i = n < m$,
- $i < \min\{m, n\}$ and $\alpha_{i+1} \prec_0 \beta_{i+1}$
- $i < \min\{m, n\}$ and $\alpha_{i+1} = \beta_{i+1}$ and $a_i < b_i$.

Using some efficient method of sequence encoding, it is possible to arithmetize the set $\mathcal{O}_0$ and the predicate $\prec_0$. It is also possible to implement the arithmetization in a way that the length of the Gödel number of $\alpha \in \mathcal{O}_0$ is proportional to the number of symbols in the expression α. By this fact, Beckmann et al. ([2003]) shows that both $\mathcal{O}_0$ and $\prec_0$ are polynomial time computable and hence formalizable in PV. (Technically, it uses a conservative extension of PV, but the difference does not affect us here.) We fix *quantifier-free* predicates $\mathcal{O}_0(x)$ and $x \prec_0 y$ to denote the formalized versions in the language $\mathcal{L}_{PV}$. In Beckmann et al. ([2003]), it is shown that PV proves that $\prec$ is a total ordering on $\mathcal{O}_0$. It is clear that PV also proves that **0** is the minimum element of $\mathcal{O}_0$. Define **1** as $\omega^{\mathbf{0}} 1$ and for o, consider the function that maps the number n to $\omega^{\mathbf{0}} n$. Denote $\omega^{o(1)}$ by ω. Then, we have $\omega^{\mathbf{0}} = \mathbf{1}$ and $\omega^{\mathbf{1}} = \omega$. The map o is ptime, and it is easy to prove in PV that o is an order-isomorphism, that is, $PV \vdash \forall x [\mathcal{O}_0(o(x)) \land o(x) \prec \omega]$, $PV \vdash \forall \alpha \prec \omega \exists! y \, o(y) = \alpha$ and

$PV \vdash x < y \leftrightarrow o(x) \prec_0 o(y)$. For $x \mapsto \omega^x$, use the evident function mapping the expression β to the expression ω^β and note that it is clearly primitive recursive.

In the rest of this subsection, we will explain how to formalize the basic ordinal arithmetic in PV, using the aforementioned representation. For that purpose, first consider the following equalities over the real ordinals below ε_0. We assumed that the inputs are *nonzero* as the operations with one zero input are trivial. These equalities make the computation of the addition, multiplication, subtraction from left and division from left possible, using the Cantor normal form of the ordinals. We will not provide a proof for these equalities as they are just simple computations, see Takeuti and Zaring (1982).

$$\left(\sum_{i=1}^{n}\omega^{\alpha_i}a_i\right)+\left(\sum_{j=1}^{m}\omega^{\beta_j}b_j\right)=\begin{cases}\sum_{i=1}^{n}\omega^{\alpha_i}a_i+\sum_{j=1}^{m}\omega^{\beta_j}b_j & \alpha_n \succ \beta_1 \\ \sum_{j=1}^{m}\omega^{\beta_j}b_j & \alpha_1 \prec \beta_1 \\ \sum_{i=1}^{k}\omega^{\alpha_i}a_i+\sum_{j=1}^{m}\omega^{\beta_j}b_j & \alpha_{k+1} \prec \beta_1 \prec \alpha_k \\ \sum_{i=1}^{k-1}\omega^{\alpha_i}a_i+\omega^{\alpha_k}(a_k+b_1)+\sum_{j=2}^{m}\omega^{\beta_j}b_j & \alpha_k = \beta_1\end{cases}$$

$$\left(\sum_{i=1}^{n}\omega^{\alpha_i}a_i\right)\dot{-}\left(\sum_{j=1}^{m}\omega^{\beta_j}b_j\right)=\begin{cases}0 & \alpha_{k+1} \prec \beta_{k+1} \\ \sum_{i=k+1}^{n}\omega^{\alpha_i}a_i & \alpha_{k+1} \succ \beta_{k+1} \\ \omega^{\alpha_{k+1}}(a_{k+1}-b_{k+1})+\sum_{i=k+2}^{n}\omega^{\alpha_i}a_i & \alpha_{k+1} = \beta_{k+1}, a_{k+1} > b_{k+1} \\ 0 & \alpha_{k+1} = \beta_{k+1}, a_{k+1} < b_{k+1}\end{cases}$$

where k is the maximum i such that $\alpha_i = \beta_i$ and $a_i = b_i$, if there is any and otherwise $k = 0$,

$$\left(\sum_{i=1}^{n}\omega^{\alpha_i}a_i\right)\left(\sum_{j=1}^{m}\omega^{\beta_j}b_j\right)=\begin{cases}\sum_{j=1}^{m}\omega^{\alpha_1+\beta_j}b_j & \beta_m \succ 0 \\ \sum_{j=1}^{m-1}\omega^{\alpha_1+\beta_j}b_j+\omega^{\alpha_1}a_1 b_m+\sum_{i=2}^{n}\omega^{\alpha_i}a_i & \beta_m = 0, m > 1 \\ \omega^{\alpha_1}a_1 b_1+\sum_{i=2}^{n}\omega^{\alpha_i}a_i & \beta_m = 0, m = 1\end{cases}$$

$$d\left(\sum_{i=1}^{n}\omega^{\alpha_i}a_i,\sum_{j=1}^{m}\omega^{\beta_j}b_j\right)=\begin{cases}0 & \alpha_1 \prec \beta_1 \\ \sum_{i=1}^{k}\omega^{\alpha_i\dot{-}\beta_1}a_i & \alpha_1 \succeq \beta_1, \alpha_k \neq \beta_1 \\ \sum_{i=1}^{k-1}\omega^{\alpha_i\dot{-}\beta_1}a_i+d(a_k,b_1) & \alpha_1 \succeq \beta_1, \alpha_k = \beta_1, (*) \\ \sum_{i=1}^{k-1}\omega^{\alpha_i\dot{-}\beta_1}a_i+(d(a_k,b_1)-1) & \text{otherwise}\end{cases}$$

where k is the greatest i such that $\alpha_i \succeq \beta_1$, $d(a_k, b_1)$ is the quotient of a_k divided by b_1 and $(*)$ is the condition that $\sum_{i=k}^{n}\omega^{\alpha_i}a_i \succeq \omega^{\alpha_k}b_1 d(a_k,b_1)+\sum_{j=2}^{m}\omega^{\beta_j}b_j$. Note that to compute any of the operations, it is enough to do constant many comparisons and basic numerical computations, a search to find the maximum index that takes at most as long as the length of the inputs and at most m or n many applications of a ptime function. Hence, all the operations are ptime and hence representable in PV. It is easy to see but tedious to show that all the claimed properties in Definition 1 hold. Therefore, the described data in this subsection defines a ptime representation for ε_0 that we denote by $\mathfrak{O}_0$.

4. Ordinal Flows and Arithmetic

Let α be an ordinal and $\mathfrak{O}$ be its ptime representation. In this section, we develop a witnessing method for the theory $PA + \bigcup_{\beta \in \mathcal{O}} TI(\prec_\beta)$. The section consists of three parts. First, in Subsection 4.1, we will introduce an auxiliary theory $TI(\forall_1, \prec)$ with a transfinite induction on the universal formulas in the language of PV. The system is powerful enough to interpret $PRA + \bigcup_{\beta \in \mathcal{O}} PRWO(\prec_\beta)$ and hence proves all Π_2^0-theorems of $PA + \bigcup_{\beta \in \mathcal{O}} TI(\prec_\beta)$. Then in Subsection 4.2, we will provide a witnessing method for $TI(\forall_1, \prec)$ that transforms the provability between two universal formulas in $TI(\forall_1, \prec)$ to an ordinal-length sequence of

PV-provable implications. Finally, in Subsection 4.3, we use Herbrand's theorem, Theorem 2, to witness the implications in PV to provide a characterization for the low complexity theorems of $\mathrm{PA} + \bigcup_{\beta \in \mathcal{O}} \mathrm{TI}(\prec_\beta)$.

4.1 The system $\mathrm{TI}(\forall_1, \prec)$

This subsection is devoted to the introduction and investigation of the auxiliary theory $\mathrm{TI}(\forall_1, \prec)$.

Definition 3. *Define $\forall_1$ (resp., $\exists_1$) as the least set of $\mathcal{L}_{\mathrm{PV}}$-formulas containing all atomic formulas and their negations and closed under conjunction, disjunction, and universal (resp. existential) quantifiers.*

Let $I\forall_1$ (resp. $I\exists_1$) be the theory extending PV by the $\forall_1$-induction (resp. $\exists_1$-induction) scheme, that is, $A(0) \wedge \forall x(A(x) \rightarrow A(x+1)) \rightarrow \forall x A(x)$, for any $A(x) \in \forall_1$ (resp. $A(x) \in \exists_1$). Note that $I\exists_1 = I\forall_1$. The proof uses the usual technique of using $\forall_1$-induction on $B(x) = \neg A(y \dot- x)$ to prove $\exists_1$-induction on $A(y)$ and similarly for the other direction, see Buss (1998).

Lemma 4. *For any primitive recursive function $f : \mathbb{N}^k \rightarrow \mathbb{N}$, there is a $\exists_1$-formula $D_f(\bar{x}, y)$ such that $I\exists_1 \vdash \forall \bar{x} \exists! y D_f(\bar{x}, y)$ and $\mathbb{N} \models D_f(\bar{n}, m)$ iff $f(\bar{n}) = m$, for any $\bar{n}, m \in \mathbb{N}$.*

Proof. For any primitive recursive function f, we provide a quantifier-free formula $C_f(\bar{x}, w, y) \in \mathcal{L}_{\mathrm{PV}}$ encoding that w is a computation of f with the input $\bar{x}$ and the output y. To that aim, we use recursion on the construction of f. The cases for the basic functions and composition are easy. For the recursion case, if $f(\bar{x}, y)$ is defined via recursive equations $f(\bar{x}, 0) = g(\bar{x})$ and $f(\bar{x}, y+1) = h(\bar{x}, y, f(\bar{x}, y))$, define $C_f(\bar{x}, y, \langle u, v \rangle, z)$ as $C_g(\bar{x}, u_0, v_0) \wedge \forall i \leq l(v) C_h(\bar{x}, i, v_i, u_{i+1}, v_{i+1}) \wedge v_{l(v)} = z$, where v encodes the sequence $\{f(\bar{x}, i)\}_{i=0}^{l(v)}$, the number $l(v)$ is the length of this sequence and u encodes the sequence of computations $\{u_i\}_{i=0}^{l(v)}$, where u_0 reads $\bar{x}$ and computes $v_0 = f(\bar{x}, 0)$ and u_{i+1} reads $\bar{x}, i$ and $f(\bar{x}, i)$ and computes $f(\bar{x}, i+1)$ via the function h. Note that the predicate $\forall i \leq l(v) C_h(\bar{x}, i, v_i, u_i, v_{i+1})$ is polynomial computable, as $l(v) \leq |v|$, where $|v|$ is the binary length of v. Hence, there exists a polynomial time function symbol in PV like F such that PV proves that $F(\bar{x}, u, v) = 1$ iff $\forall i \leq l(v) C_h(\bar{x}, i, v_i, u_i, v_{i+1})$. Therefore, C_f can be written in a quantifier-free form. Now, set $D_f(\bar{x}, y) = \exists w C_f(\bar{x}, w, y)$. It is clear that $D_f \in \exists_1$ and $\mathbb{N} \models D_f(\bar{n}, m)$ iff $f(\bar{n}) = m$, for any $\bar{n}, m \in \mathbb{N}$. Finally, the proof of the claim that $I\exists_1 \vdash \forall \bar{x} \exists! y D_f(\bar{x}, y)$ is similar to the similar claim in the representation of primitive recursive functions in $I\Sigma_1$. $\square$

Definition 5. *Define the theory $\mathrm{TI}(\forall_1, \prec)$ over $\mathcal{L}_{\mathrm{PV}}$ as the theory PV extended by the transfinite induction scheme $\forall \delta (\forall \gamma \prec \delta\, A(\gamma) \rightarrow A(\delta)) \rightarrow A(\theta)$, for any $A(\gamma) \in \forall_1$ and any constant $\theta \in \mathcal{O}$.*

Note that $\mathrm{TI}(\forall_1, \prec)$ extends the theory $I\forall_1$ as $\mathrm{TI}(\forall_1, \prec)$ proves $\forall \delta \prec \omega(\forall \gamma \prec \delta\, A(\gamma) \rightarrow A(\delta)) \rightarrow \forall \delta \prec \omega A(\delta)$, for any $A \in \forall_1$. Using the function o and the fact that it is an order-isomorphism between the numbers and the ordinals below ω, we will have $\forall x(\forall y < x\, A(y) \rightarrow A(x)) \rightarrow \forall x A(x)$ which implies $A(0) \wedge \forall x(A(x) \rightarrow A(x+1)) \rightarrow \forall x A(x)$. Therefore, by Lemma 4, $\mathrm{TI}(\forall_1, \prec)$ represents any primitive recursive function with an $\exists_1$-definition. As it is routine in arithmetic (Buss 1998), this provides both $\forall_1$ and $\exists_1$ definitions for any atomic formula in $\mathcal{L}_{\mathrm{PRA}}$. Hence, it is possible to interpret any $\forall_1$-formula in $\mathcal{L}_{\mathrm{PRA}}$ as an $\forall_1$-formula in $\mathcal{L}_{\mathrm{PV}}$. Using that interpretation, we can pretend that $\mathrm{TI}(\forall_1, \prec)$ has a fresh function symbol for any primitive recursive function and the $\forall_1$-formulas in the new language are allowed in the transfinite induction. Moreover, we can also pretend that $\mathrm{TI}(\forall_1, \prec)$ extends the theory PRA. The reason simply is that the equational defining axioms in PRA are all provable in $I\forall_1 = I\exists_1$ and hence in $\mathrm{TI}(\forall_1, \prec)$, as they are actually encoded in the definition D_f of f. For the quantifier-free induction of PRA, as we have seen before, it is possible to use the isomorphism o to prove the induction in $\mathrm{TI}(\forall_1, \prec)$.

Lemma 6. *If* $\mathrm{PRA} + \bigcup_{\beta \in \mathcal{O}} \mathrm{PRWO}(\prec_\beta) \vdash A$ *then* $\mathrm{TI}(\forall_1, \prec) \vdash A$, *for any* $A \in \mathcal{L}_{\mathrm{PV}}$.

Proof. Pretend $\mathrm{TI}(\forall_1, \prec)$ has a function symbol for any primitive recursive function, allowed in the $\forall_1$-formulas. As $\mathrm{TI}(\forall_1, \prec)$ extends PRA, it is enough to prove $\mathrm{TI}(\forall_1, \prec) \vdash \mathrm{PRWO}(\prec_\beta)$, for any $\beta \in \mathcal{O}$. For the sake of contradiction, assume $\forall y[f(\bar{x}, y+1) \prec f(\bar{x}, y) \wedge \mathcal{O}(f(\bar{x}, y)) \wedge f(\bar{x}, y) \prec \beta]$. Set $B(\gamma, \bar{x}) = \forall y(f(\bar{x}, y) \neq \gamma)$ and note that $B(\gamma, \bar{x}) \in \forall_1$. By transfinite induction, we prove $\forall \gamma \prec \beta \, B(\gamma, \bar{x})$. For that purpose, assume $\forall \delta \prec \gamma [\delta \prec \beta \to B(\delta, \bar{x})]$. Then, to prove $[\gamma \prec \beta \to B(\gamma, \bar{x})]$, if $f(\bar{x}, y) = \gamma$, for some $\gamma \prec \beta$, as $f(\bar{x}, y+1) \prec f(\bar{x}, y)$, we have $f(\bar{x}, y+1) \prec \gamma \prec \beta$. On the other hand, by $\forall \delta \prec \gamma [\delta \prec \beta \to B(\delta, \bar{x})]$, we know that none of the ordinals δ below γ is in the form of $f(\bar{x}, z)$, which contradicts with $f(\bar{x}, y+1) \prec \gamma$. Hence, $[\gamma \prec \beta \to B(\gamma, \bar{x})]$. Therefore, $\forall \delta \prec \gamma [\delta \prec \beta \to B(\delta, \bar{x})]$ implies $[\gamma \prec \beta \to B(\gamma, \bar{x})]$. Hence, by transfinite induction, we have $\forall \gamma \prec \beta \, B(\gamma, \bar{x})$ which for $\gamma = f(\bar{x}, 0) \prec \beta$ implies $\forall y(f(\bar{x}, y) \neq f(\bar{x}, 0))$ which is a contradiction. $\square$

Corollary 7. $\mathrm{PA} + \bigcup_{\beta \in \mathcal{O}} \mathrm{TI}(\prec_\beta) \equiv_{\Pi_2^0} \mathrm{TI}(\forall_1, \prec)$.

Proof. One direction is a consequence of the fact that $\mathrm{PA} + \bigcup_{\beta \in \mathcal{O}} \mathrm{TI}(\prec_\beta)$ proves the transfinite induction for any formulas and hence extends the theory $\mathrm{TI}(\forall_1, \prec)$. The other direction is a consequence of Theorem 3 and Lemma 6. $\square$

4.1.1 A proof system for $\mathrm{TI}(\forall_1, \prec)$

We now present a sequent calculus for the theory $\mathrm{TI}(\forall_1, \prec)$. By a sequent over $\mathcal{L}_{\mathrm{PV}}$, we mean an expression in the form $S = \Gamma \Rightarrow \Delta$, where Γ and Δ are multisets of formulas in $\mathcal{L}_{\mathrm{PV}}$. Define **LPV** as the usual system **LK** augmented with the equality axioms for atomic formulas and their negations and all quantifier-free *theorems* of PV as the initial sequents:

Axioms:

$$\overline{\bot \Rightarrow} \qquad \overline{\Rightarrow \top}$$

$$\overline{P \Rightarrow P} \qquad \overline{\neg P \Rightarrow \neg P} \qquad \overline{P, \neg P \Rightarrow} \qquad \overline{\Rightarrow P, \neg P} \qquad \overline{\Rightarrow A}$$

$$\overline{\Rightarrow t = t} \qquad \overline{s_1 = t_1, \ldots, s_n = t_n \Rightarrow f(\bar{s}) = f(\bar{t})} \qquad \overline{s_1 = t_1, \ldots, s_n = t_n, Q(\bar{s}) \Rightarrow Q(\bar{t})}$$

where P ranges over all atomic formulas, f ranges over all function symbols in the language, Q ranges over all atmoic formulas or their negations, and A ranges over all quantifier-free theorems of PV.

Structural Rules:

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \, Lc \qquad \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta} \, Rc \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \, Lw \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta} \, Rw$$

$$\frac{\Gamma \Rightarrow A, \Delta \qquad \Pi, A \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda} \, cut$$

Logical Rules:

$$i \in \{0, 1\} \frac{\Gamma, A_i \Rightarrow \Delta}{\Gamma, A_0 \wedge A_1 \Rightarrow \Delta} L\wedge \qquad \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta} R\wedge$$

$$\frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \; L\vee \qquad i \in \{0, 1\} \; \frac{\Gamma \Rightarrow A_i, \Delta}{\Gamma \Rightarrow A_0 \vee A_1, \Delta} \; R\vee$$

$$\frac{\Gamma, A(t) \Rightarrow \Delta}{\Gamma, \forall x A(x) \Rightarrow \Delta} \; L\forall \qquad \frac{\Gamma \Rightarrow A(y), \Delta}{\Gamma \Rightarrow \forall x A(x), \Delta} \; R\forall$$

$$\frac{\Gamma, A(y) \Rightarrow \Delta}{\Gamma, \exists x A(x) \Rightarrow \Delta} \; L\exists \qquad \frac{\Gamma \Rightarrow A(t), \Delta}{\Gamma \Rightarrow \exists x A(x), \Delta} \; R\exists$$

In the rules $(R\forall)$ and $(L\exists)$, the variable y should not appear in the consequence. Adding the rule

$$\frac{\Gamma, \forall \gamma \prec \delta \, A(\gamma) \Rightarrow \Delta, A(\delta)}{\Gamma \Rightarrow \Delta, A(\theta)} \; Ind_\alpha$$

to **LPV**, we get $\mathbf{G}_0$. Note that in (Ind_α), the variable δ should not appear in the consequence. Moreover, the constant $\theta \in \mathcal{O}$ is arbitrary and can take any value. For more on the proof theory of first-order theories and specially arithmetic, see Buss (1998a,b).

By the usual cut reduction method (Buss 1998a,b), it is easy to prove that for any $\Gamma \cup \Delta \subseteq \forall_1$, if $\Gamma \Rightarrow \Delta$ is provable in $\mathrm{TI}(\forall_1, \prec)$ (resp., PV), then it has a $\mathbf{G}_0$-proof (resp. **LPV**-proof) consisting only of $\forall_1$-formulas. For some practical reasons, we simplify the system $\mathbf{G}_0$ by changing the cut and the induction rules to the weak cut and weak induction rules, respectively:

$$\frac{\Gamma \Rightarrow A \qquad A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; wCut \qquad \frac{\Gamma, \forall \gamma \prec \delta \, A(\gamma) \Rightarrow \forall \gamma \prec \delta + 1 \, A(\gamma)}{\Gamma \Rightarrow A(\theta)} wInd_\alpha$$

Denote this system by $\mathbf{G}_1$. Note that the difference between (Ind_α) and $(wInd_\alpha)$ is that in the latter Δ is omitted and $A(\delta)$ is replaced by $\forall \gamma \prec \delta + 1 \, A(\gamma)$.

Lemma 8. *For any* $\Gamma \cup \Delta \subseteq \forall_1$, *if* $\mathrm{TI}(\forall_1, \prec) \vdash \bigwedge \Gamma \to \bigvee \Delta$, *then* $\Gamma \Rightarrow \Delta$ *has a* $\mathbf{G}_1$-*proof only consisting of* $\forall_1$-*formulas.*

Proof. By a $\forall_1$-proof in $\mathbf{G}_1$ (resp. **LPV**), we mean a proof in $\mathbf{G}_1$ (resp. **LPV**) consisting only of $\forall_1$-formulas. We show that the cut rule and the induction rule (over $\forall_1$-formulas) are derivable in $\mathbf{G}_1$ (by a $\forall_1$-proof). We only investigate the harder case of $\forall_1$-proofs. The other is the same omitting the restrictions everywhere.

For cut, consider the following proof-tree in $\mathbf{G}_1$, where the double lines mean simple omitted proofs in $\mathbf{G}_1$. The tree proves $\Gamma, \Sigma \Rightarrow \Lambda, \Delta$ from $\Gamma \Rightarrow A, \Delta$ and $\Sigma, A \Rightarrow \Lambda$.

$$\frac{\dfrac{\overline{\overline{\dfrac{\overline{\overline{\Sigma \Rightarrow \bigwedge \Sigma}}}{\Gamma, \Sigma \Rightarrow \bigwedge \Sigma, \Delta}} \quad \dfrac{\Gamma \Rightarrow A, \Delta}{\Gamma, \Sigma \Rightarrow A, \Delta}}{\dfrac{\Gamma, \Sigma \Rightarrow A \wedge \bigwedge \Sigma, \Delta}{\overline{\Gamma, \Sigma \Rightarrow (A \wedge \bigwedge \Sigma) \vee \bigvee \Delta}}} \qquad \dfrac{\dfrac{\overline{\overline{\dfrac{\overline{\overline{\bigvee \Delta \Rightarrow \Delta}}}{\bigvee \Delta \Rightarrow \Lambda, \Delta}}} \quad \dfrac{\dfrac{\Sigma, A \Rightarrow \Lambda}{A \wedge \bigwedge \Sigma \Rightarrow \Lambda}}{A \wedge \bigwedge \Sigma \Rightarrow \Lambda, \Delta}}{(A \wedge \bigwedge \Sigma) \vee \bigvee \Delta \Rightarrow \Lambda, \Delta}}{\Gamma, \Sigma \Rightarrow \Lambda, \Delta} \; wCut$$

Note that the simulation of the cut rule in $\mathbf{G}_1$ implies that $\mathbf{G}_1$ is as powerful as **LPV**. It also transforms a $\forall_1$-proof in **LPV** to a $\forall_1$-proof in $\mathbf{G}_1$. For the induction rule, consider the following proof-tree proving $\Gamma \Rightarrow A(\theta), \Delta$ from $\Gamma, \forall \gamma \prec \delta \, A(\gamma) \Rightarrow A(\delta), \Delta$:

$$\dfrac{\dfrac{}{\Gamma, \bigvee \Delta \Rightarrow [A(\delta) \vee \bigvee \Delta]} \quad \dfrac{\Gamma, \forall \gamma \prec \delta\, A(\gamma) \Rightarrow A(\delta), \Delta}{\Gamma, \forall \gamma \prec \delta\, A(\gamma) \Rightarrow [A(\delta) \vee \bigvee \Delta]}}{\dfrac{\dfrac{\Gamma, [\forall \gamma \prec \delta\, A(\gamma)] \vee \bigvee \Delta \Rightarrow [A(\delta) \vee \bigvee \Delta]}{\dfrac{\Gamma, \forall \gamma \prec \delta\, [A(\gamma) \vee \bigvee \Delta] \Rightarrow [A(\delta) \vee \bigvee \Delta]}{\dfrac{\Gamma, \forall \gamma \prec \delta\, [A(\gamma) \vee \bigvee \Delta] \Rightarrow \forall \gamma \prec \delta + 1\, [A(\gamma) \vee \bigvee \Delta]}{\dfrac{\Gamma \Rightarrow A(\theta) \vee \bigvee \Delta}{\Gamma \Rightarrow A(\theta), \Delta} \; \dagger} \; wInd} \; {*}{*}} \; {*}}$$

where ($*$) is the result of a cut with the sequent $\forall \gamma \prec \delta\, [A(\gamma) \vee \bigvee \Delta] \Rightarrow [\forall \gamma \prec \delta\, A(\gamma)] \vee \bigvee \Delta$ which has a proof in **LPV** and hence a $\forall_1$-proof in **LPV** and by the observation we have just made, a $\forall_1$-proof in $\mathbf{G}_1$. Note that the use of cut is allowed as we showed its derivability in $\mathbf{G}_1$. Moreover, ($**$) is the result of a cut with the PV-provable sequent $[A(\delta) \vee \bigvee \Delta], \forall \gamma \prec \delta\, [A(\gamma) \vee \bigvee \Delta] \Rightarrow \forall \gamma \prec \delta + 1\, [A(\gamma) \vee \bigvee \Delta]$. The latter is provable in **LPV**. Therefore, it has a $\forall_1$-proof in **LPV** and hence in $\mathbf{G}_1$. Finally, $\dagger$ is the result of a cut with $A(\theta) \vee \bigvee \Delta \Rightarrow A(\theta), \Delta$ that has a trivial $\forall_1$-proof. $\qquad \square$

4.2 Ordinal flows

In this subsection, we will witness $TI(\forall_1, \prec)$-provable implications between $\forall_1$-formulas by a sequence of β many PV-provable implications, for some $\beta \in \mathcal{O}$.

Definition 9. *Let $A(\bar{x}), B(\bar{x}) \in \forall_1$. A pair $(H(\gamma, \bar{x}), \beta)$ of a $\forall_1$-formula and $\beta \in \mathcal{O}$ such that $\beta \succeq 1$ is called an α-flow from $A(\bar{x})$ to $B(\bar{x})$, if:*

- $PV \vdash A(\bar{x}) \leftrightarrow H(0, \bar{x})$.
- $PV \vdash \forall\, 1 \preceq \delta \preceq \beta\, [\forall \gamma \prec \delta\, H(\gamma, \bar{x}) \rightarrow H(\delta, \bar{x})]$.
- $PV \vdash H(\beta, \bar{x}) \leftrightarrow B(\bar{x})$.

We denote the existence of an α-flow from $A(\bar{x})$ to $B(\bar{x})$ by $A(\bar{x}) \rhd_\alpha B(\bar{x})$. For any multisets Γ and Δ of $\forall_1$-formulas, by $\Gamma \rhd_\alpha \Delta$, we mean $\bigwedge \Gamma \rhd_\alpha \bigvee \Delta$.

In order to use α-flows to witness the proofs in $TI(\forall_1, \prec)$, we will develop a high-level calculus for this new notion, implemented in the following series of lemmas.

Lemma 10. *Let $A(\bar{x}), B(\bar{x}), C(\bar{x}) \in \forall_1$. Then:*

(i) If $PV \vdash A(\bar{x}) \rightarrow B(\bar{x})$, then $A(\bar{x}) \rhd_\alpha B(\bar{x})$.

(ii) If $A(\bar{x}) \rhd_\alpha B(\bar{x})$, then $A(\bar{x}) \circ C(\bar{x}) \rhd_\alpha B(\bar{x}) \circ C(\bar{x})$, for any $\circ \in \{\wedge, \vee\}$.

Proof. For (*i*), set $\beta = 1$ and $H(\gamma, \bar{x}) = (\gamma = 0 \rightarrow A(\bar{x})) \wedge (\gamma = 1 \rightarrow B(\bar{x}))$. It is clear that $PV \vdash H(0, \bar{x}) \leftrightarrow A(\bar{x})$ and $PV \vdash H(1, \bar{x}) \leftrightarrow B(\bar{x})$. As $PV \vdash A(\bar{x}) \rightarrow B(\bar{x})$, we can see that $(H(\gamma, \bar{x}), \beta)$ is an α-flow from $A(\bar{x})$ to $B(\bar{x})$.

For (*ii*), we only prove the conjunction case. The disjunction case is similar. Since $A(\bar{x}) \rhd_\alpha B(\bar{x})$, by Definition 9, there exist an ordinal $\beta \succeq 1$ and a formula $H(\gamma, \bar{x}) \in \forall_1$ satisfying the conditions in Definition 9. Set $I(\gamma, \bar{x}) = H(\gamma, \bar{x}) \wedge C(\bar{x})$ and note that $I(\gamma, \bar{x}) \in \forall_1$. It is easy to see that the pair $(I(\gamma, \bar{x}), \beta)$ is an α-flow from $A(\bar{x}) \wedge C(\bar{x})$ to $B(\bar{x}) \wedge C(\bar{x})$, as the PV-provability of $\forall\, 1 \preceq \delta \preceq \beta\, [\forall \gamma \prec \delta\, H(\gamma, \bar{x}) \rightarrow H(\delta, \bar{x})]$ implies the PV-provability of $\forall\, 1 \preceq \delta \preceq \beta\, [\forall \gamma \prec \delta\, (H(\gamma, \bar{x}) \wedge C(\bar{x})) \rightarrow (H(\delta, \bar{x}) \wedge C(\bar{x}))]$. $\qquad \square$
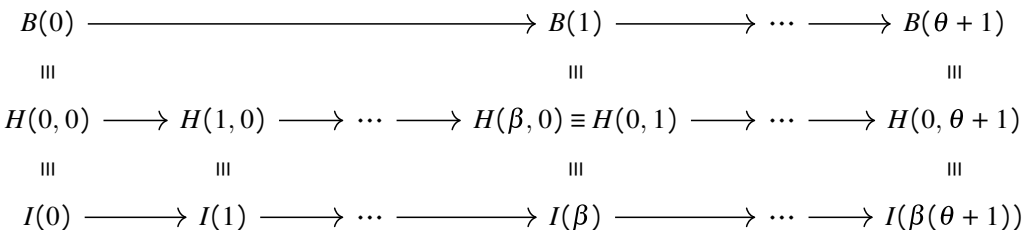
In the next lemma, we glue α-flows together to construct longer α-flows. Notice that the proof heavily uses the fact that the operations $\{+, \dot{-}, \cdot, d\}$ and their basic properties are representable in PV.

Lemma 11.

(i) *If $A(\bar{x}) \rhd_\alpha B(\bar{x})$ and $B(\bar{x}) \rhd_\alpha C(\bar{x})$, then $A(\bar{x}) \rhd_\alpha C(\bar{x})$.*

(ii) *If $\Gamma, \forall \gamma \prec \delta \, A(\gamma, \bar{x}) \rhd_\alpha \forall \gamma \prec \delta + 1 \, A(\gamma, \bar{x})$, then $\Gamma \rhd_\alpha A(\theta, \bar{x})$, for any $\theta \in \mathcal{O}$.*

Proof. For (i), as $A(\bar{x}) \rhd_\alpha B(\bar{x})$, there exists an α-flow $(H(\gamma, \bar{x}), \beta)$ from $A(\bar{x})$ to $B(\bar{x})$. Similarly, as $B(\bar{x}) \rhd_\alpha C(\bar{x})$, there is an α-flow $(H'(\gamma, \bar{x}), \beta')$ from $B(\bar{x})$ to $C(\bar{x})$. Set $\beta'' = \beta + \beta'$ and $H''(\gamma, \bar{x}) = [\gamma \preceq \beta \to H(\gamma, \bar{x})] \wedge [\beta \prec \gamma \preceq \beta + \beta' \to H'(\gamma \dot{-} \beta, \bar{x})]$. We claim that the pair $(H''(\gamma, \bar{x}), \beta'')$ is an α-flow from $A(\bar{x})$ to $C(\bar{x})$. First, note that $H''(0, \bar{x})$ is PV-equivalent to $H(0, \bar{x})$ which is PV-equivalent to $A(\bar{x})$. Similarly, as $(\beta + \beta') \dot{-} \beta = \beta'$ is provable in PV, we know that $H''(\beta + \beta', \bar{x})$ is PV-equivalent to $H'(\beta', \bar{x})$ which is PV-equivalent to $C(\bar{x})$. To prove $\text{PV} \vdash \forall 1 \preceq \delta \preceq \beta'' \, [\forall \gamma \prec \delta \, H''(\gamma, \bar{x}) \to H''(\delta, \bar{x})]$, note that if $\delta \preceq \beta$, then the claim reduces to the same claim for $H(\gamma, \bar{x})$ which is provable. If $\beta \prec \delta \preceq \beta + \beta'$, assume $\forall \gamma \prec \delta \, H''(\gamma, \bar{x})$ to prove $H''(\delta, \bar{x})$ or equivalently $H'(\delta \dot{-} \beta, \bar{x})$. Note that $\forall \gamma \prec \delta \, H''(\gamma, \bar{x})$ implies $\forall \beta \preceq \gamma \prec \delta \, H''(\gamma, \bar{x})$. As the interval $(0, \delta \dot{-} \beta)$ is isomorphic to (β, δ), by the map $\gamma \mapsto \beta + \gamma$, then $\forall \beta \preceq \gamma \prec \delta \, H''(\gamma, \bar{x})$ implies $\forall 0 \prec \gamma \prec \delta \dot{-} \beta \, H''(\beta + \gamma, \bar{x})$ which implies $\forall 0 \prec \gamma \prec \delta \dot{-} \beta \, H'(\gamma, \bar{x})$. On the other hand, $\forall \beta \preceq \gamma \prec \delta \, H''(\gamma, \bar{x})$ implies $H''(\beta, \bar{x})$ which is PV-equivalent to $H(\beta, \bar{x})$, by definition. As $H(\beta, \bar{x})$ is PV-equivalent to $B(\bar{x})$ which is also PV-equivalent to $H'(0, \bar{x})$, we can claim that $H(\beta, \bar{x})$ and $H'(0, \bar{x})$ are PV-equivalent. Hence, $\forall \beta \preceq \gamma \prec \delta \, H''(\gamma, \bar{x})$ implies $\forall \gamma \prec \delta \dot{-} \beta \, H'(\gamma, \bar{x})$ which also implies $H'(\delta \dot{-} \beta, \bar{x})$, as $(H'(\gamma, \bar{x}), \beta')$ is an α-flow.

For (ii), as $\bigwedge \Gamma \wedge \forall \gamma \prec \delta \, A(\gamma, \bar{x}) \rhd_\alpha \forall \gamma \prec \delta + 1 \, A(\gamma, \bar{x})$, by Lemma 10, we have $\bigwedge \Gamma \wedge \forall \gamma \prec \delta \, A(\gamma, \bar{x}) \rhd_\alpha \bigwedge \Gamma \wedge \forall \gamma \prec \delta + 1 \, A(\gamma, \bar{x})$. Set $B(\delta, \bar{x}) = \bigwedge \Gamma \wedge \forall \gamma \prec \delta \, A(\gamma, \bar{x})$. Therefore, $B(\delta, \bar{x}) \rhd_\alpha B(\delta + 1, \bar{x})$. Let $(H(\eta, \delta, \bar{x}), \beta)$ be the α-flow from $B(\delta, \bar{x})$ to $B(\delta + 1, \bar{x})$. Note that $H(0, \delta, \bar{x})$ is PV-equivalent to $B(\delta, \bar{x})$ and $H(\beta, \delta, \bar{x})$ is PV-equivalent to $H(0, \delta + 1, \bar{x})$, as both are PV-equivalent to $B(\delta + 1)$. Define $\beta' = \beta(\theta + 1)$ and $I(\tau, \bar{x}) = H(\tau \dot{-} \beta d(\tau, \beta), d(\tau, \beta), \bar{x})$ and note that $I(\tau, \bar{x}) \in \forall_1$. We show that $(I(\tau, \bar{x}), \beta')$ is an α-flow from $B(0, \bar{x})$ to $B(\theta + 1, \bar{x})$. Note that $(I(\tau, \bar{x}), \beta')$ is nothing but the result of gluing the α-flows $(H(\eta, \delta, \bar{x}), \beta)$, for all $\delta \prec \theta + 1$, one after another as depicted in the following figure (for simplicity, in the figures, we drop the free variables $\bar{x}$).

$$
\begin{array}{ccccccc}
B(0) & \longrightarrow & B(1) & \longrightarrow & \cdots & \longrightarrow & B(\theta+1) \\
\| \| \| & & \| \| \| & & & & \| \| \| \\
H(0,0) \longrightarrow H(1,0) \longrightarrow \cdots \longrightarrow H(\beta,0) \equiv H(0,1) & \longrightarrow \cdots \longrightarrow & H(0,\theta+1) \\
\| \| \| & & \| \| \| & & & & \| \| \| \\
I(0) \longrightarrow I(1) \longrightarrow \cdots \longrightarrow I(\beta) & \longrightarrow & \cdots \longrightarrow & I(\beta(\theta+1))
\end{array}
$$

First, as $d(0, \beta) = 0$ and $0 \dot{-} \beta d(0, \beta) = 0$, provably in PV, we know that $I(0, \bar{x})$ is PV-equivalent to $H(0, 0, \bar{x})$ which is itself PV-equivalent to $B(0, \bar{x})$. Second, as $d(\beta(\theta + 1), \beta) = \theta + 1$ and $\beta(\theta + 1) \dot{-} \beta d(\beta(\theta + 1), \beta) = 0$, provably in PV, we know that $I(\beta(\theta + 1), \bar{x})$ is PV-equivalent to $H(0, \theta + 1, \bar{x})$ which is PV-equivalent to $B(\theta + 1, \bar{x})$. For the middle condition, we must prove $\text{PV} \vdash \forall 1 \preceq \tau \preceq \beta(\theta + 1) \, [\forall \zeta \prec \tau \, I(\zeta, \bar{x}) \to I(\tau, \bar{x})]$. There are two cases to consider, either $\beta d(\tau, \beta) \prec \tau$ or $\beta d(\tau, \beta) = \tau$. If $\beta d(\tau, \beta) \prec \tau$, then $\beta d(\tau, \beta) + 1 \preceq \tau$ which implies $\tau = \beta d(\tau, \beta) + \mu$ for $\mu = \tau \dot{-} \beta d(\tau, \beta) \succeq 1$. As for any $\eta \prec \mu$, we have $\beta d(\tau, \beta) + \eta \prec \tau$, we know that $\forall \zeta \prec \tau \, I(\zeta, \bar{x})$

implies $\forall \eta \prec \mu\, H(\eta, d(\tau, \beta), \bar{x})$. As we have $\mu \succeq 1$, the latter proves $H(\mu, d(\tau, \beta), \bar{x})$ which is PV-equivalent to $I(\tau, \bar{x})$.

$$\cdots \longrightarrow H(0, d(\tau, \beta)) \longrightarrow H(1, d(\tau, \beta)) \longrightarrow \cdots \longrightarrow H(\mu, d(\tau, \beta)) \longrightarrow \cdots$$
$$\|\| \qquad\qquad \|\| \qquad\qquad \|\|$$
$$\cdots \longrightarrow I(\beta d(\tau, \beta)) \longrightarrow I(\beta d(\tau, \beta) + 1) \longrightarrow \cdots \longrightarrow I(\tau) \longrightarrow \cdots$$

For the other case, if $\beta d(\tau, \beta) = \tau$, we should use $\forall \zeta \prec \tau\, I(\zeta, \bar{x})$ to prove the formula $I(\tau, \bar{x}) = H(0, d(\tau, \beta), \bar{x})$. Again, there are two cases to consider: either $d(\tau, \beta)$ is a successor or a limit ordinal. If $d(\tau, \beta) = \rho + 1$, for some ρ, as $H(0, \rho + 1, \bar{x})$ is PV-equivalent to $H(\beta, \rho, \bar{x})$, it is enough to prove $H(\beta, \rho, \bar{x})$. As $\beta\rho + \eta \prec \beta\rho + \beta = \beta(\rho + 1) = \tau$, for any $\eta \prec \beta$, we know that $\forall \zeta \prec \tau\, I(\zeta, \bar{x})$ implies $\forall \eta \prec \beta\, H(\eta, \rho, \bar{x})$ which implies $H(\beta, \rho, \bar{x})$.

$$\cdots \longrightarrow H(0, \rho) \longrightarrow H(1, \rho) \longrightarrow \cdots \longrightarrow H(\beta, \rho) \equiv H(0, \rho + 1) \longrightarrow \cdots$$
$$\|\| \qquad\qquad \|\| \qquad\qquad \|\|$$
$$\cdots \longrightarrow I(\beta\rho) \longrightarrow I(\beta\rho + 1) \longrightarrow \cdots \longrightarrow I(\beta(\rho + 1)) \longrightarrow \cdots$$

If $d(\tau, \beta)$ is a limit ordinal, then $\forall \zeta \prec \beta d(\tau, \beta)\, I(\zeta, \bar{x})$ implies the formula $\forall \delta \prec d(\tau, \beta)\, H(0, \delta, \bar{x})$ which implies $\forall \delta \prec d(\tau, \beta) B(\delta, \bar{x})$. The latter is $\forall \delta \prec d(\tau, \beta)[\bigwedge \Gamma \wedge \forall \gamma \prec \delta\, A(\gamma, \bar{x})]$ that implies $\bigwedge \Gamma \wedge \forall \gamma \prec d(\tau, \beta)\, A(\gamma, \bar{x})$, as $d(\tau, \beta)$ is a limit ordinal. The latter is PV-equivalent to $H(0, d(\tau, \beta), \bar{x}) = I(\tau, \bar{x})$. This completes the proof of the claim and shows that $B(0, \bar{x}) \rhd_\alpha B(\theta + 1, \bar{x})$. Now, as $\text{PV} \vdash \bigwedge \Gamma \to (\bigwedge \Gamma \wedge \forall \gamma \prec 0\, A(\gamma, \bar{x}))$ and $\text{PV} \vdash (\bigwedge \Gamma \wedge \forall \gamma \prec \theta + 1\, A(\gamma, \bar{x})) \to A(\theta, \bar{x})$, by Lemma 10, we have $\bigwedge \Gamma \rhd_\alpha \bigwedge \Gamma \wedge \forall \gamma \prec 0\, A(\gamma, \bar{x})$ and $\bigwedge \Gamma \wedge \forall \gamma \prec \theta + 1\, A(\gamma, \bar{x}) \rhd_\alpha A(\theta, \bar{x})$. Hence, by part (i), we have $\bigwedge \Gamma \rhd_\alpha A(\theta, \bar{x})$ which completes the proof. □

Lemma 12. *(Conjunction and Disjunction Rules)*

(i) *If* $\Gamma, A \rhd_\alpha \Delta$ *or* $\Gamma, B \rhd_\alpha \Delta$, *then* $\Gamma, A \wedge B \rhd_\alpha \Delta$.

(ii) *If* $\Gamma \rhd_\alpha A, \Delta$ *and* $\Gamma \rhd_\alpha B, \Delta$, *then* $\Gamma \rhd_\alpha A \wedge B, \Delta$.

(iii) *If* $\Gamma \rhd_\alpha A, \Delta$ *or* $\Gamma \rhd_\alpha B, \Delta$, *then* $\Gamma \rhd_\alpha A \vee B, \Delta$.

(iv) *If* $\Gamma, A \rhd_\alpha \Delta$ *and* $\Gamma, B \rhd_\alpha \Delta$, *then* $\Gamma, A \vee B \rhd_\alpha \Delta$.

Proof. For (i) and (iii), as the implications $[(\bigwedge \Gamma \wedge (A \wedge B)) \to (\bigwedge \Gamma \wedge A)]$, $[(\bigwedge \Gamma \wedge (A \wedge B)) \to (\bigwedge \Gamma \wedge B)]$, $[(\bigvee \Delta \vee A) \to (\bigvee \Delta \vee (A \vee B))]$ and $[(\bigvee \Delta \vee B) \to (\bigvee \Delta \vee (A \vee B))]$ are all provable in PV, using Lemmas 10 and 11, we reach what we wanted. For (ii), if $\Gamma \rhd_\alpha \Delta, A$ then $\bigwedge \Gamma \rhd_\alpha \bigvee \Delta \vee A$, by definition. By Lemma 10, we reach $\bigwedge \Gamma \rhd_\alpha (\bigvee \Delta \vee A) \wedge \bigwedge \Gamma$. Similarly, we have $\bigwedge \Gamma \rhd_\alpha \bigvee \Delta \vee B$ and by Lemma 10, we reach $\bigwedge \Gamma \wedge (\bigvee \Delta \vee A) \rhd_\alpha (\bigvee \Delta \vee B) \wedge (\bigvee \Delta \vee A)$. Therefore, $\bigwedge \Gamma \rhd_\alpha (\bigvee \Delta \vee B) \wedge (\bigvee \Delta \vee A)$, by part (i) in Lemma 11. Finally, as $(\bigvee \Delta \vee B) \wedge (\bigvee \Delta \vee A) \to \bigvee \Delta \vee (A \wedge B)$ is provable in PV, by Lemmas 10 and 11, we reach $\bigwedge \Gamma \rhd_\alpha \bigvee \Delta \vee (A \wedge B)$. The proof for (iv) is similar. □

Having the required lemmas, we are now ready to prove the following theorem as the main extraction technique that witnesses the proofs in $\text{TI}(\forall_1, \prec)$ by α-flows.

Theorem. *Let* $\Gamma(\bar{x}) \cup \Delta(\bar{x}) \subseteq \forall_1$. *Then,* $\text{TI}(\forall_1, \prec) \vdash \bigwedge \Gamma(\bar{x}) \to \bigvee \Delta(\bar{x})$ *iff* $\Gamma(\bar{x}) \rhd_\alpha \Delta(\bar{x})$.

Proof. We first prove the easier direction. Assume $\Gamma(\bar{x}) \rhd_\alpha \Delta(\bar{x})$ and the pair $(H(\gamma, \bar{x}), \beta)$ is an α-flow from $\bigwedge \Gamma(\bar{x})$ to $\bigvee \Delta(\bar{x})$. As $\mathrm{PV} \vdash \forall\, 1 \leq \delta \leq \beta\, [\forall\gamma \prec \delta\, H(\gamma, \bar{x}) \to H(\delta, \bar{x})]$ and $\mathrm{TI}(\forall_1, \prec)$ extends PV, we have

$$\mathrm{TI}(\forall_1, \prec) \vdash \forall\, 1 \leq \delta \leq \beta\, [\forall\gamma \prec \delta\, H(\gamma, \bar{x}) \to H(\delta, \bar{x})].$$

Then, as $H(\gamma, \bar{x}) \in \forall_1$, by the transfinite induction in $\mathrm{TI}(\forall_1, \prec)$, we reach $\mathrm{TI}(\forall_1, \prec) \vdash H(0, \bar{x}) \to H(\beta, \bar{x})$. Finally, using the PV-provable equivalences $\bigwedge \Gamma(\bar{x}) \leftrightarrow H(0, \bar{x})$ and $H(\beta, \bar{x}) \leftrightarrow \bigvee \Delta(\bar{x})$, we reach $\mathrm{TI}(\forall_1, \prec) \vdash \bigwedge \Gamma(\bar{x}) \to \bigvee \Delta(\bar{x})$.

For the other direction, assume $\mathrm{TI}(\forall_1, \prec) \vdash \bigwedge \Gamma(\bar{x}) \to \bigvee \Delta(\bar{x})$. By Lemma 8, $\Gamma(\bar{x}) \Rightarrow \Delta(\bar{x})$ has a $\mathbf{G}_1$-proof only consisting of $\forall_1$-formulas. By induction on this proof, we show that for any sequent $\Sigma \Rightarrow \Lambda$ in the proof, we have $\Sigma \rhd_\alpha \Lambda$.

For the axioms, as they are provable in PV, using Lemma 10, there is nothing to prove. The case of structural rules (except for the weak cut) is easy. Weak cut and weak induction are addressed in Lemma 11. The conjunction and disjunction rules are proved in Lemma 12. For the right universal quantifier rule, if $\Sigma(\bar{x}) \Rightarrow \Lambda(\bar{x}), \forall z B(\bar{x}, z)$ is proved from $\Sigma(\bar{x}) \Rightarrow \Lambda(\bar{x}), B(\bar{x}, z)$, then by induction hypothesis, $\Sigma(\bar{x}) \rhd_\alpha \Lambda(\bar{x}), B(\bar{x}, z)$. Therefore, there exists an α-flow $(H(\gamma, \bar{x}, z), \beta)$ from $\bigwedge \Sigma(\bar{x})$ to $B(\bar{x}, z) \vee \bigvee \Lambda(\bar{x})$. Define $I(\gamma, \bar{x}) = \forall z H(\gamma, \bar{x}, z)$ and note that $I(\bar{x}, z) \in \forall_1$. It is easy to see that $(I(\gamma, \bar{x}), \beta)$ is an α-flow from $\forall z[\bigwedge \Sigma(\bar{x})]$ to $\forall z[B(\bar{x}, z) \vee \bigvee \Lambda(\bar{x})]$, as PV-provability of $\forall\gamma \prec \delta H(\gamma, z, \bar{x}) \to H(\delta, z, \bar{x})$ implies the PV-provability of $\forall\gamma \prec \delta \forall z H(\gamma, z, \bar{x}) \to \forall z H(\delta, z, \bar{x})$. Finally, as z does not occur as a free variable in $\Sigma(\bar{x}) \cup \Lambda(\bar{x})$, we have the PV-equivalence between $\forall z[\bigwedge \Sigma(\bar{x})]$ and $\bigwedge \Sigma(\bar{x})$ and similarly between $\forall z[B(\bar{x}, z) \vee \bigvee \Lambda(\bar{x})]$ and $\bigvee \Lambda(\bar{x}) \vee \forall z B(\bar{x}, z)$. Using Lemmas 10 and 11, we can prove $\bigwedge \Sigma(\bar{x}) \rhd_\alpha \bigvee \Lambda(\bar{x}) \vee \forall z B(\bar{x}, z)$. For the left universal quantifier rule, if $\Sigma(\bar{x}), \forall z B(\bar{x}, z) \Rightarrow \Lambda(\bar{x})$ is proved from $\Sigma(\bar{x}), B(\bar{x}, s(\bar{x})) \Rightarrow \Lambda(\bar{x})$, then by induction hypothesis $\Sigma(\bar{x}), B(\bar{x}, s(\bar{x})) \rhd_\alpha \Lambda(\bar{x})$. Since $\mathrm{PV} \vdash \bigwedge \Sigma(\bar{x}) \wedge \forall z B(\bar{x}, z) \to \bigwedge \Sigma(\bar{x}) \wedge B(\bar{x}, s(\bar{x}))$, by Lemmas 10 and 11, we reach $\Sigma(\bar{x}), \forall z B(\bar{x}, z) \rhd_\alpha \Lambda(\bar{x})$. $\qquad\square$

Corollary 13. *Let α be an ordinal with the ptime representation $\mathfrak{O}$. Then,* $\mathrm{PA} + \bigcup_{\beta \in \mathcal{O}} \mathrm{TI}(\prec_\beta) \vdash \bigwedge \Gamma(\bar{x}) \to \bigvee \Delta(\bar{x})$ *iff* $\Gamma(\bar{x}) \rhd_\alpha \Delta(\bar{x})$, *for* $\Gamma(\bar{x}) \cup \Delta(\bar{x}) \subseteq \forall_1$.

Proof. As any implication in the form $\bigwedge \Gamma(\bar{x}) \to \bigvee \Delta(\bar{x})$ is logically equivalent to a Π_2^0-formula, the claim is a consequence of Theorem 4 and Corollary 7. $\qquad\square$

Corollary 14. *Let $\mathfrak{O}_0$ be the ptime representation for ε_0 introduced in Subsection 3.1. Then,* $\mathrm{PA} \vdash \bigwedge \Gamma(\bar{x}) \to \bigvee \Delta(\bar{x})$ *iff* $\Gamma(\bar{x}) \rhd_{\varepsilon_0} \Delta(\bar{x})$, *for* $\Gamma(\bar{x}) \cup \Delta(\bar{x}) \subseteq \forall_1$.

4.3 Ordinal local search programs

In this subsection, we will first introduce the notion of an ordinal local search program as a formalized version of the transfinite ptime modifications over an initial ptime value that we explained before. We will then use these programs to witness some provable statements in the theory $\mathrm{PA} + \bigcup_{\beta \in \mathcal{O}} \mathrm{TI}(\prec_\beta)$.

Definition 15. *Let T be a theory over the language $\mathcal{L}_{\mathrm{PV}}$. A total search problem of T is a quantifier-free formula $A(\bar{x}, \bar{y})$ such that $T \vdash \forall \bar{x} \exists \bar{y} A(\bar{x}, \bar{y})$. A total search problem is called an NP-search problem, if there are sequences of polynomials $\bar{r}$ such that $\mathrm{PV} \vdash A(\bar{x}, \bar{y}) \to |\bar{y}| \leq \bar{r}(|\bar{x}|)$, where $|\bar{y}| \leq \bar{r}(|\bar{x}|)$ is an abbreviation for $\bigwedge_i (|y_i| \leq r_i(|\bar{x}|))$. We denote the class of all these total search (resp., NP-search) problems of T by $\mathrm{TSP}(T)$ (resp. $\mathrm{TFNP}(T)$).*

Definition 16. *Let α be an ordinal, $\mathfrak{O}$ be its ptime representation, $A(\bar{x}, \bar{y})$ be a quantifier-free formula in $\mathcal{L}_{\mathrm{PV}}$ and $\beta \in \mathcal{O}$. By an $\mathrm{LS}(\preceq_\beta)$-program for $A(\bar{x}, \bar{y})$, we mean the following data: an initial*

sequence of $\mathcal{L}_{PV}$-terms $\bar{i}(\bar{x})$, a quantifier-free $\mathcal{L}_{PV}$-formula $G(\gamma, \bar{x}, \bar{z})$, a sequence of $\mathcal{L}_{PV}$-terms $\bar{N}(\gamma, \bar{x}, \bar{z})$, an $\mathcal{L}_{PV}$-term $q(\gamma, \bar{x}, \bar{z})$, a sequence of $\mathcal{L}_{PV}$-terms $\bar{p}(\bar{x}, \bar{z})$, such that:

- $PV \vdash G(\beta, \bar{x}, \bar{i}(\bar{x}))$,
- $PV \vdash \gamma \neq 0 \rightarrow q(\gamma, \bar{x}, \bar{z}) \prec \gamma$,
- $PV \vdash \gamma \neq 0 \rightarrow [G(\gamma, \bar{x}, \bar{z}) \rightarrow G(q(\gamma, \bar{x}, \bar{z}), \bar{x}, \bar{N}(\gamma, \bar{x}, \bar{z}))]$,
- $PV \vdash G(0, \bar{x}, \bar{z}) \rightarrow A(\bar{x}, \bar{p}(\bar{x}, \bar{z}))$.

By LS($\preceq_\beta$), we mean the class of all formulas $A(\bar{x}, \bar{y})$ for which there exists a LS($\preceq_\beta$)-program. By PLS($\preceq_\beta$), we mean the class LS($\preceq_\beta$) $\cap$ TFNP($Th(\mathbb{N})$).

Membership $A(\bar{x}, \bar{y}) \in$ LS($\preceq_\beta$) implies $\forall \bar{x} \exists \bar{y} A(\bar{x}, \bar{y})$ and the LS($\preceq_\beta$)-program actually provides an algorithm to compute $\bar{y}$ from $\bar{x}$. To see this, denote $G(\gamma, \bar{x}, \bar{z})$ by G_γ. The algorithm starts at the level β with an initial value $\bar{i}(\bar{x})$ satisfying the property G_β. Then, using the feasible function q, it finds a lower level to go to and uses the modification $\bar{N}$ to update any value with the property G_γ to a value satisfying the property $G_{q(\gamma)}$. Finally, reaching the zeroth level, the algorithm uses $\bar{p}$ to compute $\bar{y}$ satisfying A from any value with the property G_0.

The next theorem uses LS($\preceq_\beta$)-programs (PLS($\preceq_\beta$)-programs) to witness the total search (NP-search) problems of PA $+ \bigcup_{\beta \in \mathcal{O}}$ TI($\prec_\beta$). The idea is using Herbrand's theorem, Theorem 2, applied on PV to push the data extraction of Corollary 13 a bit further to reach an ordinal local search program for total search problems.

Theorem. *Let α be an ordinal with the ptime representation $\mathfrak{O}$. Then TSP(PA $+ \bigcup_{\beta \in \mathcal{O}}$ TI($\prec_\beta$)) $= \bigcup_{\beta \in \mathcal{O}}$ LS($\preceq_\beta$) and TFNP(PA $+ \bigcup_{\beta \in \mathcal{O}}$ TI($\prec_\beta$)) $= \bigcup_{\beta \in \mathcal{O}}$ PLS($\preceq_\beta$).*

Proof. We only prove the first equality. The second is just a consequence. For the first direction, assume that $A(\bar{x}, \bar{y})$ has a LS($\preceq_\beta$)-program. Set $H(\gamma, \bar{x}) = \forall \bar{z} \neg G(\gamma, \bar{x}, \bar{z}) \wedge \forall \bar{y} \neg A(\bar{x}, \bar{y})$ and note that $H \in \forall_1$. We claim that $(H(\gamma, \bar{x}), \beta)$ is an α-flow from $\forall \bar{y} \neg A(\bar{x}, \bar{y})$ to $\perp$. First, as $PV \vdash G(0, \bar{x}, \bar{z}) \rightarrow A(\bar{x}, \bar{p}(\bar{x}, \bar{z}))$, we have $PV \vdash \forall \bar{y} \neg A(\bar{x}, \bar{y}) \rightarrow \forall \bar{z} \neg G(0, \bar{x}, \bar{z})$ and hence $PV \vdash \forall \bar{y} \neg A(\bar{x}, \bar{y}) \leftrightarrow H(0, \bar{x})$. Second, as $PV \vdash G(\beta, \bar{x}, \bar{i}(\bar{x}))$, we reach $PV \vdash \forall \bar{z} \neg G(\beta, \bar{x}, \bar{z}) \leftrightarrow \perp$ and hence $PV \vdash \perp \leftrightarrow H(\beta, \bar{x})$. Finally, using $PV \vdash \gamma \neq 0 \rightarrow q(\gamma, \bar{x}, \bar{z}) \prec \gamma$, and

$$PV \vdash \gamma \neq 0 \rightarrow [G(\gamma, \bar{x}, \bar{z}) \rightarrow G(q(\gamma, \bar{x}, \bar{z}), \bar{x}, \bar{N}(\gamma, \bar{x}, \bar{z}))],$$

it is easy to see that

$$PV \vdash \forall 1 \preceq \delta \preceq \beta \, [\neg G(q(\delta, \bar{x}, \bar{z}), \bar{x}, \bar{N}(\delta, \bar{x}, \bar{z})) \rightarrow \neg G(\delta, \bar{x}, \bar{z})]$$

and hence we reach

$$PV \vdash \forall 1 \preceq \delta \preceq \beta \, [\forall \gamma \prec \delta \, \forall \bar{z} \neg G(\gamma, \bar{x}, \bar{z}) \rightarrow \forall \bar{z} \neg G(\delta, \bar{x}, \bar{z})].$$

The latter implies $PV \vdash \forall 1 \preceq \delta \preceq \beta \, [\forall \gamma \prec \delta \, H(\gamma, \bar{x}) \rightarrow H(\delta, \bar{x})]$. Therefore, $(H(\gamma, \bar{x}), \beta)$ is an α-flow from $\forall \bar{y} \neg A(\bar{x}, \bar{y})$ to $\perp$. Hence, PA $+ \bigcup_{\beta \in \mathcal{O}}$ TI($\prec_\beta$) $\vdash \forall \bar{y} \neg A(\bar{x}, \bar{y}) \rightarrow \perp$, by Corollary 13 and thus, we reach PA $+ \bigcup_{\beta \in \mathcal{O}}$ TI($\prec_\beta$) $\vdash \forall \bar{x} \exists \bar{y} A(\bar{x}, \bar{y})$. For the converse, assume that PA $+ \bigcup_{\beta \in \mathcal{O}}$ TI($\prec_\beta$) $\vdash \forall \bar{x} \exists \bar{y} A(\bar{x}, \bar{y})$, where $A(\bar{x}, \bar{y}) \in \mathcal{L}_{PV}$ is quantifier-free. As PA $+ \bigcup_{\beta \in \mathcal{O}}$ TI($\prec_\beta$) $\vdash \forall \bar{y} \neg A(\bar{x}, \bar{y}) \rightarrow \perp$, by Corollary 13, $\forall \bar{y} \neg A(\bar{x}, \bar{y}) \rhd_\alpha \perp$. Hence, there exist $H(\gamma, \bar{x}) \in \forall_1$ and $\beta \in \mathcal{O}$ such that $PV \vdash \forall \bar{y} \neg A(\bar{x}, \bar{y}) \leftrightarrow H(0, \bar{x})$, $PV \vdash H(\beta, \bar{x}) \leftrightarrow \perp$ and

$$PV \vdash \forall 1 \preceq \delta \preceq \beta \, [\forall \gamma \prec \delta \, H(\gamma, \bar{x}) \rightarrow H(\delta, \bar{x})].$$

As $H \in \forall_1$, there exists a quantifier-free formula $I(\gamma, \bar{x}, \bar{z})$ such that $H(\gamma, \bar{x})$ and $\forall \bar{z} I(\gamma, \bar{x}, \bar{z})$ are equivalent over PV. On the other hand, as the implications are provable in PV, we can witness

the existential quantifiers by ptime functions. Hence, there are $\mathcal{L}_{\mathrm{PV}}$-terms $\bar{Y}(\bar{x}, \bar{z})$, $\bar{Z}(\gamma, \bar{x}, \bar{z})$, $\Delta(\gamma, \bar{x}, \bar{z})$, and $\bar{W}(\bar{x})$ such that

- $\mathrm{PV} \vdash \neg A(\bar{x}, \bar{Y}(\bar{x}, \bar{z})) \rightarrow I(0, \bar{x}, \bar{z})$,
- $\mathrm{PV} \vdash I(\beta, \bar{x}, \bar{W}(\bar{x})) \rightarrow \bot$,
- $\mathrm{PV} \vdash \forall 1 \preceq \delta \preceq \beta \, [[(\Delta(\delta, \bar{x}, \bar{z}) \prec \delta \rightarrow I(\Delta(\delta, \bar{x}, \bar{z}), \bar{x}, \bar{Z}(\delta, \bar{x}, \bar{z}))] \rightarrow I(\delta, \bar{x}, \bar{z})]$.

Define $G(\delta, \bar{x}, \bar{z}) = \neg I(\delta, \bar{x}, \bar{z}) \wedge (\delta \preceq \beta)$,

$$q(\delta, \bar{x}, \bar{z}) = \begin{cases} \Delta(\delta, \bar{x}, \bar{z}) & \neg I(\delta, \bar{x}, \bar{z}) \wedge (\delta \preceq \beta) \\ 0 & \text{otherwise} \end{cases}$$

$\bar{i}(\bar{x}) = \bar{W}(\bar{x})$ and $\bar{p}(\bar{x}, \bar{z}) = \bar{Y}(\bar{x}, \bar{z})$. It is easy to see that this new data is an LS($\preceq_\beta$)-program for $A(\bar{x}, \bar{y})$. $\square$

Applying Theorem 5 to $\alpha = \varepsilon_0$, we reach the following Corollary, originally proved in Beckmann (2009).

Corollary 17. *Let $\mathfrak{O}_0$ be the ptime representation of the ordinal ε_0 introduced in Subsection 3.1. Then* $\mathrm{TSP(PA)} = \bigcup_{\beta \in \mathcal{O}_0} \mathrm{LS}(\preceq_\beta)$ *and* $\mathrm{TFNP(PA)} = \bigcup_{\beta \in \mathcal{O}_0} \mathrm{PLS}(\preceq_\beta)$.

5. *k*-Flows and Bounded Arithmetic

In this section, we will modify the method developed for the strong theories of arithmetic in Section 4 to also cover the bounded and hence weaker theories of arithmetic. The structure of the present section is similar to that of Section 4. After recalling the usual sequent calculi for the theories S_2^k and T_2^k in Subsection 5.1, the next subsection, Subsection 5.2 will be devoted to investigate a suitable version of a flow for bounded arithmetic called a *k*-flow. Roughly speaking, a *k*-flow is an exponentially long uniform sequence of PV-provable implications between $\mathcal{L}_{\mathrm{PV}}$-formulas in the class $\hat{\Pi}_k^b$. After proving some basic properties of *k*-flows, we will conclude the subsection by proving a witnessing theorem, transforming the proofs of the implications between $\hat{\Pi}_k^b$-formulas in S_2^k and T_2^k to some types of *k*-flows. Finally, in Subsection 5.3, we will introduce the appropriate notion of a local search program to witness the PV-provable implications further and find a complete witnessing for the theories S_2^k and T_2^k.

5.1 Sequent calculi for bounded arithmetic

To recall the usual sequent calculi for S_2^k and T_2^k, introduced in Buss (1986), first consider the following rules:

Bounded Quantifier Rules:

$$\frac{\Gamma, A(s) \Rightarrow \Delta}{\Gamma, s \leq t, \forall y \leq t \, A(y) \Rightarrow \Delta} \, L\forall^{\leq} \qquad \frac{\Gamma, z \leq t \Rightarrow A(z), \Delta}{\Gamma \Rightarrow \forall y \leq t \, A(y), \Delta} \, R\forall^{\leq}$$

$$\frac{\Gamma, z \leq t, A(z) \Rightarrow \Delta}{\Gamma, \exists y \leq t \, A(y) \Rightarrow \Delta} \, L\exists^{\leq} \qquad \frac{\Gamma \Rightarrow A(s), \Delta}{\Gamma, s \leq t, \Rightarrow \exists y \leq t \, A(y), \Delta} \, R\exists^{\leq}$$

Induction Rules:

$$\frac{\Gamma, A(\lfloor\frac{z}{2}\rfloor) \Rightarrow A(z), \Delta}{\Gamma, A(0) \Rightarrow A(t), \Delta} \ {}_{\text{PInd}_k} \qquad \frac{\Gamma, A(z) \Rightarrow A(z+1), \Delta}{\Gamma, A(0) \Rightarrow A(t), \Delta} \ {}_{\text{Ind}_k}$$

In the rules $(R\forall^{\leq})$ and $(L\exists^{\leq})$ as well as in the induction rules, the variable z should not appear in the consequence of the rule. Moreover, in the induction rules (PInd_k) and (Ind_k), the index k means that the formula $A(z)$ is restricted to the class $\hat{\Pi}_k^b$.

The system $\mathbf{LS_2^k}$ (resp. $\mathbf{LT_2^k}$) for S_2^k (resp. T_2^k) is defined as the system $\mathbf{LPV}$ plus the bounded quantifier rules and the rule (PInd_k) (resp. (Ind_k)). For some technical reasons, we prefer to work with the alternative systems where the cut and the induction rules are weakened. Define the system $\mathbf{wLS_2^k}$ (resp. $\mathbf{wLT_2^k}$) similar to $\mathbf{LS_2^k}$ (resp. $\mathbf{LT_2^k}$) with the difference that in the former the quantifier rules in $\mathbf{LPV}$ are *omitted* and the cut and the induction rule (PInd_k) (resp. (Ind_k)) are *replaced* by the weak cut and the weak induction rule (wPInd_k) (resp. (wInd_k)) depicted below:

$$\frac{\Gamma \Rightarrow A \qquad A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \ wCut \qquad \frac{\Gamma, A(\lfloor\frac{z}{2}\rfloor) \Rightarrow A(z)}{\Gamma, A(0) \Rightarrow A(s)} \ {}_{\text{wPInd}_k} \qquad \frac{\Gamma, A(z) \Rightarrow A(z+1)}{\Gamma, A(0) \Rightarrow A(s)} \ {}_{\text{wInd}_k}$$

In the weak induction rules, we have the similar constraints as before, namely that $A \in \hat{\Pi}_k^b$ and z does not appear in the consequence of the rules. Note that the only point modified in the weak induction rules is the missing context Δ.

The following theorem ensures that the system $\mathbf{wLS_2^k}$ (resp. $\mathbf{wLT_2^k}$) is complete for the sequents of $\hat{\Pi}_k^b$-formulas. Notice that the lemma does not claim the full completeness as the system $\mathbf{wLS_2^k}$ (resp. $\mathbf{wLT_2^k}$) is clearly weak to introduce any unbounded quantifier.

Lemma 18. *For any $\Gamma \cup \Delta \subseteq \hat{\Pi}_k^b$:*

- *If $S_2^k \vdash \bigwedge \Gamma \to \bigvee \Delta$, then $\Gamma \Rightarrow \Delta$ has a $\mathbf{wLS_2^k}$-proof only consisting of $\hat{\Pi}_k^b$-formulas.*
- *If $T_2^k \vdash \bigwedge \Gamma \to \bigvee \Delta$, then $\Gamma \Rightarrow \Delta$ has a $\mathbf{wLT_2^k}$-proof only consisting of $\hat{\Pi}_k^b$-formulas.*

Proof. It is a well-known consequence of the cut reduction theorem for $\mathbf{LS_2^k}$ (resp. $\mathbf{LT_2^k}$) that if $\bigwedge \Gamma \to \bigvee \Delta$ is provable in S_2^k (resp. T_2^k), it has a proof in $\mathbf{LS_2^k}$ (resp. $\mathbf{LT_2^k}$) only consisting of $\hat{\Pi}_k^b$-formulas and only using bounded quantifier rules instead of the usual unbounded quantifier rules in $\mathbf{LPV}$ (Buss 1986; Krajíček 1995). Therefore, the only thing remained to prove is simulating the cut and the induction rules over $\hat{\Pi}_k^b$-formulas by their weak versions applied over the same family of formulas. This simulation is almost identical to the one presented in the proof of Lemma 8 and hence will be skipped here. □

5.2 k-flows

In this subsection, we will first introduce a k-flow as a uniform term-length sequence of PV-provable implications between $\hat{\Pi}_k^b$-formulas. Then, we will develop a high-level calculus for k-flows to witness the provability in theories S_2^k and T_2^k.

Definition 19. *Let $A(\bar{x}), B(\bar{x}) \in \hat{\Pi}_k^b$ be two $\mathcal{L}_{\text{PV}}$-formulas and $t(\bar{x})$ be an $\mathcal{L}_{\text{PV}}$-term. A k-flow from $A(\bar{x})$ to $B(\bar{x})$ with the length $t(\bar{x})$ is a pair $(H(u, \bar{x}), t(\bar{x}))$, where $H(u, \bar{x}) \in \hat{\Pi}_k^b$ and:*

- $\text{PV} \vdash H(0, \bar{x}) \leftrightarrow A(\bar{x})$.
- $\text{PV} \vdash H(t(\bar{x}), \bar{x}) \leftrightarrow B(\bar{x})$.
- $\text{PV} \vdash \forall u < t(\bar{x}) \, [H(u, \bar{x}) \to H(u+1, \bar{x})]$.

A k-flow is called polynomial if $t(\bar{x}) = q(|\bar{x}|)$, for some polynomial q, where by equality, we mean the syntactical equality between the terms. If there exists a k-flow from $A(\bar{x})$ to $B(\bar{x})$ with the length $t(\bar{x})$, we write $A(\bar{x}) \triangleright_k^{t(\bar{x})} B(\bar{x})$. If we intend to emphasize on the existence of the k-flow regardless of its length, we write $A(\bar{x}) \triangleright_k B(\bar{x})$ and if the k-flow is polynomial $A(\bar{x}) \triangleright_k^p B(\bar{x})$. Moreover, if $\Gamma \cup \Delta \subseteq \hat{\Pi}_k^b$, by $\Gamma \triangleright_k \Delta$ (resp. $\Gamma \triangleright_k^p \Delta$), we mean $\bigwedge \Gamma \triangleright_k \bigvee \Delta$ (resp. $\bigwedge \Gamma \triangleright_k^p \bigvee \Delta$).

Similar to the situation with the ordinal flows, it is also reasonable to provide a high-level calculus to work with the *k*-flows. The following series of lemmas realize this goal.

Lemma 20. *(Padding) Let $A(\bar{x}), B(\bar{x}) \in \hat{\Pi}_k^b$ and $t(\bar{x}), s(\bar{x})$ be two $\mathcal{L}_{PV}$-terms such that $PV \vdash t(\bar{x}) \le s(\bar{x})$. If $A(\bar{x}) \triangleright_k^{t(\bar{x})} B(\bar{x})$, then $A(\bar{x}) \triangleright_k^{s(\bar{x})} B(\bar{x})$. Therefore, without loss of generality, we can always assume that the length $t(\bar{x})$ of a k-flow is PV-monotone, that is, $PV \vdash \bigwedge_{i=1}^n (x_i \le y_i) \to t(\bar{x}) \le t(\bar{y})$.*

Proof. Let $(H(u, \bar{x}), t(\bar{x}))$ be a *k*-flow from $A(\bar{x})$ to $B(\bar{x})$. Then, define

$$H'(u, \bar{x}) = \begin{cases} H(u, \bar{x}) & u \le t(\bar{x}) \\ B(\bar{x}) & u > t(\bar{x}) \end{cases}$$

Notice that $H'(u, \bar{x}) \in \hat{\Pi}_k^b$. It is easy to prove that $(H'(u, \bar{x}), s(\bar{x}))$ is a *k*-flow from $A(\bar{x})$ to $B(\bar{x})$. The only thing worth emphasizing is the role of the assumption $PV \vdash t(\bar{x}) \le s(\bar{x})$ in the proof. This assumption together with the definition of $H'(u, \bar{x})$ shows $PV \vdash H'(s(\bar{x}), \bar{x}) \leftrightarrow B(\bar{x})$ which is one of the conditions of being a *k*-flow. This observation completes the proof of the first part of the claim. For its second part, note that for any term $t(\bar{x})$, there exists a polynomial q such that $PV \vdash t(\bar{x}) \le 2^{q(|\bar{x}|)}$ (Buss 1986; Krajíček 1995). As $2^{q(|\bar{x}|)}$ is PV-monotone, it is enough to use the first part to extend a *k*-flow with the length $t(\bar{x})$ to a *k*-flow with the length $2^{q(|\bar{x}|)}$. For polynomial *k*-flows, as the length $t(\bar{x})$ is in the form $q(|\bar{x}|)$, for some polynomial q, it is already PV-monotone and hence there is nothing to prove. □

Lemma 21. *Let $A(\bar{x}), B(\bar{x}), C(\bar{x}) \in \hat{\Pi}_k^b$. Then:*

(i) If $PV \vdash A(\bar{x}) \to B(\bar{x})$, then $A(\bar{x}) \triangleright_k^p B(\bar{x})$.

(ii) If $A(\bar{x}) \triangleright_k B(\bar{x})$, then $A(\bar{x}) \circ C(\bar{x}) \triangleright_k B(\bar{x}) \circ C(\bar{x})$, for any $\circ \in \{\wedge, \vee\}$. A similar claim also holds for $\triangleright_k^p$.

Proof. The proof is similar to that of Lemma 10. □

Lemma 22. *(Bounded variables) Let $A(\bar{x}, y), B(\bar{x}, y) \in \hat{\Pi}_k^b$ be two $\mathcal{L}_{PV}$-formulas and $s(\bar{x})$ be an $\mathcal{L}_{PV}$-term (not depending on y). If $A(\bar{x}, y) \triangleright_k B(\bar{x}, y)$, then there exists a formula $I(u, y, \bar{x}) \in \hat{\Pi}_k^b$ and an $\mathcal{L}_{PV}$-term $r(\bar{x})$ (not depending on y) such that:*

- $PV \vdash I(0, y, \bar{x}) \leftrightarrow A(\bar{x}, y)$.
- $PV \vdash \forall y \le s(\bar{x})[I(r(\bar{x}), y, \bar{x}) \leftrightarrow B(\bar{x}, y)]$.
- $PV \vdash I(u, y, \bar{x}) \to I(u + 1, y, \bar{x})$.
- $PV \vdash r(\bar{x}) \ge 1$.

If we also have $A(\bar{x}, y) \triangleright_k^p B(\bar{x}, y)$, then the term $r(\bar{x})$ can be chosen in the form $q(|\bar{x}|)$, for some polynomial q.

Proof. Assume $(H(u, y, \bar{x}), t(y, \bar{x}))$ is a k-flow from $A(\bar{x}, y)$ to $B(\bar{x}, y)$. Using Lemma 20, we can assume that $t(y, \bar{x})$ is PV-monotone and $\text{PV} \vdash t(y, \bar{x}) \geq 1$. Define

$$I(u, y, \bar{x}) = \begin{cases} H(u, y, \bar{x}) & u \leq t(y, \bar{x}) \\ B(y, \bar{x}) & u > t(y, \bar{x}) \end{cases}$$

and notice that $I(u, y, \bar{x}) \in \hat{\Pi}_k^b$. Recall from the basic facts in bounded arithmetic that for the term $s(\bar{x})$, there is a polynomial q_s such that $\text{PV} \vdash |s(\bar{x})| \leq q_s(|\bar{x}|)$ (Buss 1986; Krajíček 1995). Define $r(\bar{x}) = t(2^{q_s(|\bar{x}|)}, \bar{x})$ and note that $\text{PV} \vdash y \leq s(\bar{x}) \to t(y, \bar{x}) \leq r(\bar{x})$, as $t(y, \bar{x})$ is PV-monotone and $\text{PV} \vdash r(\bar{x}) \geq 1$. We claim that $I(u, y, \bar{x})$ and $r(\bar{x})$ work. The first and the third claims in the statement of the lemma are the trivial consequences of the fact that $(H(u, y, \bar{x}), t(y, \bar{x}))$ is a k-flow from $A(\bar{x}, y)$ to $B(\bar{x}, y)$. For the second, notice that as $\text{PV} \vdash y \leq s(\bar{x}) \to t(y, \bar{x}) \leq r(\bar{x})$, we can use the definition of $I(y, \bar{x})$ to see that the formula $I(r(\bar{x}), \bar{x})$ is PV-equivalent to $B(y, \bar{x})$.

For the polynomial case, if $(H(u, y, \bar{x}), t(y, \bar{x}))$ is a polynomial k-flow from $A(\bar{x}, y)$ to $B(\bar{x}, y)$, then there is a polynomial q_t such that $t(y, \bar{x}) = q_t(|y|, |\bar{x}|)$. Therefore, $r(\bar{x}) = q_t(q_s(|x|) + 1, |\bar{x}|)$ which implies that $r(\bar{x})$ is in the form $q_r(|\bar{x}|)$, for some poynomial q_r. □

Lemma 23. *Let* $\Gamma(\bar{x}) \cup \{A(\bar{x}), B(\bar{x}), C(\bar{x}), D(y, \bar{x})\} \subseteq \hat{\Pi}_k^b$. *Then:*

(i) *(weak gluing) If* $A(\bar{x}) \rhd_k B(\bar{x})$ *and* $B(\bar{x}) \rhd_k C(\bar{x})$ *then* $A(\bar{x}) \rhd_k C(\bar{x})$. *A similar claim also holds for* $\rhd_k^p$.

(ii) *(polynomial strong gluing) If* $\Gamma(\bar{x}), D(\lfloor \frac{y}{2} \rfloor, \bar{x}) \rhd_k^p D(y, \bar{x})$, *then we have* $\Gamma(\bar{x}), D(0, \bar{x}) \rhd_k^p D(s(\bar{x}), \bar{x})$, *for any* $\mathcal{L}_{\text{PV}}$-*term* $s(\bar{x})$.

(iii) *(strong gluing) If* $\Gamma(\bar{x}), D(y, \bar{x}) \rhd_k D(y + 1, \bar{x})$, *then* $\Gamma(\bar{x}), D(0, \bar{x}) \rhd_k D(s(\bar{x}), \bar{x})$, *for any* $\mathcal{L}_{\text{PV}}$-*term* $s(\bar{x})$.

Proof. For (i), as $A(\bar{x}) \rhd_k B(\bar{x})$ and $B(\bar{x}) \rhd_k C(\bar{x})$, there exist k-flows $(H(u, \bar{x}), t(\bar{x}))$ and $(H'(u, \bar{x}), t'(\bar{x}))$, from $A(\bar{x})$ to $B(\bar{x})$ and from $B(\bar{x})$ to $C(\bar{x})$, respectively. Set $t''(\bar{x}) = t(\bar{x}) + t'(\bar{x}) + 1$ and

$$H''(u, \bar{x}) = \begin{cases} H(u, \bar{x}) & u \leq t(\bar{x}) \\ H'(u \dot{-} (t(\bar{x}) + 1), \bar{x}) & u > t(\bar{x}) \end{cases}$$

Notice that $H''(u, \bar{x})$ is clearly a $\hat{\Pi}_k^b$-formula. We claim that $(H''(u, \bar{x}), t''(\bar{x}))$ is a k-flow from $A(\bar{x})$ to $C(\bar{x})$ as depicted in the following figure (for simplicity, in the figure, we sometimes drop the free variables $\bar{x}$):
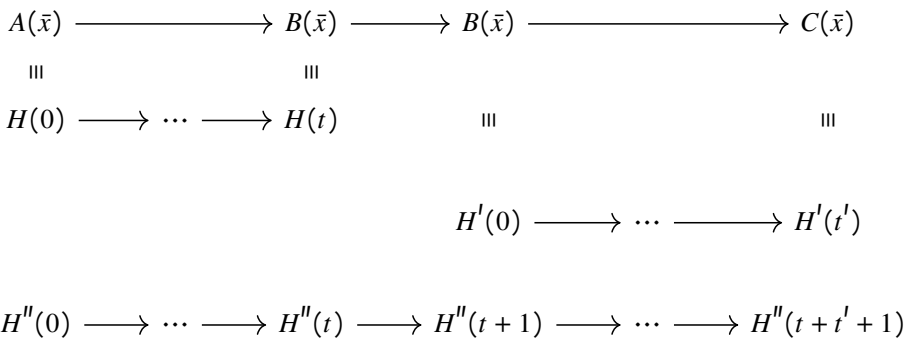
First, it is trivial that $H''(0, \bar{x})$ is PV-equivalent to $H(0, \bar{x})$ which is PV-equivalent to $A(\bar{x})$. Similarly, $H''(t''(\bar{x}), \bar{x})$ is PV-equivalent to $H'(t'(\bar{x}), \bar{x})$ which is PV-equivalent to $C(\bar{x})$. To prove

$PV \vdash \forall u < t''(\bar{x}) [H''(u, \bar{x}) \to H''(u+1, \bar{x})]$, the cases $u < t(\bar{x})$ and $t(\bar{x}) < u < t''(\bar{x})$ are reduced to a similar claim for H and H'. For $u = t(\bar{x})$, note that $H''(t(\bar{x}), \bar{x})$ is PV-equivalent to $H(t(\bar{x}), \bar{x})$ and $H''(t(\bar{x}) + 1, \bar{x})$ is PV-equivalent to $H'(0, \bar{x})$. As both formulas are PV-equivalent to $B(\bar{x})$, the proof is complete. Finally, note that if the k-flows $(H(u, \bar{x}), t(\bar{x}))$ and $(H'(u, \bar{x}), t'(\bar{x}))$ are polynomial, there are polynomials q and q' such that $t(\bar{x}) = q(|\bar{x}|)$ and $t'(\bar{x}) = q'(|\bar{x}|)$. Hence, $t''(\bar{x}) = q(|\bar{x}|) + q'(|\bar{x}|) + 1$. Therefore, the k-flow $(H''(u, \bar{x}), t''(\bar{x}))$ is also polynomial.

For (*ii*), as $\Gamma(\bar{x}), D(\lfloor \frac{y}{2} \rfloor, \bar{x}) \rhd_k^p D(y, \bar{x})$, by Lemma 21, we have $\bigwedge \Gamma(\bar{x}) \wedge D(\lfloor \frac{y}{2} \rfloor, \bar{x}) \rhd_k^p \bigwedge \Gamma \wedge D(y, \bar{x})$. For simplicity, denote $\bigwedge \Gamma(\bar{x}) \wedge D(y, \bar{x})$ by $E(y, \bar{x})$. Therefore, we have $E(\lfloor \frac{y}{2} \rfloor, \bar{x}) \rhd_k^p E(y, \bar{x})$. First, we want to prove $E(0, \bar{x}) \rhd_k^p E(s(\bar{x}), \bar{x})$. Roughly speaking, the idea is gluing the polynomial k-flows from $E(\lfloor \frac{y}{2} \rfloor, \bar{x})$ to $E(y, \bar{x})$, one after another, starting from $y = s(\bar{x})$ till reaching $E(0, \bar{x})$:

$$ E(0, \bar{x}) \longrightarrow \cdots \longrightarrow E(\lfloor \tfrac{\lfloor \frac{s(\bar{x})}{2} \rfloor}{2} \rfloor, \bar{x}) \longrightarrow E(\lfloor \tfrac{s(\bar{x})}{2} \rfloor, \bar{x}) \longrightarrow E(s(\bar{x}), \bar{x}) $$

Notice that the result of this gluing extends the length of the k-flow by $|s(\bar{x})|$ which is bounded by a polynomial and hence acceptable. More formally, using Lemma 22 for the formulas $E(\lfloor \frac{y}{2} \rfloor, \bar{x})$ and $E(y, \bar{x})$ and the term $2s(\bar{x})$ (the choice of $2s(\bar{x})$ instead of $s(\bar{x})$ is rather technical) and using the fact that $E(\lfloor \frac{y}{2} \rfloor, \bar{x}) \rhd_k^p E(y, \bar{x})$, we reach a pair $(H'(u, y, \bar{x}), t'(\bar{x}))$ such that:

(1) $PV \vdash H'(0, y, \bar{x}) \leftrightarrow E(\lfloor \frac{y}{2} \rfloor, \bar{x})$,
(2) $PV \vdash \forall y \leq 2s(\bar{x})[H'(t'(\bar{x}), y, \bar{x}) \leftrightarrow E(y, \bar{x})]$,
(3) $PV \vdash H'(u, y, \bar{x}) \to H'(u+1, y, \bar{x})$,
(4) $PV \vdash t'(\bar{x}) \geq 1$,

and $t'(\bar{x}) = q_{t'}(|\bar{x}|)$, for some polynomial $q_{t'}$. Define the function $Y(z, \bar{x})$ as the result of $|s(\bar{x})| + 1 \dot- z$ many iterations of the operation $n \mapsto \lfloor \frac{n}{2} \rfloor$ on $2s(\bar{x})$. Note that the function is clearly polynomial time computable. Therefore, we can define it recursively in PV and represent it by an $\mathcal{L}_{PV}$-term. This term is PV-provably bounded by $2s(\bar{x})$, that is, $PV \vdash Y(z, \bar{x}) \leq 2s(\bar{x})$ and we have $Y(0, \bar{x}) = 0$, $Y(|s(\bar{x})|, \bar{x}) = s(\bar{x})$ and if $z \leq |s(\bar{x})|$, then $Y(z, \bar{x}) = \lfloor \frac{Y(z+1, \bar{x})}{2} \rfloor$, all provable in PV. Now, define

$$ I(u, \bar{x}) = H'(u \dot- t'(\bar{x}) \lfloor \tfrac{u}{t'(\bar{x})} \rfloor, Y(\lfloor \tfrac{u}{t'(\bar{x})} \rfloor + 1), \bar{x}). $$

Note that $I(u, \bar{x})$ is well defined as $t'(\bar{x})$ is greater than zero, provably in PV. It is trivial that $I(u, \bar{x}) \in \hat{\Pi}_k^b$. Set $r(\bar{x}) = t'(\bar{x})|s(\bar{x})|$. We claim that the pair $(I(u, \bar{x}), r(\bar{x}))$ is a k-flow from $E(0, \bar{x})$ to $E(s(\bar{x}), \bar{x})$ as depicted in the following figure. For simplicity, we drop the free variables $\bar{x}$ in the figure.
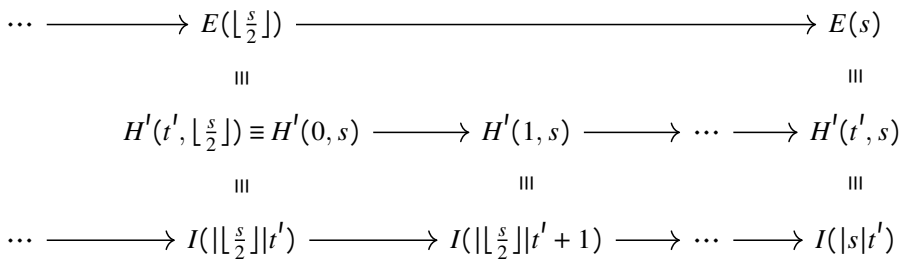
To prove, we first claim that

$$ PV \vdash \forall z \leq |s(\bar{x})| [I(t'(\bar{x})z, \bar{x}) \leftrightarrow E(Y(z, \bar{x}), \bar{x})] \qquad (*) $$

The reason is that by definition, $I(t'(\bar{x})z, \bar{x}) = H'(0, Y(z+1, \bar{x}), \bar{x})$ and the latter is PV-equivalent to $E(\lfloor \frac{Y(z+1,\bar{x})}{2} \rfloor, \bar{x})$, by the property (1) above. Finally, since for any $z \leq |s(\bar{x})|$, we have $Y(z, \bar{x}) = \lfloor \frac{Y(z+1,\bar{x})}{2} \rfloor$ provably in PV, we reach the PV-equivalence with $E(Y(z, \bar{x}), \bar{x})$.

Now, we prove that $(I(u, \bar{x}), r(\bar{x}))$ is a k-flow from $E(0, \bar{x})$ to $E(s(\bar{x}), \bar{x})$. First, note that $I(0, \bar{x})$ is PV-equivalent to $E(0, \bar{x})$, by substituting $z = 0$ in ($*$) and using the PV-provable fact that $Y(0, \bar{x}) = 0$. Second, note that $I(r(\bar{x}), \bar{x})$ is PV-equivalent to $E(s(\bar{x}), \bar{x})$, by substituting $z = |s(\bar{x})|$ in ($*$) and using the PV-provable fact that $Y(|s(\bar{x})|, \bar{x}) = s(\bar{x})$. Third, to prove $\mathrm{PV} \vdash \forall u < t'(\bar{x}) \, [I(u, \bar{x}) \to I(u+1, \bar{x})]$, there are two cases to consider: Either $u+1$ divides $t'(\bar{x})$ or not. In the latter case, we have $\lfloor \frac{u+1}{t'(\bar{x})} \rfloor = \lfloor \frac{u}{t'(\bar{x})} \rfloor$. By definition $I(u, \bar{x})$ is $H'(u \dot{-} t'(\bar{x}) \lfloor \frac{u}{t'(\bar{x})} \rfloor, Y(\lfloor \frac{u}{t'(\bar{x})} \rfloor + 1), \bar{x})$ while $I(u+1, \bar{x})$ is $H'(u+1 \dot{-} t'(\bar{x}) \lfloor \frac{u+1}{t'(\bar{x})} \rfloor, Y(\lfloor \frac{u+1}{t'(\bar{x})} \rfloor + 1), \bar{x})$. Therefore, the former proves the latter by property (3) above. For the first case, if $t'(\bar{x})|u+1$, then there exists $z \leq |s(\bar{x})|$ such that $u+1 = t'(\bar{x})z$. Therefore, $I(u+1, \bar{x})$ is $I(t'(\bar{x})z, \bar{x})$ which is PV-equivalent to $E(Y(z, \bar{x}), \bar{x})$ by ($*$), and hence PV-equivalent to $H'(t'(\bar{x}), Y(z, \bar{x}), \bar{x})$ by (2), as $Y(z, \bar{x})$ is PV-provably bounded by $2s(\bar{x})$. As $I(u, \bar{x})$ is $H'(t'(\bar{x}) \dot{-} 1, Y(z, \bar{x}), \bar{x})$ by definition, by (3), the formula $I(u, \bar{x})$ implies $I(u+1, \bar{x})$ in PV.

So far, we showed that $(I(u, \bar{x}), r(\bar{x}))$ is a k-flow from $E(0, \bar{x})$ to $E(s(\bar{x}), \bar{x})$. Again, recall that for the term $s(\bar{x})$, there is a polynomial q_s such that $\mathrm{PV} \vdash |s(\bar{x})| \leq q_s(|\bar{x}|)$ (Buss 1986; Krajíček 1995). Hence, $\mathrm{PV} \vdash r(\bar{x}) \leq q_s(|\bar{x}|)q_{t'}(|\bar{x}|)$. Therefore, using Lemma 20, we can prove the existence of a k-flow with the length $q_s(|\bar{x}|)q_{t'}(|\bar{x}|)$ from $E(0, \bar{x})$ to $E(s(\bar{x}), \bar{x})$ which implies $E(0, \bar{x}) \rhd_k^p E(s(\bar{x}), \bar{x})$. Now, to complete the proof of (ii), by the definition of $E(y, \bar{x})$, we have $\bigwedge \Gamma(\bar{x}) \wedge D(0, \bar{x}) \rhd_k^p \bigwedge \Gamma(\bar{x}) \wedge D(s(\bar{x}), \bar{x})$. As $\mathrm{PV} \vdash \bigwedge \Gamma(\bar{x}) \wedge D(s(\bar{x}), \bar{x}) \to D(s(\bar{x}), \bar{x})$, by Lemma 21, we have $\bigwedge \Gamma(\bar{x}) \wedge D(s(\bar{x}), \bar{x}) \rhd_k^p D(s(\bar{x}), \bar{x})$. Hence, by the weak gluing, the part (i) in the present lemma, we reach $\bigwedge \Gamma(\bar{x}) \wedge D(0, \bar{x}) \rhd_k^p D(s(\bar{x}), \bar{x})$.

The proof of (iii) is similar to that of (ii) and even easier. In this case, one must again define $E(y, \bar{x})$ as $\bigwedge \Gamma(\bar{x}) \wedge D(y, \bar{x})$ and then glue the k-flows from $E(y, \bar{x})$ to $E(y+1, \bar{x})$, one after another, for all $0 \leq y < s(\bar{x})$. $\qquad\square$

Lemma 24. *(Conjunction and Disjunction Rules) Let $\Gamma \cup \Delta \cup \{A, B\} \subseteq \hat{\Pi}_k^b$. Then:*

(i) *If $\Gamma, A \rhd_k \Delta$ or $\Gamma, B \rhd_k \Delta$ then $\Gamma, A \wedge B \rhd_k \Delta$.*

(ii) *If $\Gamma \rhd_k A, \Delta$ and $\Gamma \rhd_k B, \Delta$ then $\Gamma \rhd_k A \wedge B, \Delta$.*

(iii) *If $\Gamma \rhd_k A, \Delta$ or $\Gamma \rhd_k B, \Delta$ then $\Gamma \rhd_k A \vee B, \Delta$.*

(iv) *If $\Gamma, A \rhd_k \Delta$ and $\Gamma, B \rhd_k \Delta$ then $\Gamma, A \vee B \rhd_k \Delta$.*

A similar claim also holds for $\rhd_k^p$.

Proof. The argument is identical to that of Lemma 12 claiming the same fact for the ordinal flows. $\qquad\square$

Lemma 25. *(Negation Rules) If $\Gamma \cup \Delta \subseteq \hat{\Pi}_k^b$ and $A, \neg A \in \hat{\Pi}_k^b$, then:*

(i) *If $\Gamma, A \rhd_k \Delta$ then $\Gamma \rhd_k \neg A, \Delta$.*

(ii) *If $\Gamma \rhd_k A, \Delta$ then $\Gamma, \neg A \rhd_k \Delta$.*

A similar claim also holds for $\rhd_k^p$.

Proof. We only prove the claim for $\rhd_k$. The case for $\rhd_k^p$ is identical. For (i), assume $\Gamma, A \rhd_k \Delta$ which means $\bigwedge \Gamma \wedge A \rhd_k \bigvee \Delta$. As $\neg A \in \hat{\Pi}_k^b$, by Lemma 21, we have $(\bigwedge \Gamma \wedge A) \vee \neg A \rhd_k \bigvee \Delta \vee \neg A$. Since $\mathrm{PV} \vdash \bigwedge \Gamma \to (\bigwedge \Gamma \wedge A) \vee \neg A$, by Lemma 21, we have $\bigwedge \Gamma \rhd_k (\bigwedge \Gamma \wedge A) \vee \neg A$. Hence, by weak gluing, Lemma 23, we have $\bigwedge \Gamma \rhd_k \bigvee \Delta \vee \neg A$. The proof for (ii) is similar. $\qquad\square$

Lemma 26. *(Bounded Universal Quantifier) Let $A(\bar{x}), B(\bar{x}, y) \in \hat{\Pi}_k^b$ and $s(\bar{x})$ be an $\mathcal{L}_{\mathrm{PV}}$-term. If $A(\bar{x}), (y \leq s(\bar{x})) \rhd_k B(\bar{x}, y)$, then $A(\bar{x}) \rhd_k \forall y \leq s(\bar{x}) B(y, \bar{x})$. The same also holds for $\rhd_k^p$.*

Proof. Again, we only prove the claim for $\rhd_k$. The proof for $\rhd_k^p$ is identical. Since $A(\bar{x}), (y \leq s(\bar{x})) \rhd_k B(\bar{x}, y)$ and $y \leq s(\bar{x})$ is quantifier-free, by Lemma 25, we have $A(\bar{x}) \rhd_k (y \leq s(\bar{x}) \rightarrow B(\bar{x}, y))$. Note that $(y \leq s(\bar{x}) \rightarrow B(\bar{x}, y))$ is defined as $\neg(y \leq s(\bar{x})) \vee B(\bar{x}, y)$, as the negation is not primitive in the language. Use Lemma 22 for the formulas $A(\bar{x})$ and $(y \leq s(\bar{x}) \rightarrow B(\bar{x}, y))$ and the term $s(\bar{x})$. Therefore, we have a formula $I(u, y, \bar{x}) \in \hat{\Pi}_k^b$ and a term $r(\bar{x})$ such that:

- $\mathrm{PV} \vdash I(0, y, \bar{x}) \leftrightarrow A(\bar{x})$.
- $\mathrm{PV} \vdash \forall y \leq s(\bar{x})[I(r(\bar{x}), y, \bar{x}) \leftrightarrow (y \leq s(\bar{x}) \rightarrow B(\bar{x}, y))]$.
- $\mathrm{PV} \vdash I(u, y, \bar{x}) \rightarrow I(u + 1, y, \bar{x})$.

It is easy to see that the pair $\big(\forall y \leq s(\bar{x}) I(u, y, \bar{x}), r(\bar{x})\big)$ is a k-flow from $A(\bar{x})$ to $\forall y \leq s(\bar{x}) B(\bar{x}, y)$. $\square$

Now we are ready to use k-flows to witness the provable implications between $\hat{\Pi}_k^b$-formulas in S_2^k and T_2^k.

Theorem. *(Soundness and Completeness) Let $\Gamma(\bar{x}) \cup \Delta(\bar{x}) \subseteq \hat{\Pi}_k^b$. Then:*

(i) $S_2^k \vdash \bigwedge \Gamma(\bar{x}) \rightarrow \bigvee \Delta(\bar{x})$ *iff* $\Gamma(\bar{x}) \rhd_k^p \Delta(\bar{x})$.

(ii) $T_2^k \vdash \bigwedge \Gamma(\bar{x}) \rightarrow \bigvee \Delta(\bar{x})$ *iff* $\Gamma(\bar{x}) \rhd_k \Delta(\bar{x})$.

Proof. We only prove *(i)*. The proof of *(ii)* is similar. First, we prove the easier completeness part. If $\Gamma(\bar{x}) \rhd_k^p \Delta(\bar{x})$, then by Definition 19, there exist a polynomial q, and a formula $H(u, \bar{x}) \in \hat{\Pi}_k^b$ such that:

- $\mathrm{PV} \vdash H(0, \bar{x}) \leftrightarrow \bigwedge \Gamma(\bar{x})$,
- $\mathrm{PV} \vdash H(q(|\bar{x}|), \bar{x}) \leftrightarrow \bigvee \Delta(\bar{x})$,
- $\mathrm{PV} \vdash \forall u < q(|\bar{x}|)\, [H(u, \bar{x}) \rightarrow H(u + 1, \bar{x})]$.

Using Lemma 20, without loss of generality, we can also assume that $\mathrm{PV} \vdash q(|\bar{x}|) \geq 1$. As PV is a subtheory of S_2^k, we also have all the above provabilities for S_2^k. Hence, $S_2^k \vdash \forall u < q(|\bar{x}|)\, [H(u, \bar{x}) \rightarrow H(u + 1, \bar{x})]$. Since $H(u, \bar{x}) \in \hat{\Pi}_2^k$, by the $\hat{\Pi}_k^b - \mathrm{LInd}$ axiom, we have, $S_2^k \vdash H(0, \bar{x}) \rightarrow H(|2^{q(|\bar{x}|) \dot{-} 1}|, \bar{x})$. As $\mathrm{PV} \vdash q(|\bar{x}|) \geq 1$, we have $\mathrm{PV} \vdash |2^{q(|\bar{x}|) \dot{-} 1}| = |q(\bar{x})|$. Hence, $S_2^k \vdash H(0, \bar{x}) \rightarrow H(q(|\bar{x}|), \bar{x})$. Therefore, $S_2^k \vdash \bigwedge \Gamma(\bar{x}) \rightarrow \bigvee \Delta(\bar{x})$.

For soundness, assume $S_2^k \vdash \bigwedge \Gamma(\bar{x}) \rightarrow \bigvee \Delta(\bar{x})$. By Lemma 18, $\Gamma(\bar{x}) \Rightarrow \Delta(\bar{x})$ has a $\mathbf{wLS_2^k}$-proof only consisting of $\hat{\Pi}_k^b$-formulas. By induction on this proof, we show that for any sequent $\Sigma \Rightarrow \Lambda$ in the proof, we have $\Sigma \rhd_k^p \Lambda$. For the axioms, as they are provable in PV, using Lemma 21, there is nothing to prove. The case of structural rules (except for the weak cut) is easy. Weak cut and (wPInd_k) are addressed in Lemma 23. The conjunction and disjunction rules are proved in Lemma 24 and the rule $(R\forall^{\leq})$ is addressed in Lemma 26. Therefore, there are only three cases to consider. If the last rule is

$$\frac{\Sigma(\bar{x}, y), B(\bar{x}, s(\bar{x}, y)) \Rightarrow \Lambda(\bar{x}, y)}{\Sigma(\bar{x}, y), s(\bar{x}, y) \leq t(\bar{x}, y), \forall y \leq t(\bar{x}, y) B(\bar{x}, y) \Rightarrow \Lambda(\bar{x}, y)} \, L\forall^{\leq}$$

by the induction hypothesis, we have $\Sigma(\bar{x}, y), B(\bar{x}, s(\bar{x}, y)) \rhd_k^p \Lambda(\bar{x}, y)$. Since

$$\bigwedge \Sigma(\bar{x}, y) \wedge (s(\bar{x}, y) \leq t(\bar{x}, y)) \wedge \forall y \leq t(\bar{x}, y) B(\bar{x}, y)$$

implies $\bigwedge \Sigma(\bar{x}, y) \wedge B(\bar{x}, s(\bar{x}, y))$ in PV, by Lemma 21 and weak gluing, Lemma 23, we have

$$\Sigma(\bar{x}, y), s(\bar{x}, y) \le t(\bar{x}, y), \forall y \le t(\bar{x}, y) B(\bar{x}, y) \rhd_k^p \Lambda(\bar{x}, y).$$

The case for the rule $R\exists^{\le}$ is similar to the previous case. Finally, if the last rule is

$$\frac{\Sigma(\bar{x}), z \le s(\bar{x}), B(\bar{x}, z) \Rightarrow \Lambda(\bar{x})}{\Sigma(\bar{x}), \exists y \le s(\bar{x}) B(\bar{x}, y) \Rightarrow \Lambda(\bar{x})} \; L\exists^{\le}$$

by the induction hypothesis, we have $\Sigma(\bar{x}), z \le s(\bar{x}), B(\bar{x}, z) \rhd_k^p \Lambda(\bar{x})$. Since $\exists y \le s(\bar{x}) B(\bar{x}, y) \in \hat{\Pi}_k^b$ and it starts with an existential quantifier, it must belong to $\hat{\Sigma}_{k-1}^b$. Hence, both $\neg B(\bar{x}, z)$ and $\neg \exists y \le s(\bar{x}) B(\bar{x}, y) = \forall y \le s(\bar{x}) \neg B(\bar{x}, y)$ are in $\hat{\Pi}_k^b$. Therefore, by Lemma 25,

$$\Sigma(\bar{x}), z \le s(\bar{x}) \rhd_k^p \Lambda(\bar{x}), \neg B(\bar{x}, z).$$

By using the fact that the names of the parameters are not important in k-flows and employing Lemma 26, we have $\Sigma(\bar{x}) \rhd_k^p \Lambda(\bar{x}), \forall y \le s(\bar{x}) \neg B(\bar{x}, y)$. Finally again by Lemma 25, we reach $\Sigma(\bar{x}), \exists y \le s(\bar{x}) B(\bar{x}, y) \rhd_k^p \Lambda(\bar{x})$. $\qquad \square$

5.3 Reductions and $\mathrm{PLS}_{(k,l)}$-programs

In Subsection 5.2, we transformed the S_2^k-provable (resp. T_2^k-provable) implications between $\hat{\Pi}_k^b$-formulas into exponentially (resp., polynomially) long uniform sequences of PV-provable implications between $\hat{\Pi}_k^b$-formulas. Having that characterization at hand, one can use the universality of PV to employ generalized Herbrand's theorem and push the characterization of Theorem 6 even further to witness *all* essentially existential quantifiers in the PV-provable implications by polynomial-time computable functions. Instead of following this rather *absolute* approach, in this subsection, we will employ a *relative* approach to witness all the essentially existential quantifiers up to a given level $l \le k$. The idea is simple. First, by moving the PV-provable implications from PV to PV_{k-l+1}, we will pretend that all the $\mathcal{L}_{\mathrm{PV}}$-formulas in $\hat{\Pi}_{k-l}^b \cup \hat{\Sigma}_{k-l}^b$ are quantifier-free in $\mathcal{L}_{\mathrm{PV}_{k-l+1}}$. Therefore, only l many alternating quantifiers are left to peel off for which we use the generalized Herbrand's theorem. In choosing the right value for l, there is a clear trade-off between the complexity of the witnessing functions on the one hand and the complexity of the witnessing more alternating quantifiers, on the other. For the smaller values of l, the latter would be quite easy as evidenced by Theorem 2. However, the cost to pay is the higher complexity of the witnessing functions that now live in the higher level of the polynomial hierarchy, that is, in the class $\square_{k-l+1}^p$. For the higher values of l, the situation is reverse. For instance, if $l = k$, then all the witnessing functions are polynomial time as they live in $\mathrm{PV}_{k-k+1} = \mathrm{PV}$. However, the generalized Herbrand's theorem must witness k many quantifier alternations that is combinatorially too complex to deal with. In the present subsection, we will lean toward the lower values for l and will only apply the relative approach to two instances of $l = 1$ and $l = 2$ to avoid the high witnessing complexity. However, it is worth emphasizing that the main base, that is, Theorem 6 is there and one can use it for any value of l by employing the right instance of Herbrand's theorem. We only cover these two cases to show that how interesting the concrete consequences can be. For $l = 1$, we will show that some well-known witnessing theorems in bounded arithmetic are just special cases of our witnessing theorem. For $l = 2$, the witnessing results are all new.

5.3.1 The game interpretation

Let $k \ge l \ge 1$ be two numbers, $G(\bar{x}, y_1, y_2, \ldots, y_l)$ be a quantifier-free $\mathcal{L}_{\mathrm{PV}_k}$-formula and $t(\bar{x})$ be an $\mathcal{L}_{\mathrm{PV}_k}$-term. We call the pair $(G(\bar{x}, y_1, y_2, \ldots, y_l), t(\bar{x}))$ a (k, l)-*game* (a game, for short), and we

interpret it as a uniform family of l-turn games between two players parameterized by the variables $\bar{x}$. To emphasize this parameter role, we sometimes write $G_{\bar{x}}(y_1, y_2, \ldots, y_l)$ for $G(\bar{x}, y_1, y_2, \ldots, y_l)$ and if the variables are clear from the context, we use the shorthand $(G_{\bar{x}}, t(\bar{x}))$ for an instance of the game and (G, t) for the uniform family itself. Given the value $\bar{a}$ for $\bar{x}$, the game $G_{\bar{a}}$ starts with the first player, denoted by I, playing the number $b_1 \leq t(\bar{a})$ for y_1. Then, the second player, denoted by II, plays $b_2 \leq t(\bar{a})$ for y_2 and so on. The resulting tuple $(b_1, b_2, \ldots, b_l)$ is called a *play* of the game. For a play, $(b_1, b_2, \ldots, b_l)$, if $G(\bar{a}, b_1, b_2, \ldots, b_l)$ holds, the *first player* wins the game and otherwise, the second player is the winner. A play $(b_1, \ldots, b_l)$ is called a *winning play* for the first (second) player, if it makes the first (second) player wins. It is an easy and well-known fact that the first player has a *winning strategy* in $G_{\bar{a}}$ iff $\exists y_1 \leq t(\bar{a}) \forall y_2 \leq t(\bar{a}) \exists y_3 \leq t(\bar{a}) \ldots G(\bar{a}, y_1, \ldots, y_l)$ holds. As we are always interested in the first player in this subsection, by a winning play and a winning strategy, we always mean them for the first player. Having two (k, l)-games $(G(\bar{x}, y_1, y_2, \ldots, y_l), t(\bar{x}))$ and $(H(\bar{x}, z_1, z_2, \ldots, z_l), s(\bar{x}))$, a natural question to ask is the following. Let the existence of a winning strategy in $(G_{\bar{x}}, t(\bar{x}))$ implies the existence of a winning strategy in $(H_{\bar{x}}, s(\bar{x}))$, for any $\bar{x}$, that is, the implication

$$\exists y_1 \leq t(\bar{x}) \forall y_2 \leq t(\bar{x}) \ldots G(\bar{x}, y_1, \ldots, y_l) \rightarrow \exists z_1 \leq s(\bar{x}) \forall z_2 \leq s(\bar{x}) \ldots H(\bar{x}, z_1, \ldots, z_l) \qquad (\dagger)$$

hold. Then, does it mean that we can find an *explicit* way to use a winning strategy for $(G_{\bar{x}}, t(\bar{x}))$ to design a winning strategy for $(H_{\bar{x}}, s(\bar{x}))$? One can even sharpen the question by asking if having a proof of the implication $(\dagger)$ in the theory PV_k helps to provide an explicit and *relatively simple* transformation between the winning strategies. Fortunately, as PV_k is a universal theory, the extraction of the explicit transformation between the winning strategies is possible, and it is simply the content of Herbrand's theorem, Theorem 2 (up to some small modifications). We will explain the details for the two case $l = 1$ and $l = 2$, below.

5.3.2 The case $l = 1$

Let $(G(\bar{x}, y), t(\bar{x}))$ and $(H(\bar{x}, z), s(\bar{x}))$ be two $(k, 1)$-games. The most trivial way to reduce the winning strategy of the latter to that of the former is via a function $f(\bar{x}, y)$ that maps any move $y \leq t(\bar{x})$ in (G, t) to a move $z \leq s(\bar{x})$ in (H, s) such that if the play y is a winning play in (G, t), then the play $z = f(\bar{x}, y)$ is a winning play in (H, s). Moreover, as we expect the reduction to be simple and verifiable, we expect that everything happens inside a base theory, in our case PV_k. More formally:

Definition 27. *Let* $(G(\bar{x}, y), t(\bar{x}))$ *and* $(H(\bar{x}, z), s(\bar{x}))$ *be two* $(k, 1)$-*games. A* $(k, 1)$-*reduction from* $(H(\bar{x}, z), s(\bar{x}))$ *to* $(G(\bar{x}, y), t(\bar{x}))$ *is an* $\mathcal{L}_{\mathrm{PV}_k}$-*term* $f(\bar{x}, y)$ *such that:*

- $\mathrm{PV}_k \vdash \forall y \leq t(\bar{x})[f(\bar{x}, y) \leq s(\bar{x})]$,
- $\mathrm{PV}_k \vdash \forall y \leq t(\bar{x})[G(\bar{x}, y) \rightarrow H(\bar{x}, f(\bar{x}, y))]$.

Naturally, we expect a connection between the provability of

$$\exists y \leq t(\bar{x}) G(\bar{x}, y) \rightarrow \exists z \leq s(\bar{x}) H(\bar{x}, z)$$

in PV_k and the existence of a $(k, 1)$-reduction. This is the content of the following modification of Herbrand's theorem.

Theorem. *For any two* $(k, 1)$-*games* $(G(\bar{x}, y), t(\bar{x}))$ *and* $(H(\bar{x}, z), s(\bar{x}))$, *the following are equivalent:*

- $\mathrm{PV}_k \vdash \exists y \leq t(\bar{x}) G(\bar{x}, y) \rightarrow \exists z \leq s(\bar{x}) H(\bar{x}, z)$
- *There is a* $(k, 1)$-*reduction from* $(H(\bar{x}, z), s(\bar{x}))$ *to* $(G(\bar{x}, y), t(\bar{x}))$.

Proof. One direction is trivial. For the other, assume

$$PV_k \vdash \exists y \leq t(\bar{x})G(\bar{x}, y) \to \exists z \leq s(\bar{x})H(\bar{x}, z).$$

Therefore, $PV_k \vdash \forall y \exists z \left[(y \leq t(\bar{x}) \wedge G(\bar{x}, y)) \to (z \leq s(\bar{x}) \wedge H(\bar{x}, z))\right]$. By Herbrand's theorem, Theorem 2, there exists an $\mathcal{L}_{PV_k}$-term $g(\bar{x}, y)$ such that

$$PV_k \vdash [y \leq t(\bar{x}) \wedge G(\bar{x}, y)] \to [g(\bar{x}, y) \leq s(\bar{x}) \wedge H(\bar{x}, g(\bar{x}, y))].$$

Define

$$f(\bar{x}, y) = \begin{cases} g(\bar{x}, y) & g(\bar{x}, y) \leq s(\bar{x}) \\ 0 & g(\bar{x}, y) > s(\bar{x}) \end{cases}$$

It is easy to represent $f(\bar{x}, y)$ as an $\mathcal{L}_{PV_k}$-term. By definition, it is clear that $PV_k \vdash \forall y \leq t(\bar{x})[f(\bar{x}, y) \leq s(\bar{x})]$. Moreover, it is easy to see that $PV_k \vdash \forall y \leq t(\bar{x})[G(\bar{x}, y) \to H(\bar{x}, f(\bar{x}, y))]$. $\square$

As explained in the opening of this subsection, for $l = 1$, the combination of witnessing by k-flows, moving from PV to PV_k and using Theorem 7 provides an explicit witnessing theorem for theories S_2^k and T_2^k. This is what we will come back to in Corollary 30. However, as the combination has a natural form itself, it is worth defining it directly.

Definition 28. *Let $A(\bar{x}, y) \in \hat{\Pi}_{k-1}^b$ be an $\mathcal{L}_{PV}$-formula and $t(\bar{x})$ and $r(\bar{x})$ be two $\mathcal{L}_{PV}$-terms. By a $PLS_{(k,1)}$-program for $(A(\bar{x}, y), r(\bar{x}))$ with the length $t(\bar{x})$, we mean the following data: an initial $\mathcal{L}_{PV_k}$-term $i(\bar{x})$, a quantifier-free $\mathcal{L}_{PV_k}$-formula $G(\bar{x}, u, z)$, an $\mathcal{L}_{PV_k}$-term $N(\bar{x}, u, z)$, and an $\mathcal{L}_{PV_k}$-term $p(\bar{x}, z)$, such that:*

- $PV_k \vdash i(\bar{x}) \leq s(\bar{x})$,
- $PV_k \vdash G(\bar{x}, 0, i(\bar{x}))$,
- $PV_k \vdash \forall z \leq s(\bar{x})[N(\bar{x}, u, z) \leq s(\bar{x})]$,
- $PV_k \vdash \forall z \leq s(\bar{x}) [G(\bar{x}, u, z) \to G(\bar{x}, u+1, N(\bar{x}, u, z))]$,
- $PV_k \vdash \forall z \leq s(\bar{x})[p(\bar{x}, z) \leq r(\bar{x})]$,
- $PV_k \vdash \forall z \leq s(\bar{x})[G(\bar{x}, t(\bar{x}), z) \to A(\bar{x}, p(\bar{x}, z))]$.

By $PLS_{(k,1)}$, we mean the class of all the pairs $(A(\bar{x}, y), r(\bar{x}))$ for which there exists a $PLS_{(k,1)}$-program. By $PLS_{(k,1)}^p$, we mean the class of all the pairs $(A(\bar{x}, y), r(\bar{x}))$ for which there exists a $PLS_{(k,1)}$-program with a polynomial length, that is, $t(\bar{x}) = q(|\bar{x}|)$, for some polynomial q.

It is easy to see that if $(A(\bar{x}, y), r(\bar{x})) \in PLS_{(k,1)}$ then $\forall \bar{x} \exists y \leq r(\bar{x})A(\bar{x}, y)$ holds and the $PLS_{(k,1)}$-program actually provides an algorithm to compute $y \leq r(\bar{x})$ from $\bar{x}$. Denoting $G(\bar{x}, u, z)$ by G^u, the algorithm starts at the zeroth level with an initial value $i(\bar{x})$ bounded by $s(\bar{x})$ satisfying the property G^0. Then, using the modification N, it goes from one level to the next updating any value $z \leq s(\bar{x})$ with the property G^u to a value satisfying the property G^{u+1}. Note that the modification always respects the bound $s(\bar{x})$. Finally, reaching the level $t(\bar{x})$, the algorithm uses p to compute $y \leq r(\bar{x})$ satisfying A from any value $z \leq s(\bar{x})$ with the property $G^{t(\bar{x})}$.

There are two points to emphasize here. First, the case $k = 1$, where the predicate $G(\bar{x}, y, z)$ and all the functions $i(\bar{x})$, $N(\bar{x}, u, z)$, and $p(\bar{x}, z)$ are polynomial time computable is just another presentation of the well-known polynomial local search problems, (PLS for short), see Buss and Krajíček (1994); Krajíček (1995). Therefore, one can simply read $PLS_{(k,1)}$-programs as a generalization of PLS from polynomial time to the kth level of the polynomial hierarchy, where the predicate $G(\bar{x}, u, z)$ and the functions $i(\bar{x})$, $N(\bar{x}, u, z)$, and $p(\bar{x}, z)$ are all allowed to be on the k-th level of the hierarchy. It is also worth mentioning that our $PLS_{(k,1)}$-programs are similar to but weaker than $\Pi_k^b - PLS$ problems with Π_l^b-goals defined in Beckmann and Buss (2009), where the

functions $i(\bar{x})$, $N(\bar{x}, u, z)$, and $p(\bar{x}, z)$ (and not the predicate $G(\bar{x}, u, z)$) must be polynomial-time computable and everything must be provable in S_2^1 rather than in PV_k. The second point is about the $\mathrm{PLS}_{(k,1)}^p$-programs with a polynomial length. For these programs, the algorithm we just provided can efficiently (relative to the level of the polynomial hierarchy, of course) compute the value of y as it only needs to iterate the modification function N for polynomially many times. In other words, we can pack the whole algorithm in one single $\mathcal{L}_{\mathrm{PV}_k}$-term as a formalized version of a $\square_k^p$-function that computes y. We will come back to this observation in Corollary 31, where we reprove a well-known witnessing theorem for S_2^k, characterizing the $\hat{\Sigma}_k^b$-definable functions of S_2^k as the ones in the kth level of the polynomial hierarchy.

Remark 29. Employing the game interpretation we explained before, a $\mathrm{PLS}_{(k,1)}$ program for $(A(\bar{x}, y), r(\bar{x}))$ with the length $t(\bar{x})$ is nothing but the following three $(k, 1)$-reductions:

- $i(\bar{x})$ as a $(k, 1)$-reduction from $(G(\bar{x}, 0, z), s(\bar{x}))$ to $(\top, s(\bar{x}))$.
- $N(\bar{x}, u, z)$ as a $(k, 1)$-reduction from the game $(G(\bar{x}, u + 1, z), s(\bar{x}))$ to the game $(G(\bar{x}, u, z), s(\bar{x}))$. Notice that u is also a parameter here.
- $p(\bar{x}, z)$ as a $(k, 1)$-reduction from the game $(A(\bar{x}, y), r(\bar{x}))$ to the game $(G(\bar{x}, t(\bar{x}), z), s(\bar{x}))$.

Notice that the formula $A(\bar{x}, y)$ is not quantifier-free in $\mathcal{L}_{\mathrm{PV}_k}$ and hence we cannot read the pair $(A(\bar{x}, y), r(\bar{x}))$ as a $(k, 1)$-game. However, as $A(\bar{x}, y)$ is in $\hat{\Pi}_{k-1}^b$, it is PV_k-equivalent to a quantifier-free formula and hence we can pretend that it is quantifier-free. Having that observation, we can use Theorem 7 to see that there is a $\mathrm{PLS}_{(k,1)}$ program for $(A(\bar{x}, y), r(\bar{x}))$ with the length $t(\bar{x})$ iff there exist a quantifier-free $\mathcal{L}_{\mathrm{PV}_k}$-formula $G(\bar{x}, u, z)$ and an $\mathcal{L}_{\mathrm{PV}}$-term $s(\bar{x})$ such that:

- $\mathrm{PV}_k \vdash \exists z \le s(\bar{x}) G(\bar{x}, 0, z)$.
- $\mathrm{PV}_k \vdash \exists z \le s(\bar{x}) G(\bar{x}, u, z) \to \exists z \le s(\bar{x}) G(\bar{x}, u + 1, z)$.
- $\mathrm{PV}_k \vdash \exists z \le s(\bar{x}) G(\bar{x}, t(\bar{x}), z) \to \exists y \le r(\bar{x}) A(\bar{x}, y)$.

The next Corollary uses $\mathrm{PLS}_{(k,1)}$-programs (resp. $\mathrm{PLS}_{(k,1)}^p$-programs) to witness the theorems of S_2^k (resp. T_2^k) as promised before.

Corollary 30. *Let $k \ge 1$, $A(\bar{x}, y) \in \hat{\Pi}_{k-1}^b$ and $r(\bar{x})$ be an $\mathcal{L}_{\mathrm{PV}}$-term:*

(i) $S_2^k \vdash \forall \bar{x} \exists y \le r(\bar{x}) A(\bar{x}, y)$ iff $(A(\bar{x}, y), r(\bar{x})) \in \mathrm{PLS}_{(k,1)}^p$.

(ii) $T_2^k \vdash \forall \bar{x} \exists y \le r(\bar{x}) A(\bar{x}, y)$ iff $(A(\bar{x}, y), r(\bar{x})) \in \mathrm{PLS}_{(k,1)}$.

Proof. We only prove (*i*). The proof of (*ii*) is similar. For the right to left direction, if there exists a $\mathrm{PLS}_{(k,1)}$-program for $(A(\bar{x}, y), r(\bar{x}))$ with the length $q(|\bar{x}|)$, for some polynomial q, using Remark 29, there are quantifier-free $\mathcal{L}_{\mathrm{PV}_k}$-formula $G(\bar{x}, u, z)$ and an $\mathcal{L}_{\mathrm{PV}}$-term $s(\bar{x})$ such that:

- $\mathrm{PV}_k \vdash \exists z \le s(\bar{x}) G(\bar{x}, 0, z)$.
- $\mathrm{PV}_k \vdash \exists z \le s(\bar{x}) G(\bar{x}, u, z) \to \exists z \le s(\bar{x}) G(\bar{x}, u + 1, z)$.
- $\mathrm{PV}_k \vdash \exists z \le s(\bar{x}) G(\bar{x}, q(|\bar{x}|), z) \to \exists y \le r(\bar{x}) A(\bar{x}, y)$.

Since PV_k is interpretable in S_2^k, mapping all quantifier-free $\mathcal{L}_{\mathrm{PV}_k}$-formulas to $\mathcal{L}_{\mathrm{PV}}$-formulas in $\hat{\Sigma}_k^b$, we can pretend that $G(\bar{x}, u, z) \in \hat{\Sigma}_k^b$ and all the above formulas are also provable in S_2^k. Finally, since the theory S_2^k has the axiom $\hat{\Sigma}_k^b - \mathrm{LInd}$ and $\exists z \le s(\bar{x}) G(\bar{x}, u, z) \in \hat{\Sigma}_k^b$, we have $S_2^k \vdash \exists y \le r(\bar{x}) A(\bar{x}, y)$. For the other direction, assume $S_2^k \vdash \forall \bar{x} \exists y \le r(\bar{x}) A(\bar{x}, y)$. Hence, $S_2^k \vdash \forall y \le$

$r(\bar{x}) \neg A(\bar{x}, y) \to \bot$. By Theorem 6, $\forall y \leq r(\bar{x}) \neg A(\bar{x}, y) \rhd_k^p \bot$. Therefore, there exist a polynomial q and a formula $H(u, \bar{x}) \in \hat{\Pi}_k^b$ such that:

- $PV \vdash H(0, \bar{x}) \leftrightarrow [\forall y \leq r(\bar{x}) \neg A(\bar{x}, \bar{y})]$.
- $PV \vdash H(q(|\bar{x}|), \bar{x}) \leftrightarrow \bot$.
- $PV \vdash \forall u < q(|\bar{x}|) [H(u, \bar{x}) \to H(u+1, \bar{x})]$.

Define $H'(u, \bar{x})$ as $[(u \leq q(|\bar{x}|)) \to H(u, \bar{x})]$. It is easy to see that

- $PV \vdash [\forall y \leq r(\bar{x}) \neg A(\bar{x}, \bar{y})] \to H'(0, \bar{x})$.
- $PV \vdash H'(q(|\bar{x}|), \bar{x}) \to \bot$.
- $PV \vdash H'(u, \bar{x}) \to H'(u+1, \bar{x})$.

As PV has the pairing function, it can encode finite many bounded variables as one bounded variable. Hence, without loss of generality, we can assume that H' is in the prenex bounded form starting with one universal quantifier on z, that is, $H'(u, \bar{x}) = \forall z \leq s'(\bar{x}, u) I(\bar{x}, u, z)$, where s' is PV-monotone and $I \in \hat{\Sigma}_{k-1}^b$. Define $s(\bar{x})$ as $s'(q(|\bar{x}|), \bar{x})$. Then, it is easy to see that

$$PV \vdash H'(u, \bar{x}) \leftrightarrow [\forall z \leq s(\bar{x})[(z \leq s'(u, \bar{x})) \wedge (u \leq q(|\bar{x}|)) \to I(\bar{x}, u, z)]].$$

Hence, without loss of generality, we can assume that H' is in the form $\forall z \leq s(\bar{x}) J(\bar{x}, u, z)$, where J is in $\hat{\Sigma}_{k-1}^b$. Since PV is a subtheory of PV_k and in PV_k any formula in $\hat{\Sigma}_{k-1}^b$ is equivalent to a quantifier-free formula, we can assume that J is quantifier-free in the language of PV_k and we have

- $PV_k \vdash [\forall y \leq r(\bar{x}) \neg A(\bar{x}, y)] \to \forall z \leq s(\bar{x}) J(\bar{x}, 0, z)$.
- $PV_k \vdash \forall z \leq s(\bar{x}) J(\bar{x}, q(|\bar{x}|), z) \to \bot$.
- $PV_k \vdash \forall z \leq s(\bar{x}) J(\bar{x}, u, z) \to \forall z \leq s(\bar{x}) J(\bar{x}, u+1, z)$.

Define $G(\bar{x}, u, z)$ as $\neg J(\bar{x}, q(|\bar{x}|) \dot{-} u, z)$ and note that it is a quantifier-free $\mathcal{L}_{PV_k}$-formula. Therefore, we have

- $PV_k \vdash \exists z \leq s(\bar{x}) G(\bar{x}, q(|\bar{x}|), z) \to \exists y \leq r(\bar{x}) A(\bar{x}, y)$.
- $PV_k \vdash \exists z \leq s(\bar{x}) G(\bar{x}, 0, z)$.
- $PV_k \vdash \exists z \leq s(\bar{x}) G(\bar{x}, u, z) \to \exists z \leq s(\bar{x}) G(\bar{x}, u+1, z)$.

Finally, it is enough to use Remark 29 to get a $PLS_{(k,1)}$-program for the pair $(A(\bar{x}, y), r(\bar{x}))$ with the length $q(|\bar{x}|)$. Hence, $(A(\bar{x}, y), r(\bar{x})) \in PLS_{(k,1)}^p$. $\qquad \square$

Note that the second part in Corollary 30, when applied on $k = 1$, reproves the well-known characterization of the T_2^1-provable formulas of the form $\forall \bar{x} \exists y \leq r(\bar{x}) A(\bar{x}, y)$, where $A(\bar{x}, y) \in \hat{\Pi}_0^b$, in terms of the usual PLS problems (Buss and Krajíček 1994; Krajíček 1995). Our result, however, seems a bit weaker than the one proved in Buss and Krajíček (1994), Krajíček (1995), as in the latter y is not assumed to be bounded and $A(\bar{x}, y)$ can be in $\hat{\Sigma}_1^b$ rather than in our lower class of $\hat{\Pi}_0^b$. However, proving the stronger form from the one we provided is just a standard technique. First, notice that the presence of $r(\bar{x})$ is no restriction, thanks to Parikh theorem. Second, to reduce the complexity of $A(\bar{x}, y)$, it is enough to write $A(\bar{x}, y)$ in the form $\exists \bar{z} \leq \bar{x}(\bar{x}) B(\bar{x}, y, \bar{z})$, where $B(\bar{x}, y, \bar{z}) \in \hat{\Pi}_0^b$. Then, using the pairing function available in PV, we can make y and all the variables $\bar{z}$ into one bounded variable $w \leq t(\bar{x})$. Now, we can apply Corollary 30 to compute w by a $PLS_{(k,1)}$-program. With this technique, the $PLS_{(k,1)}$-program not only computes the intended variable y but it also finds a value for the variables $\bar{z}$. To retrieve our formula $A(\bar{x}, y)$, we can simply keep the computation for y and forget the other values for $\bar{z}$ by reintroducing their existential

quantifiers. Having this observation about the usual PLS, one can read Corollary 30 as a generalization of the mentioned characterization for T_2^1 to cover both T_2^k and S_2^k, for any $k \geq 1$. However, the latter case can be strengthened even further as the polynomial $\text{PLS}_{(k,1)}$-program provided in Corollary 30 can be simplified to one single $\mathcal{L}_{\text{PV}_k}$-term. This reproves the following well-known witnessing theorem for S_2^k (Buss 1986; Krajíček 1995).

Corollary 31. *The provably $\hat{\Sigma}_k^b$-definable functions of S_2^k are in $\Box_k^p$. Even better, if $S_2^k \vdash \forall \bar{x} \exists y A(\bar{x}, y)$, where $A(\bar{x}, y) \in \hat{\Sigma}_k^b$, then there exists a function $f \in \Box_k^p$ represented as an $\mathcal{L}_{\text{PV}_k}$-term such that $\text{PV}_k \vdash \forall \bar{x}\, A(\bar{x}, f(\bar{x}))$.*

Proof. Following the technique we just described, without loss of generality, we can assume that $A(\bar{x}, y)$ has no existential quantifier in its front and hence it is actually in $\hat{\Pi}_{k-1}^b$. By Parikh theorem, there exists an $\mathcal{L}_{\text{PV}}$-term $r(\bar{x})$ such that $S_2^k \vdash \forall \bar{x} \exists y \leq r(\bar{x}) A(\bar{x}, y)$. By Corollary 30, there exists a $\text{PLS}_{(k,1)}$-program for $(A(\bar{x}, y), r(\bar{x}))$ with the length $q(|\bar{x}|)$, for a polynomial q. Let $G(\bar{x}, u, z)$, $i(\bar{x})$, $N(\bar{x}, u, z)$, and $p(\bar{x}, z)$ be the data of the $\text{PLS}_{(k,1)}$-program. By recursion on notation on w, define the function $M(w, \bar{x})$ as

$$\begin{cases} M(0, \bar{x}) = i(\bar{x}) \\ M(w, \bar{x}) = N(\bar{x}, |w| \dot{-} 1, M(\lfloor \frac{w}{2} \rfloor, \bar{x})) \quad w > 0 \end{cases}$$

Recall that the $\mathcal{L}_{\text{PV}_k}$-terms are closed under bounded recursion on notation. As both i and N are $\mathcal{L}_{\text{PV}_k}$-terms and $i(\bar{x})$ is bounded by $s(\bar{x})$ and $N(\bar{x}, u, z)$ maps any $z \leq s(\bar{x})$ to something below $s(\bar{x})$, we can make sure that the function $M(w, \bar{x})$ is also representable as an $\mathcal{L}_{\text{PV}_k}$-term. Now, define $I(\bar{x}, w, z)$ as $G(\bar{x}, |w|, z)$. Using the properties of the $\text{PLS}_{(k,1)}$-program, it is clear that

- $\text{PV}_k \vdash I(\bar{x}, 0, i(\bar{x}))$,
- $\text{PV}_k \vdash \forall z \leq s(\bar{x}) \forall w > 0 \, [I(\bar{x}, \lfloor \frac{w}{2} \rfloor, z) \rightarrow I(\bar{x}, w, N(\bar{x}, |w| \dot{-} 1, z))]$.

Therefore, as $\text{PV}_k \vdash M(w, \bar{x}) \leq s(\bar{x})$, by using the axiom PInd on the quantifier-free formula $I(\bar{x}, w, M(w, \bar{x}))$, we can prove $\text{PV}_k \vdash I(\bar{x}, w, M(w, \bar{x}))$. Substituting $w = \lfloor \frac{2^{q(|\bar{x}|)}}{2} \rfloor$, we reach

$$\text{PV}_k \vdash I(\bar{x}, |\lfloor \tfrac{2^{q(|\bar{x}|)}}{2} \rfloor|, M(\lfloor \tfrac{2^{q(|\bar{x}|)}}{2} \rfloor, \bar{x})).$$

Using the fact that $\text{PV}_k \vdash |\lfloor \frac{2^{q(|\bar{x}|)}}{2} \rfloor| = q(|\bar{x}|)$, we have

$$\text{PV}_k \vdash G(\bar{x}, q(|\bar{x}|), M(\lfloor \tfrac{2^{q(|\bar{x}|)}}{2} \rfloor, \bar{x})).$$

Therefore, as $p(\bar{x}, z)$ is an $\mathcal{L}_{\text{PV}_k}$-term and it has the property

$$\text{PV}_k \vdash \forall z \leq s(\bar{x})[G(\bar{x}, q(|\bar{x}|), z) \rightarrow A(\bar{x}, p(\bar{z}, z))],$$

we can define $f(\bar{x}) = p(\bar{x}, M(\lfloor \frac{2^{q(|\bar{x}|)}}{2} \rfloor, \bar{x}))$ as an $\mathcal{L}_{\text{PV}_k}$-term. Therefore, $\text{PV}_k \vdash \forall \bar{x} A(\bar{x}, f(\bar{x}))$. $\qquad \Box$

5.3.3 The case $l = 2$

Let $(G(\bar{x}, y, z), t(\bar{x}))$ and $(H(\bar{x}, v, w), s(\bar{x}))$ be two $(k, 2)$-games. To use a winning strategy for $(G(\bar{x}, y, z), t(\bar{x}))$ to design one for $(H(\bar{x}, v, w), s(\bar{x}))$, the most trivial way is using two functions $f(\bar{x}, y)$ and $g(\bar{x}, y, w)$, where $f(\bar{x}, y)$ reads the first move $y \leq t(\bar{x})$ in (G, t) and computes a first move $v \leq s(\bar{x})$ in (H, s). Then, $g(\bar{x}, y, w)$ reads the second move $w \leq s(\bar{x})$ in (H, s) and computes a second move $z \leq t(\bar{x})$ in (G, t). These computations must be in a way that if the play (y, z) is winning

in (G, t), the play (v, w) is winning in (H, t). Expecting the whole reduction process to be simple relative to PV_k, we have:

Definition 32. *Let $(G(\bar{x}, y, z), t(\bar{x}))$ and $(H(\bar{x}, v, w), s(\bar{x}))$ be two $(k, 2)$-games. By a* deterministic $(k, 2)$-reduction *from (H, s) to (G, t), we mean two $\mathcal{L}_{PV_k}$-terms $f(\bar{x}, y)$ and $g(\bar{x}, y, w)$ such that:*

- $PV_k \vdash \forall y \leq t(\bar{x})[f(\bar{x}, y) \leq s(\bar{x})]$.
- $PV_k \vdash \forall w \leq s(\bar{x}) \forall y \leq t(\bar{x})[g(\bar{x}, y, w) \leq t(\bar{x})]$.
- $PV_k \vdash \forall w \leq s(\bar{x}) \forall y \leq t(\bar{x})[G(\bar{x}, y, g(\bar{x}, y, w)) \rightarrow H(\bar{x}, f(\bar{x}, y), w)]$.

In a similar fashion to what we had n the previous subsubsection, we expect an equivalence between the provability of

$$\exists y \leq t(\bar{x}) \forall z \leq t(\bar{x}) G(\bar{x}, y, z) \rightarrow \exists v \leq s(\bar{x}) \forall w \leq s(\bar{x}) H(\bar{x}, v, w)$$

in PV_k and the existence of a deterministic $(k, 2)$-reduction from (H, s) to (G, t). Unfortunately, this expected equivalence does not exist, unless a hardness conjecture in complexity theory fails. Let us first explain this conjecture.

Let $U, V \subseteq \mathbb{N}$ be two disjoint **NP**-sets. We call a polynomial time computable $S \subseteq \mathbb{N}$ a *separator* for U and V, if $U \subseteq S$ and $S \cap V = \varnothing$. The hardness conjecture we want to use states that there are two disjoint **NP**-sets U and V that have no separator.

Example 33. *Let U and V be two disjoint* **NP***-sets that have no separator and represent them by the $\mathcal{L}_{PV}$-formulas $\exists y \leq s_B(x) B(x, y)$ and $\exists y \leq s_C(x) C(x, y)$, respectively, where $B(x, y)$ and $C(x, z)$ are two quantifier-free $\mathcal{L}_{PV}$-formulas and $s_B(x)$ and $s_C(x)$ are two $\mathcal{L}_{PV}$-terms. First, notice that without loss of generality, we can always assume that $s_B(x) = s_C(x)$ and $\forall x(s_B(x) > 0)$ holds in the standard model. The reason is that we can replace $\exists y \leq s_B(x) B(x, y)$ by*

$$\exists y \leq s_B(x) + s_C(x) + 1 \, [y \leq s_B(x) \wedge B(x, y)]$$

and similarly for $\exists z \leq s_C(x) C(x, z)$. From now on, denote both $s_B(x)$ and $s_C(x)$, by the common name $s(x)$. Moreover, notice that as $U \cap V = \varnothing$, the formula $\forall y \leq s(x) \neg B(x, y) \vee \forall z \leq s(x) \neg C(x, z)$ is true, for any value for x. Now, let

$$A(x, w, y, z) = (w = 0 \rightarrow \neg B(x, y)) \wedge (w \neq 0 \rightarrow \neg C(x, z)).$$

It is clear that the formula

$$\exists w_0 w_1 \leq s(x) \forall y_0 y_1 z_0 z_1 \leq s(x) \, [A(x, w_0, y_0, z_0) \vee A(x, w_1, y_1, z_1)]$$

logically implies $\exists w \leq s(x) \forall y z \leq s(x) \, A(x, w, y, z)$ and hence the implication is provable in PV. Unfortunately, in both formulas, some of the quantifier blocks have more than one bounded quantifiers, and hence, the formulas cannot be read as $(k, 2)$-games. However, using the pairing function and its projections available in PV, it is not hard to change the formulas to PV-equivalent formulas in the right form. We will avoid applying this change here as it makes everything unnecessarily complicated. Instead, we keep working with the original formulas as the one and the only exception in this paper. However, let us emphasize that whatever we claim in this example can be rewritten in a precise way using the mentioned encoding. Having said that, in the rest of this example, we pretend that we are working with the two $(k, 2)$-games $(A(x, w, y, z), s(x))$ and $(A(x, w_0, y_0, z_0) \vee A(x, w_1, y_1, z_1), s(x))$ and we show that there is no deterministic $(k, 2)$-reduction from the $(k, 2)$-game $(A(x, w, y, z), s(x))$ to the $(k, 2)$-game $(A(x, w_0, y_0, z_0) \vee A(x, w_1, y_1, z_1), s(x))$. For the sake of contradiction, assume that there are polynomial time computable functions $f(x, w_0, w_1)$, $g_0(x, w_0, w_1, y, z)$, $g_1(x, w_0, w_1, y, z)$, $h_0(x, w_0, w_1, y, z)$, and finally $h_1(x, w_0, w_1, y, z)$, all represented as $\mathcal{L}_{PV}$-terms such that they read w_0, w_1, y, and z below $s(x)$ and compute w, y_0, y_1,

z_0, and z_1 all below $s(x)$, respectively, satisfying the property

$$\text{PV} \vdash \forall w_0 w_1 y z \le s[(A(x, w_0, g_0, h_0) \vee A(x, w_1, g_1, h_1)) \to A(x, f, y, z)].$$

(The arguments of the functions are omitted, for simplicity). Therefore, the formula

$$\forall w_0 w_1 y z \le s[(A(x, w_0, g_0, h_0) \vee A(x, w_1, g_1, h_1)) \to A(x, f, y, z)]$$

is true in the standard model. Substitute $w_0 = 0$ and $w_1 = 1$ and notice that the condition $s(x) \ge 1$ allows such a substitution. We see that $A(x, 0, g_0, h_0)$ is equivalent to $\neg B(x, g_0)$ and $A(x, 1, g_1, h_1)$ is equivalent to $\neg C(x, h_1)$. Therefore, the following formula is true:

$$\forall y z \le s(x)[(\neg B(x, g_0) \vee \neg C(x, h_1)) \to A(x, f, y, z)].$$

Therefore, we have

$$[\forall y \le s(x)\neg B(x, y) \vee \forall z \le s(x)\neg C(x, z)] \to \forall y z \le s(x) A(x, f, y, z).$$

Recall that as $U \cap V = \varnothing$, we know $\forall y \le s(x)\neg B(x, y) \vee \forall z \le s(x)\neg C(x, z)$ is true. Therefore, we reach $\forall y z \le s(x) A(x, f, y, z)$. Now, note that if $f(x, 0, 1) = 0$, the formula $A(x, f, y, z)$ is equivalent to $\neg B(x, y)$ and if $f(x, 0, 1) \ne 0$, it is equivalent to $\neg C(x, z)$. Therefore, if $f(x, 0, 1) = 0$, we have $\forall y \le s(x)\neg B(x, y)$ and if $f(x, 0, 1) \ne 0$, we have $\forall z \le s(x)\neg C(x, z)$. We claim that the set $S = \{x \in \mathbb{N} \mid f(x) \ne 0\}$ is a separator. First, note that S is polynomial computable as $f(x, w_0, w_1)$ is a polynomial-time computable function. Second, it is clear that S is disjoint from V. To show that it includes U, assume $x \in U$ and $f(x, 0, 1) = 0$. Then, $\forall y \le s(x)\neg B(x, y)$ which means that $x \notin U$. Therefore, we found a separator which is impossible. Hence, the claimed deterministic $(k, 2)$-reduction does not exist.

As we observed in Example 33, deterministic $(k, 2)$-reductions are not even powerful enough to capture the pure logical implications between the existence of the winning strategies. To solve the problem, in the following, we strengthen the notion by relaxing the determinism in the definition.

Definition 34. *Let $(G(\bar{x}, y, z), t(\bar{x}))$ and $(H(\bar{x}, v, w), s(\bar{x}))$ be two $(k, 2)$-games. By a $(k, 2)$-reduction from (H, s) to (G, t), we mean a finite sequence of $\mathcal{L}_{\text{PV}_k}$-terms $f_0(\bar{x}, y), f_1(\bar{x}, y, w_0), \ldots, f_m(\bar{x}, y, w_0, \ldots, w_{m-1})$ together with an $\mathcal{L}_{\text{PV}_k}$-term $g(\bar{x}, y, w_0, \ldots, w_m)$ such that all the following are provable in PV_k:*

- *$\forall \bar{w} \le s(\bar{x}) \forall y \le t(\bar{x})[f_i(\bar{x}, y, w_0, \ldots, w_{i-1}) \le s(\bar{x})]$, for any $0 \le i \le m$.*
- *$\forall \bar{w} \le s(\bar{x}) \forall y \le t(\bar{x})[g(\bar{x}, y, w_0, \ldots, w_m) \le t(\bar{x})]$.*
- *$\forall \bar{w} \le s(\bar{x}) \forall y \le t(\bar{x})[G(\bar{x}, y, g(\bar{x}, y, w_0, \ldots, w_m)) \to \bar{H}(\bar{x}, y, w_0, \ldots, w_m)]$, where the formula $\bar{H}(\bar{x}, y, w_0, \ldots, w_m)$ is $\bigvee_{i=0}^{m} H(\bar{x}, f_i(\bar{x}, y, w_0, \ldots, w_{i-1}), w_i)$.*

Remark 35. Here is a computational interpretation of a $(k, 2)$-reduction as a nondeterministic version of the deterministic $(k, 2)$-reductions we had before. A $(k, 2)$-reduction starts with reading the first move $y \le t(\bar{x})$ in (G, t) and uses f_0 to transform it to a first move $v_0 \le s(\bar{x})$ in (H, s). Then, as before it reads the second move $w_0 \le s(\bar{x})$ in (H, s). However, instead of using it to find a second move in (G, t), it uses f_1 to come up with another possible first move $v_1 \le s(\bar{x})$ in (H, s) and asks about its second move w_1. It keeps repeating this procedure to finally after $m + 1$ many enquiries, it uses g to compute the second move in (G, t). These computations are in a way that if the produced play for (G, t) is winning, then one of the produced plays for (H, s) is winning.

The following theorem slightly modifies Herbrand's theorem, Theorem 2, to connect the PV_k-provability of the implication between the existence of the strategies and the existence of $(k, 2)$-reductions.

Theorem. *Let $(G(\bar{x}, y, z), t(\bar{x}))$ and $(H(\bar{x}, v, w), s(\bar{x}))$ be two $(k, 2)$-games. Then*

$$\mathrm{PV}_k \vdash \exists y \leq t(\bar{x}) \forall z \leq t(\bar{x}) G(\bar{x}, y, z) \to \exists v \leq s(\bar{x}) \forall w \leq s(\bar{x}) H(\bar{x}, v, w)$$

iff there exists a $(k, 2)$-reduction from (H, s) to (G, t).

Proof. One direction is clear. For the other, assume

$$\mathrm{PV}_k \vdash \exists y \leq t(\bar{x}) \forall z \leq t(\bar{x}) G(\bar{x}, y, z) \to \exists v \leq s(\bar{x}) \forall w \leq s(\bar{x}) H(\bar{x}, v, w).$$

Define $\tilde{G}(\bar{x}, y, z)$ as $[y \leq t(\bar{x}) \wedge (z \leq t(\bar{x}) \to G(\bar{x}, y, z))]$ and $\tilde{H}(\bar{x}, v, w)$ as $[v \leq s(\bar{x}) \wedge (w \leq s(\bar{x}) \to H(\bar{x}, v, w))]$. Now, by moving the quantifiers, we have

$$\mathrm{PV}_k \vdash \forall y \exists v z \forall w [\tilde{G}(\bar{x}, y, z) \to \tilde{H}(\bar{x}, v, w)].$$

Using the pairing function available in PV, we can make two variables v and z into one variable, apply Herbrand's theorem, Theorem 2 and then retrieve y and z again, by projections. Therefore, there are $\mathcal{L}_{\mathrm{PV}_k}$-terms $g_0(\bar{x}, y)$, $h_0(\bar{x}, y)$, $g_1(\bar{x}, y, w_0)$, $h_1(\bar{x}, y, w_0)$, ..., $g_m(\bar{x}, y, w_0, \ldots, w_{m-1})$ and $h_m(\bar{x}, y, w_0, \ldots, w_{m-1})$ such that

$$\mathrm{PV}_k \vdash \bigvee_{i=0}^{m} [\tilde{G}(\bar{x}, y, g_i(\bar{x}, y, w_0, \ldots, w_{i-1})) \to \tilde{H}(\bar{x}, h_i(\bar{x}, y, w_0, \ldots, w_{i-1}), w_i)].$$

Define $g'(\bar{x}, y, w_0, \ldots, w_m)$ by cases: if $\tilde{G}(\bar{x}, y, g_0(\bar{x}, y))$ is false, define g' as $g_0(\bar{x}, y)$; if $\tilde{G}(\bar{x}, y, g_0(\bar{x}, y))$ is true but $\tilde{G}(\bar{x}, y, g_1(\bar{x}, y, w_0))$ is false, define g' as $g_1(\bar{x}, y, w_0)$; if both $\tilde{G}(\bar{x}, y, g_0(\bar{x}, y))$ and $\tilde{G}(\bar{x}, y, g_1(\bar{x}, y, w_0))$ are true but $\tilde{G}(\bar{x}, y, g_2(\bar{x}, y, w_0, w_1))$ is false, define g' as $g_2(\bar{x}, y, w_0, w_1)$ and so on. Finally, if all of $G(\bar{x}, y, g_i(\bar{x}, y, w_0, \ldots, w_{i-1}))$'s are true, define g' as 0:

$$g'(\bar{x}, y, w_0, \ldots, w_m) = \begin{cases} g_0(\bar{x}, y) & \neg\tilde{G}(\bar{x}, y, g_0(\bar{x}, y)) \\ g_1(\bar{x}, y, w_0) & \tilde{G}(\bar{x}, y, g_0(\bar{x}, y)), \neg\tilde{G}(\bar{x}, y, g_1(\bar{x}, y, w_0)) \\ \ldots & \ldots \\ 0 & o.w. \end{cases}$$

Note that g' is defined in a way that unless $\bigwedge_{i=0}^{m} \tilde{G}(\bar{x}, y, g_i(\bar{x}, y, w_0, \ldots, w_{i-1}))$ is true, we always have $\neg\tilde{G}(\bar{x}, y, g'(\bar{x}, y, w_0, \ldots, w_m))$. Therefore, it is easy to see that

$$\mathrm{PV}_k \vdash [\tilde{G}(\bar{x}, y, g'(\bar{x}, y, w_0, \ldots, w_m)) \to \bigwedge_{i=0}^{m} \tilde{G}(\bar{x}, y, g_i(\bar{x}, y, w_0, \ldots, w_{i-1}))],$$

and hence

$$\mathrm{PV}_k \vdash [\tilde{G}(\bar{x}, y, g'(\bar{x}, y, w_0, \ldots, w_m)) \to \bigvee_{i=0}^{m} \tilde{H}(\bar{x}, h_i(\bar{x}, y, w_0, \ldots, w_{i-1}), w_i)].$$

Define

$$f_i(\bar{x}, y, w_0, \ldots, w_{i-1}) = \begin{cases} h_i(\bar{x}, y, w_0, \ldots, w_{i-1}) & h_i(\bar{x}, y, w_0, \ldots, w_{i-1}) \leq s(\bar{x}) \\ 0 & h_i(\bar{x}, y, w_0, \ldots, w_{i-1}) > s(\bar{x}) \end{cases}$$

for any $0 \leq i \leq m$ and set

$$g(\bar{x}, y, w_0, \ldots, w_m) = \begin{cases} g'(\bar{x}, y, w_0, \ldots, w_m) & g'(\bar{x}, y, w_0, \ldots, w_m) \leq t(\bar{x}) \\ 0 & g'(\bar{x}, y, w_0, \ldots, w_m) > t(\bar{x}) \end{cases}$$

It is clear that $\mathrm{PV}_k \vdash f_i(\bar{x}, y, w_0, \ldots, w_{i-1}) \leq s(\bar{x})$, for any $0 \leq i \leq m$ and $\mathrm{PV}_k \vdash g(\bar{x}, y, w_0, \ldots, w_m) \leq t(\bar{x})$. It is also clear that

$$\forall y \leq t(\bar{x}) [G(\bar{x}, y, g(\bar{x}, y, w_0, \ldots, w_m)) \to \tilde{G}(\bar{x}, y, g'(\bar{x}, y, w_0, \ldots, w_m))]$$

and
$$\forall \bar{w} \leq s(\bar{x})[\tilde{H}(\bar{x}, h_i(\bar{x}, y, w_0, \ldots, w_{i-1}), w_i) \to H(\bar{x}, y, f_i(\bar{x}, y, w_0, \ldots, w_{i-1}), w_i)]$$

are provable in PV_k. Therefore, we reach the implication $\forall \bar{w} \leq s(\bar{x}) \forall y \leq t(\bar{x})[G(\bar{x}, y, g(\bar{x}, y, w_0, \ldots, w_m)) \to \bigvee_{i=0}^{m} H(\bar{x}, f_i(\bar{x}, y, w_0, \ldots, w_{i-1}), w_i)]$ in PV_k. $\square$

Definition 36. *Let $k \geq 2$ be a natural number, $A(\bar{x}, y, z) \in \hat{\Sigma}^b_{k-1}$ be an $\mathcal{L}_{\mathrm{PV}_k}$-formula and $t(\bar{x})$ and $r(\bar{x})$ be two $\mathcal{L}_{\mathrm{PV}}$-terms. By a $\mathrm{PLS}_{(k,2)}$-program for the pair $(A(\bar{x}, y, z), r(\bar{x}))$, we mean a $(k, 2)$-game $(G(\bar{x}, u, v, w), s(\bar{x}))$ (read u as a parameter) and*

- *an initial sequence $i(\bar{x}, w)$ of $\mathcal{L}_{\mathrm{PV}_k}$-terms as a $(k, 2)$-reduction from the game $(G(\bar{x}, 0, v, w), s(\bar{x}))$ to $(\top, s(\bar{x}))$,*
- *a sequence $N(\bar{x}, u, v, w)$ of $\mathcal{L}_{\mathrm{PV}_k}$-terms as a $(k, 2)$-reduction from the game $(G(\bar{x}, u + 1, v, w), s(\bar{x}))$ to $(G(\bar{x}, u, v, w), s(\bar{x}))$,*
- *a sequence $p(\bar{x}, v, z)$ of $\mathcal{L}_{\mathrm{PV}_k}$-terms as a $(k, 2)$-reduction from the game $(A(\bar{x}, y, z), r(\bar{x}))$ to $(G(\bar{x}, t(\bar{x}), v, w), s(\bar{x}))$. Here, we pretend that $A(\bar{x}, y, z)$ is a quantifier-free $\mathcal{L}_{\mathrm{PV}_k}$-formula.*

By $\mathrm{PLS}_{(k,2)}$, we mean the class of all the pairs $(A(\bar{x}, y, z), r(\bar{x}))$ for which there exists a $\mathrm{PLS}_{(k,2)}$-program. By $\mathrm{PLS}^p_{(k,2)}$, we mean the class of all the pairs $(A(\bar{x}, y, z), r(\bar{x}))$ for which there exists a $\mathrm{PLS}_{(k,2)}$-program with polynomial length, that is, $t(\bar{x}) = q(|\bar{x}|)$, for some polynomial q.

One can read a (polynomial) $\mathrm{PLS}_{(k,2)}$-program as (a polynomially) an exponentially long sequence of reductions between 2-turn games, starting with an explicit winning strategy for the first game, where all the functions and predicates live in the k-th level of the polynomial hierarchy verified in PV_k.

Similar to what we had in the last subsubsection, we can finally witness provability in T_2^k (resp. S_2^k) by (resp. polynomial) $\mathrm{PLS}_{(k,2)}$-programs.

Corollary 37. *Let $k \geq 2$, $A(\bar{x}, y, z) \in \hat{\Sigma}^b_{k-2}$ and $r(\bar{x})$ be an $\mathcal{L}_{\mathrm{PV}}$-term:*

(i) $S_2^k \vdash \forall \bar{x} \exists y \leq r(\bar{x}) \forall z \leq r(x) A(\bar{x}, y, z)$ iff $(A(\bar{x}, y, z), r(\bar{x})) \in \mathrm{PLS}^p_{(k-1,2)}$.

(ii) $T_2^k \vdash \forall \bar{x} \exists y \leq r(\bar{x}) \forall z \leq r(\bar{x}) A(\bar{x}, y, z)$ iff $(A(\bar{x}, y, z), r(\bar{x})) \in \mathrm{PLS}_{(k-1,2)}$.

Proof. The proof is similar to that of Corollary 30. Therefore, we only explain the main ingredients for *(i)*. For the right to left, assume that there is a $\mathrm{PLS}_{(k,2)}$-program for $(A(\bar{x}, y, z), r(\bar{x}))$ with the length $q(|\bar{x}|)$, for some polynomial q. We use Theorem 8 to transform the existence of the reductions in the $\mathrm{PLS}_{(k,2)}$-program to the following provable implications:

- $\mathrm{PV}_{k-1} \vdash \exists v \leq s(\bar{x}) \forall w \leq s(\bar{x}) G(\bar{x}, 0, v, w)$.
- $\mathrm{PV}_{k-1} \vdash \exists v \leq s(\bar{x}) \forall w \leq s(\bar{x}) G(\bar{x}, u, v, w) \to \exists v \leq s(\bar{x}) \forall w \leq s(\bar{x}) G(\bar{x}, u + 1, v, w)$.
- $\mathrm{PV}_{k-1} \vdash \exists v \leq s(\bar{x}) \forall w \leq s(\bar{x}) G(\bar{x}, q(|\bar{x}|), v, w) \to \exists y \leq r(\bar{x}) \forall z \leq r(\bar{x}) A(\bar{x}, y, z)$.

As any quantifier-free $\mathcal{L}_{\mathrm{PV}_{k-1}}$-formula can be interpreted as an $\mathcal{L}_{\mathrm{PV}}$-formula in $\hat{\Pi}^b_{k-1}$ and PV_{k-1} can be interpreted in S_2^{k-1}, we can pretend that all the above implications are provable in S_2^k and $G \in \hat{\Pi}^b_{k-1}$. Therefore, we can assume that $\exists v \leq s(\bar{x}) \forall w \leq s(\bar{x}) G(\bar{x}, u, v, w) \in \hat{\Sigma}^b_k$. Using LInd in S_2^k on the formula $\exists v \leq s(\bar{x}) \forall w \leq s(\bar{x}) G(\bar{x}, u, v, w)$, we reach $S_2^k \vdash \exists y \leq r(\bar{x}) \forall z \leq r(\bar{x}) A(\bar{x}, y, z)$. Conversely, we assume that $S_2^k \vdash \exists y \leq r(\bar{x}) \forall z \leq r(\bar{x}) A(\bar{x}, y, z)$. Hence, $\forall y \leq r(\bar{x}) \exists z \leq r(\bar{x}) \neg A(\bar{x}, y, z) \to \bot$ is provable in S_2^k. Since $A \in \hat{\Sigma}^b_{k-2}$, the formula $\forall y \leq r(\bar{x}) \exists z \leq r(\bar{x}) \neg A(\bar{x}, y, z)$ is in $\hat{\Pi}^b_k$. Hence, by Theorem 6, we have $\forall y \leq r(\bar{x}) \exists z \leq r(\bar{x}) \neg A(\bar{x}, y, z) \rhd^p_k \bot$. Call the k-flow

$(H(u, \bar{x}), t(\bar{x}))$, where $t(\bar{x}) = q(|\bar{x}|)$, for some polynomial q. Without loss of generality, write $H(u, \bar{x})$ in the form $\forall v \leq s(\bar{x}) \exists w \leq s(\bar{x}) J(\bar{x}, u, v, w)$, where $J \in \hat{\Pi}^b_{k-2}$. As $k \geq 2$, the theory PV is a subtheory of PV_{k-1}. Therefore, moving the implications in the definition of the k-flow from PV to PV_{k-1}, we have:

- $PV_{k-1} \vdash [\forall y \leq r(\bar{x}) \exists z \leq r(\bar{x}) \neg A(\bar{x}, y, z)] \rightarrow \forall v \leq s(\bar{x}) \exists w \leq s(\bar{x}) J(\bar{x}, 0, v, w)$.
- $PV_{k-1} \vdash \forall v \leq s(\bar{x}) \exists w \leq s(\bar{x}) J(\bar{x}, q(|\bar{x}|), v, w) \rightarrow \bot$.
- $PV_{k-1} \vdash \forall v \leq s(\bar{x}) \exists w \leq s(\bar{x}) J(\bar{x}, u, v, w) \rightarrow \forall v \leq s(\bar{x}) \exists w \leq s(\bar{x}) J(\bar{x}, u+1, v, w)$.

As $J \in \hat{\Pi}^b_{k-2}$, in PV_{k-1}, we can pretend that J is a quantifier-free $\mathcal{L}_{PV_{k-1}}$-formula. Define $G(\bar{x}, u, v, w)$ as $\neg J(\bar{x}, q(|\bar{x}|) \dot{-} u, v, w)$. Therefore, we have:

- $PV_{k-1} \vdash \exists v \leq s(\bar{x}) \forall w \leq s(\bar{x}) G(\bar{x}, q(|\bar{x}|), v, w) \rightarrow \exists y \leq r(\bar{x}) \forall z \leq r(\bar{x}) A(\bar{x}, y, z)$.
- $PV_{k-1} \vdash \exists v \leq s(\bar{x}) \forall w \leq s(\bar{x}) G(\bar{x}, 0, v, w)$.
- $PV_{k-1} \vdash \exists v \leq s(\bar{x}) \forall w \leq s(\bar{x}) G(\bar{x}, u, v, w) \rightarrow \exists v \leq s(\bar{x}) \forall w \leq s(\bar{x}) G(\bar{x}, u+1, v, w)$.

Finally, it is enough to use Theorem 8 to get a $PLS_{(k-1,2)}$-program for the pair $(A(\bar{x}, y, z), r(\bar{x}))$ with the length $q(|\bar{x}|)$. Hence, $(A(\bar{x}, y, z), r(\bar{x})) \in PLS^p_{(k-1,2)}$. $\qquad\square$

It is worth putting Corollary 37 for the concrete case $k = 2$ into plain words. Here, the corollary characterizes the T^2_2-provability (resp. S^2_2-provability) of a formula in the form $\forall \bar{x} \exists y \leq r(\bar{x}) \forall z \leq r(x) A(\bar{x}, y, z)$, where A is a polynomial time computable predicate represented as a quantifier-free $\mathcal{L}_{PV}$-formula by the existence of an exponentially (resp. polynomially) long sequence of polynomial time reductions between polynomial time games starting on an explicit polynomial time winning strategy in the first game.

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