J. Austral. Math. Soc. (*Series A*) 41 (1986), 47–50

ON THE COMMUTANTS MODULO C_p OF A^2 AND A^3

FUAD KITTANEH

(Received 7 March 1984)

Communicated by G. H. Chabrowski

Abstract

We prove the following statements about bounded linear operators on a complex separable infinite dimensional Hilbert space. (1) Let A and B^* be subnormal operators. If $A^2X = XB^2$ and $A^3X = XB^3$ for some operator X, then $AX = XB$. (2) Let A and B^* be subnormal operators. If $A^2X - XB^2 \in C_p$ and $A^3X - XB^3 \in C_p$ for some operator X, then $AX - XB \in C_{8p}$. (3) Let T be an operator such that $\mathbf{1} - T^*T \in C_p$ for some $p \geqslant 1$. If $T^2X - XT^2 \in C_p$ and $T^3X - XT^3 \in C_p$ for some operator X, then $TX - XT \in C_p$. (4) Let T be a semi-Fredholm operator with ind $T < 0$. If $T^2X - XT^2 \in C_2$ and $T^3X - XT^3 \in C_2$ for some operator X, then $TX - XT \in C_2$.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 47 B 47, 47 B 10, 47 B 20.
Keywords: commutant, normal operators, subnormal operators, isometry, Schatten p-class.

Let H denote a complex separable infinite dimensional Hilbert space and let $B(H)$ denote the algebra of all bounded linear operators acting on H. Let C_p denote the Schatten p-class with $\|\cdot\|_p$ $(1 \leqslant p \leqslant \infty)$ denoting the associated p-norm. Hence C_2 is the Hilbert-Schmidt class and C_∞ is the class of compact operators. Relations between the commutant of an operator A and the commutants of powers of A have been investigated by many authors, for example, see [1] and [2]. In [1], Al-Moajil proved the following result.

THEOREM 1. *If $N \in B(H)$ is normal and $N^2X = XN^2$ and $N^3X = XN^3$ for some $X \in B(H)$, then $NX = XN$.*

The purpose of this paper is to extend Theorem 1 to intertwining relations between a subnormal and a co-subnormal operator and to present the commutator modulo C_p versions of these results.

Using Berberian's trick we have the following.

THEOREM 2. *Let N and M be normal operators. If $N^2X = XM^2$ and $N^3X = XM^3$ for some $X \in B(H)$, then $NX = XM$.*

PROOF. On $H \oplus H$, let $L = \begin{vmatrix} N & 0 \\ 0 & M \end{vmatrix}$ and $Y = \begin{vmatrix} 0 & X \\ 0 & 0 \end{vmatrix}$. Then L is normal and $L^2Y = YL^2$ and $L^3Y = YL^3$. Therefore Theorem 1 implies that $LY = YL$, from which it follows that $NX = XM$.

COROLLARY 1. *Let A and B^* be subnormal operators. If $A^2X = XB^2$ and $A^3X = XB^3$ for some $X \in B(H)$, then $AX = XB$.*

PROOF. By assumption there exists a Hilbert space H_1 and there exist normal operators N and M on $H \oplus H_1$, such that $N = \begin{vmatrix} A & R \\ 0 & A_1 \end{vmatrix}$ and $M = \begin{vmatrix} B & 0 \\ S & B_1 \end{vmatrix}$. Let $Y = \begin{vmatrix} X & 0 \\ 0 & 0 \end{vmatrix}$. Then easy matrix calculations yield $N^2Y = YM^2$ and $N^3Y = YM^3$. Hence by Theorem 2 we have $NY = YM$, from which we get $AX = XB$ as required.

The next step we want to consider is the investigation of the same relations modulo the C_p class. For $p = \infty$, the situation is easy; just look at the Calkin algebra $B(H)/C_\infty$. For the $p \neq \infty$ case we need first to present the following simple lemma, which is due to Weiss [5].

LEMMA [5]. *Suppose N is normal in $B(H)$, $X \in B(H)$, and I is any two-sided ideal in $B(H)$. Then $NX \in I$ implies $N^*X \in I$ and $XN \in I$ implies $XN^* \in I$.*

THEOREM 3. *Let N be a normal operator. If $N^2X - XN^2 \in C_p$ and $N^3X - XN^3 \in C_p$ for some $X \in B(H)$, then $NX - XN \in C_{8p}$.*

PROOF. Let $K = NX - XN$. if $N^2X - XN^2 = K_1 \in C_p$ and $N^3X - XN^3 = K_2 \in C_p$, then $N^2K = N^2(NX - XN) = N^3X - N^2XN = N^3X - (XN^2 + K_1)N = N^3X - XN^3 - K_1N = K_2 - K_1N \in C_p$. Similarly we can show that $KN^2 \in C_p$. Applying the lemma we obtain $N^*NK \in C_p$ and $KNN^* \in C_p$, and so $(NK)^*(NK) \in C_p$ and $(KN)(KN)^* \in C_p$. Hence $NK \in C_{2p}$ and $KN \in C_{2p}$. Applying the lemma again we see that $N^*K \in C_{2p}$ and $KN^* \in C_{2p}$. Now $KK^*K = K(X^*N^* - N^*X^*)K = KX^*N^*K - KN^*X^*K \in C_{2p}$. Therefore $(KK^*)^2 \in C_{2p}$, and so $KK^* \in C_{4p}$, which is equivalent to saying that $K \in C_{8p}$, as required.

COROLLARY 2. *Let A and B^* be subnormal operators. If $A^2X - XB^2 \in C_p$ and $A^3X - XB^3 \in C_p$ for some $X \in B(H)$, then $AX - XB \in C_{8p}$.*

PROOF. Let N, M and Y be as in the proof of Corollary 1. Then $N^2Y - YM^2 \in C_p$ and $N^3Y - YM^3 \in C_p$. Theorem 3 now implies that $NY - YM \in C_{8p}$. Hence $AX - XB \in C_{8p}$.

QUESTION. Can the $8p$ in Theorem 3 be improved?

Due to the fact (consider the polar decomposition) that if $T \in B(H)$, then $T \in F(H)$ (the ideal of finite rank operators) if and only if $T^*T \in F(H)$, we can modify the proofs of Theorem 3 and Corollary 2 to get the following.

THEOREM 4. *Let A and B^* be subnormal operators. If $A^2X - XB^2 \in F(H)$ and $A^3X - XB^3 \in F(H)$ for some $X \in B(H)$, then $AX - XB \in F(H)$.*

If we replace the normal operator in Theorem 3 by an isometry modulo C_p, then we obtain the following better result.

THEOREM 5. *Let $T \in B(H)$ be such that $\mathbf{1} - T^*T \in C_p$ for some $p \geqslant 1$. If $T^2X - XT^2 \in C_p$ and $T^3X - XT^3 \in C_p$ for some $X \in B(H)$, then $TX - XT \in C_p$.*

PROOF. It has been proved in [3] that if $\mathbf{1} - T^*T \in C_\infty$, then $T = C + V$, where $C \in C_\infty$, and where V is either an isometry or a co-isometry with finite dimensional null space. But the proof applies to any $p < \infty$ as well.

Now $T^2X - XT^2 \in C_p$ and $T^3X - XT^3 \in C_p$ imply that $V^2X - XV^2 \in C_p$ and $V^3X - XV^3 \in C_p$. Let $K = VX - XV$; then, as in the proof of Theorem 3, we have $V^2K \in C_p$ and $KV^2 \in C_p$. If V is an isometry, then $K = V^{*2}V^2K \in C_p$. If V^* is an isometry, then $K = KV^2V^{*2} \in C_p$. Hence, in either case, $K \in C_p$, and this completes the proof.

The self-adjointness modulo C_p, i.e. the fact that $T - T^* \in C_p$, is not as good here as $\mathbf{1} - T^*T \in C_p$. In fact we can easily see

THEOREM 6. *Let $T \in B(H)$ be such that $T - T^* \in C_p$. If $T^2X - XT^2 \in C_p$ and $T^3X - XT^3 \in C_p$ for some $X \in B(H)$, then $TX - XT \in C_{8p}$.*

For $p = 2$, we have the following result.

THEOREM 6. *Let $T \in B(H)$ be a semi-Fredholm operator with* $\operatorname{ind} T < 0$ (ind *denotes the usual Fredholm index*). *If $T^2X - XT^2 \in C_2$ and $T^3X - XT^3 \in C_2$ for some $X \in B(H)$, then $TX - XT \in C_2$.*

PROOF. Since $\operatorname{ind} T < 0$, there exists a finite rank partial isometry F whose initial space is $\operatorname{Ker} T$ and whose range is properly contained in $\operatorname{Ker} T^*$. It follows that $T + F$ is bounded below, and that $(T + F)^*$ has a non-trivial kernel (see [4]). Let $T + F = A$ and $K = TX - XT$. Then, as we have seen in the proof of Theorem 3, $T^2K \in C_2$. Thus $(A - F)^2(AX - XA + XF - FX) \in C_2$, and from this we conclude that $A^2(AX - XA) \in C_2$. If $\{e_n\}$ is an orthonormal basis for H, then $\infty > \Sigma\|A^2(AX - XA)e_n\|^2 \geqslant b^4\Sigma\|(AX - XA)e_n\|^2$, where b is a constant such that $\|Ax\| \geqslant b\|x\|$ for all $x \in H$. Therefore $AX - XA \in C_2$. But then $TX - XT = AX - XA + XF - FX \in C_2$, which completes the proof.

References

[1] A. H. Al-Moajil, 'The commutants of relatively prime powers in Banach algebras', *Proc. Amer. Math. Soc.* **57** (1976), 243–249.
[2] M. R. Embry, '*n*th roots of operators', *Proc. Amer. Math. Soc.* **19** (1968), 63–68.
[3] P. Fillmore, J. Stampfli and J. Williams, 'On the essential numerical range, the essential spectrum, and a problem of Halmos', *Acta Sci. Math.* (*Szeged*) **33** (1972), 179–192.
[4] C. Pearcy, *Some recent developments in operator theory* (Lecture Notes No. 36, Amer. Math. Soc., Providence, R. I., 1978).
[5] G. Weiss, 'The Fuglede commutativity theorem modulo operator ideals', *Proc. Amer. Math. Soc.* **83** (1981), 113–118.

Department of Mathematics
United Arab Emirates University
P.O. Box 15551 Al-Ain
U.A.E.