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THE ARITHMETIC OF THE QUASI-UNISERIAL SEMIGROUPS WITHOUT ZERO

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An element a in a partially ordered semigroup T is called integral if

$$ax \subseteq x \text{ and } xa \subseteq x \text{ for every } x \in T$$

is valid. The integral elements form a subsemigroup S of T if they exist. Two different integral idempotents e and f in T generate different one-sided ideals, because $eT = fT$, say, implies $e = fe \subseteq f$ and $f = ef \subseteq e$.

Let M be a completely simple semigroup. M is the disjoint union of its maximal subgroups [**4**]. Their identity elements generate the minimal one-sided ideals in M. The previous paragraph suggests the introduction of the following hypothesis on M.

Hypothesis 1. Every minimal one-sided ideal in M is generated by an integral idempotent.

It is the objective of this paper to derive the S-arithmetic of M, i.e. the theory of the lattice ordered semigroup $V_S(M)$ of the S-ideals in M, in the case that the structure group G of M is the naturally ordered infinite cyclic group. Then the results of this investigation can be applied to the arithmetic of the D^*-arithmetic prime rings (see, e.g., [**2**; **3**]).

Denote by $\{e_i; i \in I\}$ the set of the integral idempotents in M. Then, by the above remark, the set of the minimal right ideals and the set of the minimal left ideals in M can be indexed by I. Therefore in the Rees matrix representation of M (see e.g., [**5**; **4**]) the maximal subgroups H_{ij} can be indexed by $I \times I$, and the sandwichmatrix P is contained in $G_{I\times I}$ where the structure group G of M is isomorphic to each H_{ij}. The partial ordering $\subseteq$ of M induces a partial ordering of $G = H_{11}$.

Now take

$$G = \mathfrak{Z} = \{\omega^z : z \in \mathbf{Z}\}, \tag{1}$$

the infinite cyclic group, (partially) ordered by

$$\omega^x \leqq \omega^y \Leftrightarrow x \geqq y \text{ for } x, y \in \mathbf{Z}. \tag{2}$$

Then by the results of Behrens [**1**] the sandwichmatrix

$$P: (i, j) \to \omega^{(ij)} \text{ for } (i, j) \in I \times I \tag{3}$$

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in the Rees matrix representation $M = M(\mathfrak{Z}; I, I; P)$ has entries with exponents $0 \leqq (ij) \in \mathbf{Z}$, which have to satisfy the conditions

$$(4) \qquad \begin{cases} 0. & (ii) = 0 \\ 1. & (ij) + (jk) \geqq (ik) \\ 2. & (ij) + (ji) > 0, \text{ if } i \neq j, \end{cases}$$

for $i, j, k \in I$, and the partial order $\subseteq$ in M is given by

$$(\omega^x; h, i) \subseteq (\omega^y; j, k) \Leftrightarrow x \geqq (hj) + y + (ki).$$

The integral idempotents are

$$e_i = (\omega^0; i, i) \text{ for } i \in I$$

and every element in M can be expressed uniquely by

$$(\omega^{z+(ij)}; i, j) = \omega^z e_i e_j,$$

defining

$$\omega^x(\omega^y; i, j) = (\omega^{x+y}; i, j).$$

The subsemigroup of the integral elements in M consists of

$$(5) \qquad S = \{\omega^s e_i e_j; 0 \leqq s \in \mathbf{Z} \text{ and } i, j \in I\}$$

and S is quasi-uniserial (see [**1**]). Conversely every quasi-uniserial semigroup without zero can be derived in this way, up to isomorphism. The partial ordering $\subseteq$ in M can be defined by

$$(6) \qquad a \subseteq b \Leftrightarrow a \in (b) = SbS \text{ for } a, b \in M$$

also, where SbS is the principal S-ideal in M, generated by the element $b \in M$. S is not commutative (with the only exception of the case $|I| = 1$), but it satisfies

$$(\text{F2}) \qquad (ab) = (a)(b).$$

Therefore it is possible to make this paper self-contained by the introduction of the expressions $\omega^z e_i e_j$ as the elements of a semigroup M, where $\omega^z \in \mathfrak{Z}$ and the multiplication in M is given by

$$(7) \qquad \omega^x e_h e_i \omega^y e_j e_k = \omega^{x+y+(hi)+(ij)+(jk)-(hk)} e_h e_k.$$

The sandwichmatrix P, determining the multiplication (7) is given in (3) and has to fulfill the conditions 0, 1 and 2 in (4). P can be assumed as 1-normalized, i.e., $(1i) = 0$ for $i \in I$, where 1 is a fixed index in I.

Now define the subsemigroup S of M, by (5). Then (6) defines a relation $\subseteq$ on M, which makes M a partially ordered semigroup with S as the set of its integral elements and satisfying hypothesis 1. The formula (F2) can be checked very easily.

Definition 1. An S-ideal in M is a subset A of M, different from M, such that $SA \leqslant A$ and $AS \leqslant A$ is valid. The S-ideal A is integral if $A \leqslant S$. The principal S-ideal, generated by $b \in M$, is denoted by (b).

The S-ideals in M form a lattice ordered semigroup $V_S(M)$ under the multiplication

$$A \cdot B = \{ab; a \in A, b \in B\}$$

for S-ideals A and B, and their set theoretic union and meet as lattice operations. The theory of $V_S(M)$ can be called the arithmetic of the quasi-uniserial semigroup S. It will be developed in this paper under the following hypothesis.

Hypothesis 2. The exponents (ij) in the entries $p_{ij} = \omega^{(ij)}$ of the sandwich-matrix P are bounded.

By the use of the metric

$$\delta : (i, j) \rightarrow (ij) + (ji) \text{ for } i, j \in I$$

on the set I, introduced in [**1**, §4] the hypothesis 2 reads: the distances of the elements in the set I, with respect to the metric δ on I, are bounded.

At first we need a method which associates with every S-ideal A a matrix A in $\mathfrak{Z}_{I\times I}$ such that the operations in $V_S(M)$ are handled in $\mathfrak{Z}_{I\times I}$ in a simple manner. Every element in A is of the form

$$\omega^z e_i e_j$$

and A contains with $\omega^\alpha e_i e_j$, the principal S-ideal $\omega^\alpha(e_i e_j)$. Therefore the numbers

$$\alpha_{ij} = \min\{\xi \in \mathbf{Z}; \omega^\xi e_i e_j \in A\} \text{ for } i, j \in I \tag{8}$$

describe the ideal A completely and A possesses the representation

$$A = \bigcup_{i,j \in I} \omega^{\alpha_{ij}}(e_i e_j), \tag{9}$$

which is *normalized* in the sense that its (i, j)-component is maximal and therefore uniquely determined by A. The number α_{ij} exists because of hypothesis 2 and the following remark. The S-ideal A contains with the element $\omega^\alpha e_i e_j$ the elements $e_h \omega^\alpha e_i e_j e_k$ for every $h, k \in I$. By (7) this is equivalent to

$$\alpha_{hk} + (hk) = \min_{i,j}\{(hi) + \alpha_{ij} + (ij) + (jk)\}, \tag{10}$$

and also to

$$\begin{aligned}\alpha_{hk} + (hk) &= \min_i\{(hi) + \alpha_{ik} + (ik)\},\\ &= \min_j\{\alpha_{hj} + (hj) + (jk)\},\end{aligned}$$

considering $j = k$ and $h = i$ respectively in the last two equations. $h = k = 1$ in (10) proves that the set of numbers $\alpha_{ij} + (ij)$ is bounded from below. Therefore the mapping

$$\varphi : A = \bigcup \omega^{\alpha_{ij}}(e_i e_j) \rightarrow \mathrm{A} = (\omega^{\alpha_{ij}+(ij)}) \tag{11}$$

maps the S-ideals A in M into a subset of the set $\mathfrak{Z}_{I\times I}{}^*$ of those matrices in $\mathfrak{Z}_{I\times I}$, the entries of which have exponents bounded from below for each matrix.

The set $\mathfrak{Z}_{I\times I}{}^*$ is a lattice, if we define the $(i, j,)$-entry of the meet of A and B as the maximum of the (i, j)-entries of A and B and the join of A and B dually. Moreover the mapping φ becomes a lattice monomorphism of $V_S(M)$ into $\mathfrak{Z}_{I\times I}{}^*$.

To exhibit the multiplication in $V_S(M)$ in $\mathfrak{Z}_{I\times I}{}^*$, combine (F2) with the formula (7) and the representations (9) for A and B in $V_S(M)$, to obtain

$$A \cdot B = \bigcup_{h,k} \bigcup_{i,j} \omega^{\alpha_{hi}+\beta_{jk}}(e_h e_i)(e_j e_k)$$
$$= \bigcup_{h,k} \omega^{\gamma_{hk}}(e_h e_k).$$

Here

$$\gamma_{hk} = \min_{i,j}\{\alpha_{hi} + (hi) + (ij) + (jk) - (hk) + \beta_{jk}\}$$

implies that

$$\gamma_{hk} + (hk) = \min_j\{\alpha_{hj} + (hj) + \beta_{jk} + (jk)\}, \tag{12}$$

because by (10)

$$\min_i\{\alpha_{hi} + (hi) + (ij)\} = \alpha_{hj} + (hj)$$

is valid.

This suggests the definition of a $\circ$-product in $\mathfrak{Z}_{I\times I}{}^*$ by

$$(\omega^{\xi_{ij}}) \circ (\omega^{\eta_{ij}}) = \omega^{\pi_{ij}} \text{ with } \pi_{ij} = \min_r\{\xi_{ir} + \eta_{rj}\}. \tag{13}$$

THEOREM 1. *$\mathfrak{Z}_{I\times I}{}^*$ is a lattice-ordered semigroup under the $\circ$-multiplication defined by* (13). *Moreover*

$$\mathrm{A} \circ [\mathrm{B} \cap \Gamma] = \mathrm{A} \circ \mathrm{B} \cap \mathrm{A} \circ \Gamma \tag{14}$$

is valid for A, B, $\Gamma \in \mathfrak{Z}_{I\times I}{}^*$.

Proof. The proof is straightforward.

THEOREM 2. *The mapping φ, defined by* (11), *is an isomorphism of the lattice-ordered semigroup $V_S(M)$ of the S-ideals in M onto the subsemigroup*

$$\varphi(V_S(M)) = \{\mathrm{A} \in \mathfrak{Z}_{I\times I}{}^*; P \circ \mathrm{A} \circ \mathrm{P} = \mathrm{A}\}, \tag{15}$$

where

$$P = \varphi S = P \circ P \tag{16}$$

is the sandwichmatrix of M.

Proof. This follows from (12), (13) and (10).

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THEOREM 3. *The S-ideals $\omega^{\alpha}(e_i e_j)$, $\alpha \in \mathbf{Z}$ and $i, j \in I$, are the only join-irreducible S-ideals in M. Each S-ideal A in M possesses exactly one representation*

$$A = \bigcup_{i,j \in I} \omega^{\alpha_{ij}}(e_i e_j) \tag{17}$$

by its irreducible join-components, which is normalized in the sense that the α_{ij} are minimal with respect to A.

Proof. The existence and the uniqueness of the representation (17) are derived in (9); (17) then shows that at most the principal S-ideals $\omega^{\alpha}(e_i e_j)$ are join-irreducible. On the other hand

$$\omega^{\alpha}(e_i e_j) = \bigcup_{\tau \in \mathrm{T}} B_{\tau}$$

implies that there exists a $\tau_0 \in \mathrm{T}$ such that the S-ideal B_{τ_0} contains the element $\omega^{\alpha} e_i e_j$ and therefore the S-ideal $\omega^{\alpha}(e_i e_j)$ also.

To construct the meet-irreducible S-ideals in M, we look at first for the greatest S-ideal (17) in M, which possesses $\omega^{\alpha}(e_h e_k)$ as its (h, k)-join-component. It has to satisfy

$$\alpha \leqq (hi) + \alpha_{ij} + (ij) + (jk) - (hk) \text{ for } i, j \in I,$$

because otherwise the (h, k)-join-component of A would be greater. This is equivalent to

$$\alpha_{ij} + (ij) \geqq \alpha + (hk) - (hi) - (jk) \text{ for } i, j \in I.$$

Now the equalities

$$\tau_{ij} + (ij) = \alpha + (hk) - (hi) - (jk) \text{ for } i, j \in I$$

define a matrix

$$\mathrm{T}^{(\alpha)}{}_{hk} = (\omega^{\tau_{ij} + (ij)}) \in \mathfrak{Z}_{I \times I}{}^{*}$$

with

$$P \circ \mathrm{T}^{(\alpha)}{}_{hk} \circ P = \mathrm{T}^{(\alpha)}{}_{hk}.$$

Set $\mathrm{T}_{hk} = \mathrm{T}_{hk}{}^{(0)}$. Then the S-ideal

$$\mathrm{T}^{(\alpha)}{}_{hk} = \bigcup_{i,j} \omega^{\alpha + (hk) - (hi) - (ij) - (jk)}(e_i e_j) = \omega^{\alpha} \mathrm{T}_{hk}$$

is mapped under φ onto $\mathrm{T}^{(\alpha)}{}_{hk}$.

THEOREM 4. *The only meet-irreducible S-ideals in M are the ideals $\omega^{\alpha}\mathrm{T}_{hk}$, where*

$$\mathrm{T}_{hk} = \bigcup_{i,j} \omega^{(hk) - (hi) - (ij) - (jk)}(e_i e_j) \text{ for } h, k \in I. \tag{18}$$

They are the greatest S-ideals in M possessing the (h, k)-join-component, $\omega^{\alpha}(e_h e_k)$. Every S-ideal A in M possesses exactly one representation

$$A = \bigcap_{i,j} \omega^{\alpha_{ij}} T_{ij} \tag{19}$$

by its irreducible meet-components, which is normalized in the sense that the α_{ij} *are maximal with respect to* (19). *The exponents* α_{ij} *in* (19) *and in the join-representation* (17) *in Theorem* 3 *are the same.*

Remark. The mapping φ can be defined also by

$$A = \bigcap \omega^{\alpha_{ij}} T_{ij} \rightarrow (\omega^{\alpha_{ij}+(ij)}).$$

Proof. Because $\omega^{\alpha_{ij}} T_{ij}$ is the greatest S-ideal with $\omega^{\alpha_{ij}}(e_i e_j)$ as its (i, j)-join-component, the (r, s)-join-component of A is contained in the (r, s)-join-component of $\omega^{\alpha_{ij}} T_{ij}$ for $r, s \in I$. Therefore the representation (19) follows from (17) immediately. At most the $\omega^\alpha T_{hk}$ are meet-irreducible. Now $\omega^\alpha T_{hk} = \bigcap B_\tau$ implies that there exists a τ_0 such that B_{τ_0} possesses $\omega^\alpha(e_h e_k)$ as its (h, k)-join-component and therefore is contained in $\omega^\alpha T_{hk}$.

Theorem 5 below will explain the rôle of the generating element ω of $\mathfrak{Z}$ in the theory of the lattice-ordered semigroup $V_S(M)$ of the S-ideals in M in a more conceptual manner.

Firstly, the ideal S is the identity element in $V_S(M)$. The S-ideals $\omega^\alpha S$, $\alpha \in \mathbf{Z}$, form a cyclic subgroup C of $V_S(M)$, generated by the S-ideal $W = \omega S$. Then the representation of an S-ideal A in M can be written in the form

$$A = \bigcap_{i,j} W^{\alpha_{ij}} T_{ij}. \tag{20}$$

THEOREM 5. *The infinite cyclic group* $C = \{W^\alpha; \alpha \in \mathbf{Z}\}$, $W = \omega S$, *is the centre of the semigroup* $V_S(M)$ *of the S-ideals in M.*

Proof. It is clear that C is contained in the centre of $V_S(M)$. An S-ideal A in M is an element of the centre of $V_S(M)$, if and only if $A \cdot (e_k) = (e_k) \cdot A$ is valid for every integral idempotent e_k, $k \in I$, because $(e_i e_j) = (e_i)(e_j)$ by (F2). Now the equality

$$e_i e_k e_j = \omega^{(ik)+(kj)-(ij)}$$

implies that

$$\mathrm{E}_k = \varphi((e_k)) = (\omega^{(ik)+(kj)}).$$

Therefore A is in the centre of $V_S(M)$ if and only if $\mathrm{A} = \varphi A$ satisfies

$$\mathrm{A} \circ \mathrm{E}_k = \mathrm{E}_k \circ \mathrm{A} \text{ for every } k \in I. \tag{21}$$

By Theorem 2, the (i, j)-entries of $\mathrm{A} \circ \mathrm{E}_k$ and of $\mathrm{E}_k \circ \mathrm{A}$ are respectively

$$\min_t\{\alpha_{it} + (tk) + (kj)\} \text{ and } \min_r\{(ik) + (kr) + \alpha_{rj}\}, \tag{22}$$

and $\mathrm{A} \circ P = \mathrm{A} = P \circ \mathrm{A}$ is valid. Set $k = i$. Then

$$\begin{aligned}\alpha_{ii} + (ij) &= \min_t\{\alpha_{it} + (ti)\} + (ij)\\ &= (ii) + \min_r\{(kr) + \alpha_{rj}\} = \alpha_{ij}\end{aligned}$$

follows from (21) and (22); analogously

$$\alpha_{ij} = (ij) + \alpha_{jj} \text{ if } k = j.$$

This proves that

$$\alpha_{ii} + (ij) = \alpha_{ij} = (ij) + \alpha_{jj} \text{ for } i, j \in I$$

and therefore

$$\alpha_{ii} = \alpha_{jj} = \alpha \text{ and } \alpha_{ij} = \alpha + (ij).$$

In other words $\mathrm{A} = \omega^\alpha P$ and $A = \omega^\alpha S$.

Perhaps it is worth comparing these results with the classical arithmetic in a commutative Dedekind domain, R. There, every fractional R-ideal $\mathfrak{a}$ in the quotient field Q of R is uniquely representable as a product of powers of prime ideals. The sequence of the exponents in such a representation determines $\mathfrak{a}$ and the multiplication of two ideals is given by the componentwise addition of their exponent sequences. In the above semigroup case, the sequence of exponents is replaced by the associated matrix $\varphi A = \mathrm{A}$, with entries in the infinite cyclic group $\mathfrak{Z} = \{\omega^z; z \in \mathbf{Z}\}$, and the multiplication of two ideals is given by the $\circ$-multiplication of their associated matrices in $\mathfrak{Z}_{I\times I}{}^*$. Exactly those matrices in $\mathfrak{Z}_{I\times I}{}^*$ are associated with S-ideals in M which satisfy $P \circ \mathrm{A} \circ P = \mathrm{A}$, where P is the sandwichmatrix of M, governing the multiplication in the completely simple semigroup M. Similar to the classical arithmetic, the prime ideals in S are linked to a localization theory. The first step consists in proving the following theorem.

THEOREM 6. *The prime ideals in S, i.e. the integral ideals $\mathfrak{P} \neq S$, which satisfy the implication*

$$(a) \cdot (b) \leqslant \mathfrak{P} \Rightarrow a \in \mathfrak{P} \text{ or } b \in \mathfrak{P} \tag{23}$$

for $a, b \in S$, $a \neq b$, are the maximal ideals in S, namely

$$\mathfrak{P}_i = S \backslash e_i \text{ for } i \in I. \tag{24}$$

Proof. $\mathfrak{P}_i$ is an ideal in S because e_i is maximal with respect to the partial order $\subseteq$ in S, defined by $a \subseteq b \Leftrightarrow a \in (b)$, and $\mathfrak{P}_i$ is maximal in $V_S(S)$ indeed. Its complement $S\backslash\mathfrak{P}_i$ is the idempotent e_i, a one element multiplicatively closed subset of S. This implies that $\mathfrak{P}_i$ is prime, because of $(a) \cdot (b) = (ab)$ in S. If the ideal A in S is different from every $\mathfrak{P}_i$ and A contains a join-component $\omega^\alpha(e_h e_k)$ with $h \neq k$ and $\alpha \geqq 1$, then the square of $\omega^{\alpha-1}e_h e_k$ is contained in A without $\omega^{\alpha-1}e_h e_k$ being so. But, if $\alpha = 0$ in every join-component with $h \neq k$, then there exist $h, k \in I$ with $h \neq k$ and $e_h e_k$ in A but neither e_h nor e_k. So A is not prime.

Definition 2. An S-order of M is a subsemigroup of M, different from M and containing S.

Remark. Every S-order $\mathfrak{O}$ is an S-ideal in M, but the converse is not true.

The context between the prime ideals in S and certain maximal S-orders is given by the following theorem.

THEOREM 7. *The sets*

$$\mathfrak{L}_k = \{x \in M; e_k x \in S\} \tag{25}$$

and

$$\mathfrak{R}_k = \{x \in M; x e_k \in S\} \tag{26}$$

are maximal S-orders in M. The ideals in the semigroups $\mathfrak{L}_k$ and $\mathfrak{R}_k$ are the powers of $\omega\mathfrak{L}_k = W\mathfrak{L}_k$ and of $\omega\mathfrak{R}_k = W\mathfrak{R}_k$ respectively.

Proof. If $x, y \in \mathfrak{L}_k$ then

$$e_k \cdot xy = e_k \cdot e_k x \cdot y \in e_k S y \leqslant (e_k y) \leqslant S,$$

because of (F2). The multiplicatively closed set $\mathfrak{L}_k$ contains S because $e_k \in S$. The equalities

$$e_k x = e_k \omega^{\xi} e_i e_j = \omega^{\xi + (ki) + (ij) - (kj)} e_k e_j$$

imply that

$$\mathfrak{L}_k = \{\omega^{\xi} e_i e_j; \xi \geqq (kj) - (ki) - (ij)\}. \tag{27}$$

Analogously

$$\mathfrak{R}_k = \{\omega^{\xi} e_i e_j; \xi \geqq (ik) - (jk) - (ij)\} \tag{28}$$

is valid. This proves that $\mathfrak{L}_k \neq M$ and $\mathfrak{R}_k \neq M$. Assume that the S-order $\mathfrak{O}$ contains $\mathfrak{L}_k$ and an element $\omega^{\gamma} e_r e_s \notin \mathfrak{L}_k$. Then

$$\gamma < (ks) - (kr) - (rs)$$

and

$$\omega^{\delta} = \omega^{\gamma} e_r e_s \omega^{(kr) - (ks) - (sr)} e_s e_r \in \mathfrak{O},$$

where

$$\delta < (ks) - (kr) - (rs) + (rs) + (sr) + (kr) - (ks) - (sr) = 0.$$

This proves that $\omega^{-1} e_r \in \mathfrak{O}$ and therefore $\omega^{-N} e_r e_s \in \mathfrak{O}$ for every $N \in \mathbf{N}$, in contradiction to $\mathfrak{O} \neq M$. Analogously, (28) implies that $\mathfrak{R}_k$ is maximal. An ideal A of the semigroup $\mathfrak{L}_k$ satisfies $SA \leqslant A$ and $AS \leqslant A$. Therefore A is an S-ideal in M also. The o-product of

$$\varphi A = (\omega^{\alpha_{ij}} + (ij)) \quad \text{and} \quad \varphi \mathfrak{L}_k = (\omega^{(kj) - (ki)}) \tag{29}$$

has the number $\min_r\{\alpha_{ir} + (ir) + (kj) - (kr)\}$ as the exponent of its (i, j)-entry. This implies that $A\mathfrak{L}_k \leqslant A$ is equivalent to

$$\alpha_{ij} + (ij) - (kj) = \min_r\{\alpha_{ir} + (ir) - (kr)\}.$$

In other words: the expression $\alpha_{ij} + (ij) - (kj)$ is independent of j. Analogously, $\mathfrak{L}_k A \leqslant A$ implies $\alpha_{ij} + (ij) + (ki)$ is independent of i. Then the equalities

$$\begin{aligned} \alpha_{ij} + (ij) + (ki) &= \alpha_{1j} + (1j) + (k1) = [\alpha_{1j} + (1j) - (kj)] + (kj) + (k1) \\ &= \alpha_{11} + (11) - (k1) + (kj) + (k1) = \alpha_{11} + (kj) \end{aligned}$$

and therefore the equality

$$\alpha_{ij} + (ij) = \alpha_{11} + (kj) - (ki)$$

prove, by comparison with (29), that

$$A = \omega^{\alpha_{11}}\mathfrak{L}_k.$$

Similarly, the $\omega^\alpha\mathfrak{R}_k$, $0 \leqq \alpha \in \mathbf{Z}$, form the monoid $V(\mathfrak{R}_k)$ of the ideals in the maximal S-order $\mathfrak{R}_k$.

The following theorem shows that S is both the meet of the maximal S-orders $\mathfrak{L}_k$, and also of the maximal orders $\mathfrak{R}_k$, $k \in I$, and it explains how the meet-irreducible S-ideals $T^{(\alpha)}{}_{ij}$ in M are linked to the orders $\mathfrak{L}_i$ and $\mathfrak{R}_j$, and it gives rise to a localization theory of $V_S(M)$, considering (19) in Theorem 4.

THEOREM 8.

$$S = \bigcap_k \mathfrak{L}_k = \bigcap_k \mathfrak{R}_k; \tag{30}$$

$$T^{(\alpha)}{}_{hk} = \omega^\alpha\mathfrak{L}_h\mathfrak{R}_k = W^\alpha\mathfrak{L}_h\mathfrak{R}_k, \tag{31}$$

where $W = \omega S$.

Proof. The first equation (30) follows from (27), applied to all $k \in I$, i.e., from

$$\xi = \max_k\{(kj) - (ki) - (ij)\} = 0, \tag{32}$$

because $(ki) + (ij) \geqq (kj)$. By (27) and (28) the exponent of the (i, j)-entry in $[\varphi\mathfrak{L}_h] \circ [\varphi\mathfrak{R}_k]$ is equal to

$$\begin{aligned}\min_r\{(hr) - (hi) + (rk) - (jk)\} &= -(hi) - (jk) + \min_r\{(hr) + (rk)\}\\ &= (hk) - (hi) - (jk)\}\end{aligned}$$

and therefore, by (18), it is equal to the (i, j)-entry in φT_{hk}.

The maximal S-orders $\mathfrak{R}_h$ and $\mathfrak{L}_k$ are linked to the join-irreducible ideals also.

THEOREM 9.

$$\mathfrak{R}_h\mathfrak{L}_k = \omega^{(hk)+\max\{(kr)+(rh)\}}(e_he_k). \tag{33}$$

Proof. The proof is similar to the proof of the last theorem.

The calculation of the $\circ$-product $[\varphi\mathfrak{L}_r] \circ [\varphi\mathfrak{L}_s] \circ [\varphi\mathfrak{L}_t]$ proves Theorem 10.

THEOREM 10. *The maximal ideals $W\mathfrak{L}_h = \omega\mathfrak{L}_h$ for $h \in I$ in the S-orders $\mathfrak{L}_h$ generate a semigroup without zero, which is dual to the quasi-uniserial semigroup S above in the following sense: by the formula*

$$\mathfrak{L}_r\mathfrak{L}_s\mathfrak{L}_t = \omega^{-(ts)-(sr)+(tr)}\mathfrak{L}_r\mathfrak{L}_t \tag{34}$$

it consists of the integral elements of the partially ordered semigroup $M(\mathfrak{Z}; I, I; P^*)$, *which is associated with the exponential sandwichmatrix*

$$\Pi^* = -\Pi^t: (i, j) \to -(ji) \text{ for } (i, j) \in I \times I \tag{35}$$

and with the dual ordering $\omega^x \leqq \omega^y \Leftrightarrow x \leqq y$ *of the structure group* $\mathfrak{Z}$.

The same is true for the orders $\mathfrak{R}_k$, $k \in I$, *if the* $\mathfrak{L}_k$ *are replaced by the* $\mathfrak{R}_k$ *in* (34).

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