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THE ISOMETRIES OF $H^p(K)$

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Abstract

Let $1 \leq p < \infty$, $p \neq 2$ and let K be any complex Hilbert space. We prove that every isometry T of $H^p(K)$ onto itself is of the form

$$(TF)(z) = U(F \circ \phi(z)) \cdot (d\phi/dz)^{1/p} \qquad (F \in H^p(K), |z| < 1),$$

where U is a unitary operator on K and ϕ is a conformal map of the unit disc onto itself.

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1. Introduction

Let D be the open unit disc in the complex plane and let E be any complex Banach space. Then the Banach space $H^p(E)$, $1 \leq p \leq \infty$, consists of all $F : D \to E$ such that $\langle F, e^* \rangle$ belongs to the Hardy class H^p for all $e^* \in E^*$, and the norm of F is given by

$$\|F\|_p = \lim_{r \to 1^-} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \|F(re^{i\theta})\|^p \, d\theta \right\} \quad \text{if } 1 \leq p < \infty,$$
$$\|F\|_\infty = \operatorname*{ess\,sup}_{|z|<1} \|F(z)\|.$$

A complex Banach space E is said to have the *analytic Radon-Nikodym property*, if for each $F \in H^p(E)$, $F(e^{i\theta}) = \lim_{r \to 1^-} F(re^{i\theta})$ exists almost everywhere (for more detail see [1] and [4]). (It is known that the L_p-spaces,

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$1 \leq p < \infty$, have the analytic Radon-Nikodym property.) If E has the analytic Radon-Nikodym property and if $F \in H^p(E)$, then the norm of F is

$$\|F\|_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \|F(e^{i\theta})\|^p \, d\theta \right\}^{1/p} \quad \text{if } 1 \leq p < \infty,$$

$$\|F\|_\infty = \operatorname*{ess\,sup}_{0 \leq \theta \leq 2\pi} \|F(e^{i\theta})\|.$$

The linear isometries of H^p were first studied by deLeeuw, Rudin, and Wermer [6]. They proved that if T is a surjective isometry on H^1 (respectively, H^∞), then there are a conformal map ϕ of the unit disk onto itself and a unimodular complex number b such that

$$Tf = b \cdot (d\phi/dz) \cdot f \circ \phi \quad (\text{respectively, } Tf = b \cdot f \circ \phi).$$

Later, F. Forelli [7] extended this result to H^p for $p \neq 2$. He proved that

THEOREM A. *If $p \neq 2$ and if T is a linear isometry of H^p onto H^p, then there is a conformal map ϕ of the unit disk onto itself and a unimodular complex number b such that*

$$Tf = b \cdot (d\phi/dz)^{1/p} \cdot f \circ \phi.$$

The isometries of the vector valued H^p function spaces were studied by M. Cambern. He [2] showed that if K is a complex Hilbert space and if T is a surjective isometry on $H^\infty(K)$, then there are a conformal map ϕ of the unit disc onto itself and a unitary operator U on K such that for any $F \in H^\infty(K)$ and any $z \in D$,

$$TF(z) = U(F \circ \phi(z)).$$

Recently, M. Cambern and K. Jarosz [3] proved a similar result holds on $H^1(K)$ if K is a finite dimensional complex Hilbert space. In this article, we extend this result to $H^p(K)$, $1 \leq p < \infty$. The main result of this article is the following theorem.

MAIN THEOREM. *Let $1 \leq p < \infty$, $p \neq 2$, and let K be any complex Hilbert space. If $T: H^p(K) \to H^p(K)$ is a surjective isometry, then there exist a unitary operator U on K, and a conformal map ϕ from the disc onto the disc such that*

$$(TF)(z) = U(F \circ \phi(z)) \cdot (d\phi/dz)^{1/p}(z) \qquad (F \in H^p(K),\ |z| < 1).$$

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Let ϕ be a conformal map of the unit disk onto itself, and let U be a unitary operator on a complex Hilbert space K. If

$$TF(z) = U(F \circ \phi(z)) \cdot (d\phi/dz)^{1/p}(z)$$

for all $F \in H^p(K)$, then T satisfies the following conditions.

(a) If $\langle e_1, e_2 \rangle = 0$, then $\langle T(fe_1)(e^{i\theta}), T(ge_2)(e^{i\theta}) \rangle = 0$ a.e. for all $f, g \in H^p$.

(a$'$) If $\langle e_1, e_2 \rangle = 0$, then $\langle T(z^n e_1)(e^{i\theta}), T(z^m e_2)(e^{i\theta}) \rangle = 0$ a.e. for all $n, m \geq 0$.

(b) $|\langle T(fe)(e^{i\theta}), T(ge)(e^{i\theta}) \rangle| = \|T(fe)(e^{i\theta})\| \cdot \|T(ge)(e^{i\theta})\|$ a.e. (that is $T(fe)(e^{i\theta})$ and $T(ge)(e^{i\theta})$ are linearly dependent a.e.) for all $e \in K$ and $f, g \in H^p$.

(b$'$) $|\langle T(z^n e)(e^{i\theta}), T(z^m e)(e^{i\theta}) \rangle| = \|T(z^n e)(e^{i\theta})\| \cdot \|T(z^m e)(e^{i\theta})\|$ a.e. for all $e \in K$ and $n, m \geq 0$

(c) For any $e \in K$, there is $e' \in K$ such that $T(H^p e) = H^p e'$. Moreover, if $T(H^p e_1) = H^p e_3$, $T(H^p e_2) = H^p e_4$, and $\langle e_1, e_2 \rangle = 0$, then $\langle e_3, e_4 \rangle = 0$.

Clearly, (a) implies (a$'$), (b) implies (b$'$), and (c) implies (a) and (b). Since $\{z^n : n \geq 0\}$ spans H^p, (a$'$) implies (a) and (b$'$) implies (b). By Theorem A, one can show that if T is a surjective isometry on $H^p(K)$ which satisfies (c), then T satisfies the conclusion of the Main Theorem (see Section 3). Hence, we only need to show every surjective isometry on $H^p(K)$ satisfies (c). However, we do not know any direct proof. In Section 2, we establish the following proposition.

Proposition 1. *Suppose that $1 \leq p < \infty$, and $p \neq 2$. If K is a complex Hilbert space and if T is a linear isometry of $H^p(K)$ onto $H^p(K)$, then T satisfies* (a$'$).

If $1 \leq p < 2$, we provide a direct proof of Proposition 1. But we do not know whether there is a direct proof if $2 < p < \infty$. In this case, we first show that T satisfies (b$'$), and then use (b) to prove Proposition 1. In Section 3, we use the conclusion of Proposition 1 to show that every surjective isometry on $H^p(K)$ satisfies (c), and then give the proof of the Main Theorem.

2. The proof of Proposition 1

In this section, we will assume that (i) $1 \leq p < \infty$ and $p \neq 2$, (ii) K is a complex Hilbert space, and (iii) T is a surjective isometry of $H^p(K)$ onto itself. Before proving Proposition 1, we need the following fact.

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FACT 1. Suppose that $1 \le p < 2$. If f_1, f_2 are any two positive functions in L^p such that $f_2 > 0$ a.e. and $\|f_1\|_p = 1 = \|f_2\|_p$, then

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{f_1^2(t)}{f_2^{2-p}(t)}\,dt \ge 1.$$

Moreover, equality holds if and only if $f_1 = f_2$ a.e.

PROOF OF PROPOSITION 1 WHEN $1 \le p < 2$. Let e_1 and e_2 be any two nonzero vectors in K. If $\|re_2\| < \|e_1\|$, then

$$\begin{aligned}
\|e_1 + re^{ix}e_2\|^p &= (\|e_1\|^2 + r^2\|e_2\|^2 + r(e^{-ix}\langle e_1, e_2\rangle + e^{ix}\langle e_2, e_1\rangle))^{p/2} \\
&= \|e_1\|^p\left(1 + \frac{r}{\|e_1\|^2}(r\|e_2\|^2 + (e^{-ix}\langle e_1, e_2\rangle + e^{ix}\langle e_2, e_1\rangle))\right)^{p/2} \\
&= \|e_1\|^p \sum_{j=0}^{\infty}\binom{p/2}{j}\left(\frac{r}{\|e_1\|^2}\right)^j (r\|e_2\|^2 + (e^{-ix}\langle e_1, e_2\rangle + e^{ix}\langle e_2, e_1\rangle))^j \\
&= \sum_{j=0}^{\infty}\binom{p/2}{j}\|e_1\|^{p-2j} r^j (r\|e_2\|^2 + (e^{-ix}\langle e_1, e_2\rangle + e^{ix}\langle e_2, e_1\rangle))^j,
\end{aligned}$$

and so

$$(*)\qquad \begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\|e_1 + re^{ix}e_2\|^p\,dx \\
&\quad = \sum_{j=0}^{\infty}\sum_{l=0}^{[\frac{j}{2}]}\binom{p/2}{j}\binom{j}{2l}\binom{2l}{l} r^{2j-2l}\|e_1\|^{p-2j}\|e_2\|^{2j-4l}|\langle e_1, e_2\rangle|^{2l}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\lim_{r\to 0}\frac{1}{r^2}\left(\frac{1}{2\pi}\int_0^{2\pi}\|e_1 + re^{ix}e_2\|^p\,dx - \|e_1\|^p\right) \\
&\quad = \frac{p}{2}\left(\|e_1\|^{p-2}\|e_2\|^2 + \frac{p-2}{2}\|e_1\|^{p-4}|\langle e_1, e_2\rangle|^2\right).
\end{aligned}$$

Let $F = T(z^m e_1)$ and $G = T(z^n e_2)$. Then $\|F(e^{i\theta})\| \ne 0$ a.e. By Fatou's Lemma and the Fubini Theorem,

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$$\frac{p}{4\pi}\int_0^{2\pi}\|F(e^{i\theta})\|^{p-2}\|G(e^{i\theta})\|^2+\frac{p-2}{2}\|F(e^{i\theta})\|^{p-4}|\langle F(e^{i\theta}),G(e^{i\theta})\rangle|^2\,d\theta$$
$$\le \liminf_{r\to 0}\frac{1}{4r^2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\|F(e^{i\theta})+re^{ix}G(e^{i\theta})\|^p-\|F(e^{i\theta})\|^p\,dx\,d\theta$$
$$=\liminf_{r\to 0}\frac{1}{4r^2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\|F(e^{i\theta})+re^{ix}G(e^{i\theta})\|^p-\|F(e^{i\theta})\|^p\,d\theta\,dx$$
$$=\liminf_{r\to 0}\frac{1}{2r^2\pi}\int_0^{2\pi}\|F+re^{ix}G\|_p^p-\|F\|_p^p\,dx$$
$$=\liminf_{r\to 0}\frac{1}{2r^2\pi}\int_0^{2\pi}\|z^me_1+re^{ix}z^ne_2\|_p^p-\|z^me_1\|_p^p\,dx$$
$$=\liminf_{r\to 0}\frac{((1+r^2)^{p/2}-1)}{r^2}=\frac{p}{2}.$$

But $\|F\|_p=\|z^me_1\|_p=1=\|z^ne_2\|_p=\|G\|_p$. By Fact 1, we have $\|F(e^{i\theta})\|=\|G(e^{i\theta})\|$ a.e. and $\langle F(e^{i\theta}),G(e^{i\theta})\rangle=0$ a.e.

Now, we assume $2<p<\infty$, and we need the following lemma.

LEMMA 2. *Let $m\neq n$, and let e be any nonzero element in K. If $F=T(z^me)$ and $G=T(z^ne)$, then $\|F(e^{i\theta})\|=\|G(e^{i\theta})\|$ a.e., and*

$$|\langle F(e^{i\theta}),G(e^{i\theta})\rangle|=\|F(e^{i\theta})\|\cdot\|G(e^{i\theta})\|\quad a.e.$$

PROOF. By $(*)$, there exists $A>0$ such that for any two nonzero vectors e_1 and e_2 in K,

$$\frac{1}{2\pi}\int_0^{2\pi}\|e_1+re^{ix}e_2\|^p-\|e_1\|^p\,dx\le Ar^2\|e_1\|^{p-2}\|e_2\|^2$$

whenever $\|e_1\|>2r\|e_2\|$. On the other hand, if $\|e_1\|\le 2r\|e_2\|$, then

$$\frac{1}{2\pi}\int_0^{2\pi}\|e_1+re^{ix}e_2\|^p-\|e_1\|^p\,dx\le(3^p+1)r^p\|e_2\|^p.$$

So

$$\frac{1}{2\pi r^2}\int_0^{2\pi}\|F(e^{i\theta})+re^{ix}G(e^{i\theta})\|^p-\|F(e^{i\theta})\|\,dx$$
$$\le\max(A,3^p+1)(\|F(e^{i\theta})\|^{p-2}\|G(e^{i\theta})\|^2+\|G(e^{i\theta})\|^p)$$

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for all $0 < r < 1$. By the dominated convergence theorem,

$$\begin{aligned}
&\frac{p}{4\pi}\int_0^{2\pi} \|F(e^{i\theta})\|^{p-2}\|G(e^{i\theta})\|^2 + \frac{p-2}{2}\|F(e^{i\theta})\|^{p-4}|\langle F(e^{i\theta}), G(e^{i\theta})\rangle|^2\,d\theta \\
&\quad= \lim_{r\to 0}\frac{1}{4r^2\pi^2}\int_0^{2\pi}\int_0^{2\pi} \|F(e^{i\theta}) + re^{ix}G(e^{i\theta})\|^p - \|F(e^{i\theta})\|^p\,dx\,d\theta \\
&\quad= \lim_{r\to 0}\frac{1}{4r^2\pi^2}\int_0^{2\pi}\int_0^{2\pi} \|F(e^{i\theta}) + re^{ix}G(e^{i\theta})\|^p - \|F(e^{i\theta})\|^p\,d\theta\,dx \\
&\quad= \lim_{r\to 0}\frac{1}{2r^2\pi}\int_0^{2\pi} \|F + re^{ix}G\|_p^p - \|F\|_p^p\,dx \\
&\quad= \lim_{r\to 0}\frac{1}{2r^2\pi}\int_0^{2\pi} \|z^m e + re^{ix}z^n e\|_p^p - \|z^m e\|_p^p\,dx \\
&\quad= \lim_{r\to 0}\frac{1}{4r^2\pi^2}\int_0^{2\pi}\int_0^{2\pi} |1 + re^{ix+i(n-m)\theta}|^p - 1\,d\theta\,dx \\
&\quad= \lim_{r\to 0}\frac{1}{4r^2\pi^2}\int_0^{2\pi}\int_0^{2\pi} |1 + re^{ix+i(n-m)\theta}|^p - 1\,dx\,d\theta = \frac{p^2}{4}.
\end{aligned}$$

This implies

$$p = \frac{1}{\pi}\int_0^{2\pi} \|F(e^{i\theta})\|^{p-2}\|G(e^{i\theta})\|^2 + \frac{p-2}{2}\|F(e^{i\theta})\|^{p-4}|\langle F(e^{i\theta}), G(e^{i\theta})\rangle|^2\,d\theta.$$

By Hölder inequality, we have $\|F(e^{i\theta})\| = \|G(e^{i\theta})\|$ a.e. and

$$|\langle G(e^{i\theta}), F(e^{i\theta})\rangle| = \|F(e^{i\theta})\|^2 \quad \text{a.e.}$$

Proof of Proposition 1 when $2 < p < \infty$. By Lemma 2, for any $e \in K$, $T(z^m e)(e^{i\theta})$ and $T(z^n e)(e^{i\theta})$ are linearly dependent for almost all $\theta \in [0, 2\pi]$. Since $\{z^n : n \geq 0\}$ spans H^p, for any $e \in K$ and any $f, g \in H^p$, $T(fe)(e^{i\theta})$ and $T(ge)(e^{i\theta})$ are linear dependent for almost all $\theta \in [0, 2\pi]$. Hence, for each $e \in K\backslash\{0\}$, $T|_{H^p e}$ induces an isometry from H^p into L^p. By the proof of [7, Theorem 1], there is a function h_e such that

(1) $|h_e(e^{i\theta})| = 1$ a.e.,

(2) for each $n \in \mathbb{N}$, $T(z^n e) = h_e^n T(1_D e)$.

Clearly, if $h_e = h_{e'}$, then $h_e = h_{\alpha e + \beta e'}$ for all $\alpha, \beta \in \mathbb{C}$.

(1) Let e, e' be any two unit elements in K. We claim that $h_e = h_{e'}$. Since

$$h_e T(1_D e) + h_{e'} T(1_D e') = T(ze + ze') = h_{e+e'} T(1_D(e + e')),$$

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and

$$h_e T(1_D e) + i h_{e'} T(1_D e') = T(ze + ize') = h_{e+ie'} T(1_D(e + ie')),$$

for almost all $\theta \in [0, 2\pi]$ we have

$$\begin{aligned}
&\|T(1_D e)(e^{i\theta})\|^2 + \|T(1_D e')(e^{i\theta})\|^2 \\
&\qquad + h_e(e^{i\theta})\overline{h}_{e'}(e^{i\theta})\langle T(1_D e)(e^{i\theta}), T(1_D e')(e^{i\theta})\rangle \\
&\qquad + h_{e'}(e^{i\theta})\overline{h}_e(e^{i\theta})\langle T(1_D e')(e^{i\theta}), T(1_D e)(e^{i\theta})\rangle \\
&\quad = \|T(1_D e)(e^{i\theta})\|^2 + \|T(1_D e')(e^{i\theta})\|^2 + \langle T(1_D e)(e^{i\theta}), T(1_D e')(e^{i\theta})\rangle \\
&\qquad + \langle T(1_D e')(e^{i\theta}), T(1_D e)(e^{i\theta})\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\|T(1_D e)(e^{i\theta})\|^2 + \|T(1_D e')(e^{i\theta})\|^2 \\
&\qquad - i h_e(e^{i\theta})\overline{h}_{e'}(e^{i\theta})\langle T(1_D e)(e^{i\theta}), T(1_D e')(e^{i\theta})\rangle \\
&\qquad + i h_{e'}(e^{i\theta})\overline{h}_e(e^{i\theta})\langle T(1_D e')(e^{i\theta}), T(1_D e)(e^{i\theta})\rangle \\
&\quad = \|T(1_D e)(e^{i\theta})\|^2 + \|T(1_D e')(e^{i\theta})\|^2 - i\langle T(1_D e)(e^{i\theta}), T(1_D e')(e^{i\theta})\rangle \\
&\qquad + i\langle T(1_D e')(e^{i\theta}), T(1_D e)(e^{i\theta})\rangle.
\end{aligned}$$

So

$$\langle T(1_D e)(e^{i\theta}), T(1_D e')(e^{i\theta})\rangle = h_e(e^{i\theta})\overline{h}_{e'}(e^{i\theta})\langle T(1_D e)(e^{i\theta}), T(1_D e')(e^{i\theta})\rangle.$$

Replacing e' by $e' + re$ for some $r \in \mathbb{R}$ if necessary, we may assume that

$$\langle T(1_D e)(e^{i\theta}), T(1_D e')(e^{i\theta})\rangle \neq 0 \quad \text{a.e.}$$

Therefore, $h_e = h_{e'}$ a.e., and if $F \in H^p(K)$, then $T(z^n F) = h_e^n T(F)$.

(2) Since T is an onto mapping, there is $F \in H^p(K)$ such that $T(F) = 1_D e$. So $T(zF) = h_e 1_D e$ and $h_e \in H^\infty$.

(3) By (2) there exist two inner functions h and h' such that for any $g \in H^\infty$ and $F \in H^p(K)$, $T(gF) = g \circ h \cdot T(F)$ and $T^{-1}(gF) = g \circ h' \cdot T^{-1}(F)$. From $TT^{-1}(F) = F = T^{-1}T(F)$, we find

$$g \circ h' \circ h \cdot F = g \cdot F = g \circ h \circ h' \cdot F.$$

So $h \circ h' = I = h' \circ h$ and h is a conformal map of the unit disk onto itself.

(4) Since h is an onto conformal mapping, for any $f \in H^\infty$ there is $g \in H^\infty$ such that $f = g \circ h$. Hence, if e, e' are any two unit vectors in K,

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and g is any function in H^∞, then we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} & |f(e^{i\theta})|^p \|T(1_D e)(e^{i\theta})\|^p\, d\theta \\ &= \|fT(1_D e)\|_p^p = \|T(ge)\|_p^p \\ &= \|T(ge')\|_p^p = \|fT(1_D e')\|_p^p \\ &= \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^p \|T(1_D e')(e^{i\theta})\|^p\, d\theta.\end{aligned}$$

Since $\{|g|: g \in H^\infty\}$ spans real L^∞, $\|T(1_D e)(e^{i\theta})\| = \|T(1_D e')(e^{i\theta})\|$ a.e.

(5) Now, suppose that $\langle e, e'\rangle = 0$. Then there exists a measure zero subset A of $[0, 2\pi]$ such that if $\theta \notin A$ and $\alpha \in \mathbb{Q}$, then

$$\|\cos\alpha T(1_D e)(e^{i\theta}) + \sin\alpha T(1_D e')(e^{i\theta})\| = \|T(1_D e)(e^{i\theta})\|.$$

By continuity,

$$\|\cos\alpha T(1_D e)(e^{i\theta}) + \sin\alpha T(1_D e')(e^{i\theta})\| = \|T(1_D e)(e^{i\theta})\|$$

whenever $\alpha \in \mathbb{R}$ and $\theta \notin A$. So we have $\langle T(1_D e)(e^{i\theta}), T(1_D e')(e^{i\theta})\rangle = 0$. □

REMARK 1. Let $T: H^p(K) \to H^p(K)$ be an onto isometry. If e_1 and e_2 are linearly independent, then $T(fe_1)(e^{i\theta})$ and $T(fe_2)(e^{i\theta})$ are linearly independent for almost for θ.

3. Proof of Theorem

Before proving the Main Theorem, we need another lemma.

LEMMA 3. *For any unit vector e in K, there exists a unit vector e' such that $T(H^p e) = H^p e'$.*

PROOF. Let $\{e_j: j \in J\}$ be an orthonormal basis of K. For any unit vector $e \in K$, there exist $F_j \in T(H^p e_j)$, such that $1_D e = \sum_{j\in J} F_j$. Clearly, $F_j = 0$ except for countably many j. Hence, we may assume that J is countable and $F_j(e^{i\theta})$'s are orthogonal. So for any $\theta \in [0, 2\pi]$, we have

(i) $\sum_{j\in J}\langle F_j(e^{i\theta}), e\rangle = 1_D$,

(ii) $\sum_{j\in J}\|F_j(e^{i\theta})\|^2 = 1$.

(1) For each $j \in J$, $\langle F_j(e^{i\theta}), e\rangle$ is an analytic function and for any $\theta \in [0, 2\pi]$

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$$1 = \left\| \sum_{j\in J} \overline{\operatorname{sgn}(\langle F_j(e^{i\theta}), e\rangle)} F_j(e^{i\theta}) \right\| \ge \sum_{j\in J} |(\langle F_j(e^{i\theta}), e\rangle)| \ge 1.$$

So $\langle F_j(e^{i\theta}), e\rangle$ is a non-negative constant function for each $j \in J$.

(2) If $\langle F_k(e^{i\theta}), e\rangle = 0$, then

$$1 \le \left\| \sum_{j\neq k} F_j \right\|_p \le \left\| \sum_{j\in J} F_j \right\|_p = 1.$$

The second inequality holds if and only if $F_k \neq 0$. So we must have $F_k = 0$ if $\langle F_k(e^{i\theta}), e\rangle = 0$.

(3) Let e' be a nonzero element in K and k be a fixed element in J. We claim that if there exists an $f \in H^p$ such that $m\{\theta\colon \|T(fe_k)(e^{i\theta}) - e'\| < 1/n\} > 0$ for every $n \in \mathbb{N}$, then $1_D e' \in T(H^p e_k)$. With loss of generality, we may assume $\|e'\| = 1$. Since there exist $F_j \in T(H^p e_j)$ such that $\sum_{j\in J} F_j = 1_D e'$, by Proposition 1, there exists a measurable set A such that

(iii) $m(A) = 0$,

(iv) if $\theta \notin A$, then $\{F_j(e^{i\theta})\colon j \in J\}$ (respectively $\{T(fe_k)(e^{i\theta})\} \cup \{F_j(e^{i\theta})\colon j \neq k\}$) is orthogonal.

Hence, if $\theta \notin A$, then there exist $1 \ge a \ge 0$, $b \in \mathbb{C}$ and $z, y \in K$ which satisfy

(v) $\langle z, e'\rangle = 0 = \langle y, e'\rangle$,

(vi) $F_k(e^{i\theta}) = ae' + z$, $\sum_{j\neq k} F_j(e^{i\theta}) = (1-a)e' - z$, and $T(fe_k)(e^{i\theta}) = be' + y$.

So we have

$$b(1-a) - \langle z, y\rangle = \left\langle \sum_{j\neq k} F_j(e^{i\theta}), F_k(e^{i\theta}) \right\rangle = 0,$$

$$a(1-a) - \|z\|^2 = \langle T(fe_k)(e^{i\theta}), f_k(e^{i\theta})\rangle = 0.$$

If $\|be' + y - e'\| < 1/n$, then $|b| \ge 1 - 1/n$, $\|y\| \le 1/n$,

$$\frac{\|z\|}{n} \ge |\langle z, y\rangle| = |b(1-a)| \ge \left(1 - \frac{1}{n}\right)(1-a),$$

$$\|z\| \ge (n-1)(1-a), \quad a(1-a) = \|z\|^2 \ge (n-1)^2(1-a)^2.$$

So we have $a \ge (n-1)^2/n^2$. But $\langle F_k, e'\rangle$ is a constant function, so $\langle F_k, e'\rangle \equiv 1$. By (1) and (2), $F_j \equiv 0$ for all $j \neq i$, and $F_k \equiv 1$.

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Suppose that there exist e_1', $e_2' \in K$ and f_1, $f_2 \in H^p$ such that

$$m\{\theta: \|T(f_1 e)(e^{i\theta}) - e_1'\| < 1/n\} > 0$$

(respectively, $m\{\theta: \|T(f_2 e)(e^{i\theta}) - e_2'\| < 1/n\} > 0$) for all $n \in \mathbb{N}$. By (3), $1_D e_1'$ and $1_D e_2'$ are in $T(H^p e)$. But T^{-1} is a surjective isometry from $H^p(K)$ onto $H^p(K)$. By Remark 1, e_1' and e_2' are linearly dependent. And we have proved the lemma.

PROOF OF THEOREM. Let e be any unit vector in K. By Lemma 3, there exists a unit vector e' such that $T(fe) = \langle T(fe), e'\rangle e'$. We define the operator (it may not be linear) U by $U(ce) = ce'$ for all $c \in \mathbb{C}$.

By Lemma 3, the restriction of T to $H^p e_1$ is a surjective isometry from $H^p e_1$ into $H^p U(e_1)$. Hence, there exist a conformal map ϕ_1 of the disc onto itself, and a unimodular complex number b_1 such that $T(fe_1) = b_1 \cdot (d\phi_1/dz)^{1/p} \cdot f \circ \phi_1 \cdot U(e_1)$. (Replacing $U(e_1)$ by $b_1 U(e_1)$, we may assume that $b_1 = 1$.) If e_2 is any other vector in K, then there exist a conformal map ϕ_2 of the disc onto itself, and a unimodular complex number b_2 such that $T(fe_2) = b_2 \cdot (d\phi_2/dz)^{1/p} \cdot f \circ \phi_2 \cdot U(e_2)$. We claim that $\phi_2 = \phi_1$. Clearly, this is true if e_1 and e_2 are linearly dependent. So we may assume that e_1 and e_2 are linearly independent. By Lemma 3,

$$\begin{aligned}(d\phi_1/dz)^{1/p} \cdot f \circ \phi_1 \cdot U(e_1) + b_2 \cdot (d\phi_2/dz)^{1/p} \cdot f \circ \phi_2 \cdot U(e_2) \\ = T(f(e_1 + e_2)) = \left\langle T(f(e_1 + e_2)), U\left(\frac{e_1 + e_2}{\|e_1 + e_2\|}\right)\right\rangle U\left(\frac{e_1 + e_2}{\|e_1 + e_2\|}\right).\end{aligned}$$

Since $U(e_1)$ and $U(e_2)$ are linearly independent (by Remark 1), we have $(d\phi_1/dz)^{1/p} f \circ \phi_1$ and $(d\phi_2/dz)^{1/p} f \circ \phi_2$ are linearly dependent. Let $f = 1$. Then we have $(d\phi_2/dz) = d_1(d\phi_1/dz)$ or $\phi_2 = d_1\phi_1 + d_2$ for some $d_1, d_2 \in \mathbb{C}$. But ϕ_1 and ϕ_2 are conformal maps from the unit disc onto itself. This implies $|d_1| = 1$ and $d_2 = 0$. Let $f = z + 1$. We have $(d\phi_1/dz)^{1/p}(\phi_1 + 1)$ and $d_1(d\phi_1/dz)^{1/p}(d_1\phi_1 + 1)$ are linearly dependent. But $\phi_1 \neq 1$. So d_1 must be 1.

Replace $U(ce_2)$ by $b_2 \cdot c \cdot U(e_2)$. Then we have $T(fe) = (d\phi_1/dz)^{1/p} \cdot f \circ \phi_1 \cdot U(e)$ for any $f \in H^p$ and $\|e\| = 1$. Hence, for any $a, b \in \mathbb{C}$

$$\begin{aligned}(d\phi_1/dz)^{1/p} \cdot a \cdot U(e_1) + (d\phi_1/dz)^{1/p} \cdot b \cdot U(e_2) \\ = T(ae_1 + be_2) = (d\phi_1/dz)^{1/p} \cdot (\|ae_1 + be_2\|) \cdot U\left(\frac{ae_1 + be_2}{\|ae_1 + be_2\|}\right) \\ = (d\phi_1/dz)^{1/p} \cdot U(ae_1 + be_2).\end{aligned}$$

This implies that U is a linear isometry. Since T is an onto mapping, U must be an onto mapping. So U is a unitary operator.

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