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A note on the space of all Toeplitz operators

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Abstract. We study Toeplitz operators on the space of all real analytic functions on the real line and the space of all holomorphic functions on finitely connected domains in the complex plane. In both cases, we show that the space of all Toeplitz operators is isomorphic, when equipped with the topology of uniform convergence on bounded sets, with the symbol algebra. This is surprising in view of our previous results, since we showed that the symbol map is not continuous in this topology on the algebra generated by all Toeplitz operators. We also show that in the case of the Fréchet space of all holomorphic functions on a finitely connected domain in the complex plane, the commutator ideal is dense in the algebra generated by all Toeplitz operators in the topology of uniform convergence on bounded sets.

1 Introduction

It is a fundamental result in the theory of Toeplitz operators on the Hardy space that the map defined in the following way

$$\mathrm{Symb}\Big(\sum_i \prod_j T_{F_{ij}}\Big) := \sum_i \prod_j F_{ij}$$

is indeed well defined and can be extended by continuity to the closed (in the algebra of all bounded operators) algebra generated by all Toeplitz operators. Here, F_{ij} are bounded measurable functions. The map Symb factorizes through the Calkin algebra. Let us recall that a Toeplitz operator on the Hardy space is the operator of the form

$$T_F\colon f \mapsto P_+(F\cdot f),$$

where P_+ is the Riesz projection and $F \in L^\infty(\mathbb{T})$. We refer the reader to [25] (especially Theorem 3.1.3) for more on this subject. We also refer the reader to [1]. We cannot give all the details here. Let us, however, quote the words of Sheldon Axler [1, p 130], who writes, *The symbol map (...) was a magical and mysterious homomorphism to me* .... In this paper, we continue our study of the symbol map in the case of all real analytic functions and start the investigation of the case of all functions holomorphic on finitely connected domains in the plane.

In [18], we investigated the case of all real analytic functions on the real line $\mathcal{A}(\mathbb{R})$. A real analytic function on $\mathbb{R}$ is a function, which locally around each point develops

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into a Taylor sequence which converges to the function – later on, we shall give more details concerning real analytic functions. This implies that a real analytic function is a germ of a holomorphic function on $\mathbb{R}$. We developed the fundamentals of the theory of Toeplitz operators on this space in [6]. Let us here only recall that a Toeplitz operators on $\mathcal{A}(\mathbb{R})$ is an operator of the form

$$T_F\colon f \mapsto \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{F(\zeta)\cdot f(\zeta)}{\zeta - z} d\zeta, \tag{1.1}$$

where $F \in \mathcal{X}(\mathbb{R})$ is essentially a function, which is holomorphic in some set $U \setminus K$, where $U \supset \mathbb{R}$ is open (and may be assumed simply connected) and $K \subset \mathbb{R}$ is compact (and may be assumed connected). The symbol γ denotes a C^∞ smooth Jordan curve in U and the domain of definition of the function f such that both the point z and the set K are enclosed by the curve γ. It is a consequence of Cauchy's theorem that the definition is correct. That is, it does not depend on the curve γ nor on the representative F of the symbol. The last statement requires an explanation. Namely, strictly speaking, the symbol algebra is

$$\mathcal{X}(\mathbb{R}) := \lim \operatorname{ind} H(U \setminus K),$$

where U runs through open neighborhoods of $\mathbb{R}$ and K through compact subsets of $\mathbb{R}$. Formula (1.1) defines a function which is holomorphic on some open neighborhood of the real line. Hence, it defines a real analytic function on $\mathbb{R}$.

Arguably, the most natural example of the Toeplitz operators which we study is given by the formula

$$(Tf)(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{1}{\prod_{i=0}^n (\zeta - x_i)} \frac{f(\zeta) d\zeta}{\zeta - z},$$

where $x_0, x_1, \dots, x_n \in \mathbb{R}$ are different and γ encloses the points x_i and the point z. This operator turns out to be just the divided difference $[z, x_0, \dots, x_n]$. We refer the reader to [21] for the definition, which was also recalled in [15, p 2]. Thus, our Toeplitz operators appear in approximation theory and number theory, since they control the convergence of the Newton series. The estimates of them are, for instance, behind the proof of the classical Lindemann's theorem [11, p 167, Satz 9], which says that π and e are transcendental numbers. We refer the reader to our previous research, especially [15, Introduction], for more on this subject.

In [18], we proved that the map Symb is also well defined in the case of the space $\mathcal{A}(\mathbb{R})$. However, we also showed the following result.

Theorem 1.1 *The map Symb: $\mathrm{Alg}\,\mathcal{T}(\mathbb{R}) \to \mathcal{X}(\mathbb{R})$ is not continuous when $\mathrm{Alg}\,\mathcal{T}(\mathbb{R})$ is equipped with the topology of uniform convergence on bounded subset of $\mathcal{A}(\mathbb{R})$. In fact, there is no multiplicative linear map $\varphi\colon \mathrm{Alg}\,\mathcal{T}(\mathbb{R}) \to \mathcal{X}(\mathbb{R})$ with $\varphi(T_F) = F$ which is continuous with respect to the topology of uniform convergence on bounded subsets of $\mathcal{A}(\mathbb{R})$.*

The symbol $\mathrm{Alg}\,\mathcal{T}(\mathbb{R})$ stands for the algebra of all Toeplitz operators on the space $\mathcal{A}(\mathbb{R})$. Arguably, the topology of uniform convergence on bounded sets of $\mathcal{A}(\mathbb{R})$ is a most natural topology on the algebra of all continuous linear operators on $\mathcal{A}(\mathbb{R})$. We

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emphasize that in general, unlike in the Banach space case, there is no distinguished topology on the algebra of all continuous linear operators on a locally convex space. We refer the reader to [20, Chapter Eight] for more information on this subject. We remark that in [18, Theorem 2], we proved that there is a (Hausdorff) locally convex topology t on the quotient algebra

$$\mathrm{Alg}\,\mathcal{T}(\mathbb{R})/\mathfrak{C}(\mathbb{R}),$$

where $\mathfrak{C}(\mathbb{R})$ is the ideal generated by all commutators, such that

$$(\mathrm{Alg}\,\mathcal{T}(\mathbb{R})/\mathfrak{C}(\mathbb{R}), t)$$

is isomorphic as a locally convex space and as an algebra with the symbol algebra $\mathcal{X}(\mathbb{R})$. This space carries a natural (Hausdorff) inductive locally convex topology. This readily implies that the symbol map is indeed well defined.

In view of Theorem 1.1, it seems of interest that we prove in this paper the following theorem.

Theorem 1.2 *Let $\mathcal{T}(\mathbb{R})$ denote the space of all Toeplitz operators on the space of all real analytic functions on the real line. The space $\mathcal{T}(\mathbb{R})$, when equipped with the topology of uniform convergence on bounded sets, is isomorphic with the symbol space $\mathcal{X}(\mathbb{R})$.*

Next, we turn our attention to the Fréchet space of all holomorphic functions on a finitely connected domain in $\mathbb{C}$. We recall the setting from [16].

The Jordan curve theorem [21, Theorem 4.14, p 70] says that the complement of any (closed) Jordan curve has exactly two components, with γ as their common boundary. One of these components $I(\gamma)$, called the interior of γ, is bounded, and the other component $E(\gamma)$, called the exterior of γ, is unbounded. We shall use this notation throughout the paper.

Definition 1.1 (Definition 1, [16]) Let $\gamma_0, \gamma_1, \dots, \gamma_n \subset \mathbb{C}$ be C^∞ smooth Jordan curves such that $\overline{I(\gamma_i)} \cap \overline{I(\gamma_j)} = \varnothing$ for $i, j = 1, \dots, n$ with $i \neq j$ and

$$\gamma_1, \dots, \gamma_n \subset I(\gamma_0).$$

Then

$$D := D(\gamma_0; \gamma_1, \dots, \gamma_n) := I(\gamma_0) \cap E(\gamma_1) \cap \dots \cap E(\gamma_n),$$

and X denotes the Fréchet space $H(D)$ of all functions holomorphic in D.

Let $\gamma_1, \dots \gamma_n \subset \mathbb{C}$ be C^∞ smooth Jordan curves such that $\overline{I(\gamma_i)} \cap \overline{I(\gamma_j)} = \varnothing$, $i \neq j$. Then

$$D := D(\gamma_1, \dots, \gamma_n) := E(\gamma_1) \cap \dots \cap E(\gamma_n),$$

and X denotes either the Fréchet space $H_0(D)$ of all functions holomorphic in D which vanish at ∞ or the space $H(D)$ of all functions holomorphic in D (in general, with no value at ∞).

A Toeplitz operator on the space X is the operator of the form

$$(T_F f)(z) := \frac{1}{2\pi \mathrm{i}} \sum_{i=0}^{n} \int_{(\gamma_i)_\varepsilon} \frac{F_i(\zeta)\cdot f(\zeta)}{\zeta - z} d\zeta, \tag{1.2}$$

if $X = H(D)$, D bounded or $X = H_0(D)$, $D = E(\gamma_1) \cap \cdots \cap E(\gamma_n)$ – in this case, the sum starts with $i = 1$. The symbol $(\gamma_i)_\varepsilon$ stands for an appropriate dilatation of the curve γ_i, $i = 0, 1, \ldots, n$, and $E(\gamma_i)$ is the exterior of the curve γ_i – recall again Jordan's theorem. The symbol space in these cases is

$$\mathfrak{S}(D) := \bigoplus_{i=0}^{n} \lim\,\mathrm{ind}\, H(U_i \cap D),$$

where U_i are open neighborhoods of the curves γ_i (again, if D is unbounded, then the sum starts with $i = 1$). Thus, each F_i is actually holomorphic in some neighborhood in D of γ_i.

If $X = H(D)$, $D = E(\gamma_1) \cap \cdots \cap E(\gamma_n)$, then the symbol space is

$$\mathfrak{S}(D) := \bigoplus_{i=0}^{n} \lim\,\mathrm{ind}\, H(U_i \cap D) \oplus \mathfrak{S}_\infty,$$

where

$$\mathfrak{S}_\infty := \lim\,\mathrm{ind}\, H(U \setminus \{\infty\})$$

and U run through open neighborhoods of ∞ in the Riemann sphere. For

$$F = F_1 \oplus \cdots \oplus F_n \oplus F_\infty \in \mathfrak{S}(D),$$

the Toeplitz operator $T_F\colon H(D) \to H(D)$ is defined in the following way:

$$(T_F f)(z) := \frac{1}{2\pi \mathrm{i}} \sum_{i=1}^{n} \int_{(\gamma_i)_\varepsilon} \frac{F_i(\zeta)\cdot f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=R} \frac{F_\infty(\zeta)\cdot f(\zeta)}{\zeta - z} d\zeta,$$

where F_∞ is holomorphic in some punctured neighborhood of ∞ and R is sufficiently large.

We cannot repeat all the information concerning the Toeplitz operators on the space $X = H_0(D), H(D)$. We refer the reader to our previous paper [16] for the details. Let us only mention here that in order to motivate our study in [16], we introduced an analog of the Riemann-Hilbert problem in the space $X = H(D)$ and described the role in it of the Toeplitz operators (1.2).

In this paper, we prove the following result.

Theorem 1.3 *Let $\mathfrak{T}(D)$ denote the space of all Toeplitz operators on the space X. The space $\mathfrak{T}(D)$, when equipped with the topology of uniform convergence on bounded sets, is isomorphic with the symbol space $\mathfrak{S}(D)$.*

We also obtain the following theorem, which we think serves as a motivation for further study.

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Theorem 1.4 *Let X be a space from Definition 1.1. The commutator ideal $\mathfrak{C}(D)$ is dense in the algebra generated by all Toeplitz operators on the space X in the topology of uniform convergence on bounded sets.*

Recall that the commutator ideal in $\operatorname{Alg} \mathcal{T}(D)$ is the ideal generated by commutators

$$[T_F, T_G] = T_F \circ T_G - T_G \circ T_F$$

with $F, G \in \mathfrak{S}(D)$.

We intend this paper to be rather short. This is why we chose not to provide a separate background section and rather give extensive explanations in the proofs. We refer the reader to our previous papers, especially [6], [15], and [18], for more information. Let us here, however, recall the definition of a real analytic function. We say that $f: \mathbb{R} \to \mathbb{C}$ is real analytic if for every x_0, the function f can be developed into a Taylor series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \tag{1.3}$$

convergent for $|x - x_0| < \delta_{x_0}$, $\delta_{x_0} > 0$ to the function f. The series converges for complex numbers $|z - x_0| < \delta_{x_0}$. Thus, every real analytic function is actually holomorphic in some open neighborhood of the real line. Formally,

$$\mathcal{A}(\mathbb{R}) = \lim \operatorname{ind} H(U), \tag{1.4}$$

where $\mathcal{A}(\mathbb{R})$ denotes the space of all real analytic functions on the real line and $H(U)$ is the Fréchet space of all functions holomorphic in the set U. We remark that in PDE's, real analytic functions are real-valued, since the equations have real coefficients, but in view of (1.4), it is natural to assume that they are complex-valued. Equation (1.4) is used to equip the space $\mathcal{A}(\mathbb{R})$ with a locally convex topology. This is the strongest locally convex topology which makes all inclusions

$$H(U) \hookrightarrow \mathcal{A}(\mathbb{R})$$

continuous. This topology is called the inductive topology. We remark that there is also the projective topology. This is the weakest locally convex topology which makes every restriction

$$\mathcal{A}(\mathbb{R}) \to H(K)$$

continuous. The symbol $H(K)$ stands for the space of all germs of holomorphic functions on the compact set $K \subset \mathbb{R}$. It is a fundamental result of Martineau [22, Proposition 1.9, Theorem 1.2] that the inductive and the projective topology are equal. The space $\mathcal{A}(\mathbb{R})$ with this topology is a locally convex space. Although this topology is not metrizable, the most important tools of functional analysis, such as the Hahn-Banach theorem, the open mapping/closed graph theorem, and the uniform boundedness principle, are available. We refer the reader to [5] for a very intuitive introduction to the theory of real analytic functions and operators on the spaces of these functions. We emphasize that the Toeplitz operators are not the only operators

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studied on this space. In fact, a lot is known about the composition operators and, especially, the differential operators on this space. Probably the most important result is the one of Hörmander [12], which characterizes the differential operators with constant coefficients which are surjective on the spaces of real analytic functions. We also refer the reader to [8] for a more formal introduction to real analytic functions.

As for the case of the Toeplitz operators on finitely connected domains in $\mathbb{C}$, we refer the reader to our paper [16].

The paper is a part of the project the aim of which is to study classical operators on locally convex spaces of analytic functions. We defined the Toeplitz operators on $\mathcal{A}(\mathbb{R})$ in [6]. In the consecutive papers [13], [14], and [15], we studied the Fredholm property, invertibility, and one-sided invertibility of these operators. In this paper, and also [18], we turn our attention to the space of all Toeplitz operators and the algebra generated by all Toeplitz operators. The motivation of course comes from the Hardy space case and the fundamental theorem of Douglas [10, Theorem 7.11]. Essentially, this theorem says that the symbol map is well defined in the Hardy space case and can be extended by continuity to the closed algebra generated by all Toeplitz operators.

The starting point for our study was the research of Domański and Langenbruch [7]. The authors considered the operators on the space of all real analytic functions the eigenvalues of which are just monomials. These operators are called the Hadamard multipliers. In some sense, the Hadamard multipliers are the operators the matrix of which is diagonal. Let us remark here only that the situation is more complicated, since by the fundamental result of Domański and Vogt [9], the space $\mathcal{A}(\mathbb{R})$ has no basis. The idea to study Hadamard multipliers turned out be very fruitful. We refer the reader to our previous papers for the extensive bibliography. Let us write, however, that it is just one step from diagonal matrices to Toeplitz matrices, and we made this step in our papers and studied Toeplitz operators on the space of all real analytic functions and entire functions and on the space of all functions holomorphic on finitely connected domains in $\mathbb{C}$. Let us also remark that we proved an analog of Theorem 1.2 in the cases of all entire functions in [17, Main Theorem 2]. The method of the proof is, however, different. The one which we present in the current paper is in some sense more natural and straightforward, since we do not use the results of Mujica [24] and Vogt [27], which give the explicit form of seminorms on the spaces $H(K)$.

This note is divided into three sections. In the next one, we consider the case of the space $\mathcal{A}(\mathbb{R})$, and we prove Theorem 1.2. The third section is devoted to the spaces $H_0(D), H(D)$ on finitely connected domains in $\mathbb{C}$. We prove therein Theorem 1.3 and Theorem 1.4.

As far as the functional analysis background is concerned, we refer the reader to the fundamental book by Meise and Vogt [23]. The beautiful theory of the Toeplitz operators on the Hardy spaces is presented in the classical monograph by Böttcher and Silbermann [4] and more recent books by Nikolski [25] and [26]. We do not need to say that we are motivated by the classical theory. The subject of locally convex spaces of holomorphic functions and operators between them is a well-established topic. We refer the reader to the recent book of Bonet, Jornet, and Sevilla-Peris [3] for the panorama of research.

2 The case of real analytic functions

Our first aim is to give the proof of Theorem 1.2. We recall also the necessary definitions.

Proof of Theorem 1.2 Recall that the symbol space $\mathcal{X}(\mathbb{R})$ is the inductive limit of the spaces $H(U \setminus K)$,

$$\lim \operatorname{ind} H(U \setminus K),$$

where U run through open neighborhoods of the real line and K are compact subsets of $\mathbb{R}$ and the linking maps are just the inclusions. We remark that when we restrict attention to simply connected open neighborhoods of $\mathbb{R}$ and to connected compact subsets, we obtain the same symbol space $\mathcal{X}(\mathbb{R})$. In [6, p 12], we showed that the inductive locally convex topology of this system exists; that is, it is Hausdorff. Not very precise, we may write

$$\mathcal{X}(\mathbb{R}) = \bigcup_{U,K} H(U \setminus K).$$

That is, roughly speaking, an element of $\mathcal{X}(\mathbb{R})$ is a function F holomorphic in some set $U \setminus K$, $U \supset \mathbb{R}$ open, $K \subset \mathbb{R}$ compact. It follows from Cauchy's integral formula that

$$F(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{F(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{F(\zeta)}{\zeta - z} d\zeta := F_+(z) + F_-(z),$$

where Γ is a C^∞ smooth Jordan curve such that both z and K are contained in the interior $I(\Gamma)$ and γ is a C^∞ smooth Jordan curve such that $K \subset I(\gamma)$ and $z \in E(\gamma)$ – the exterior of the curve γ. Observe that F_+ defines a function holomorphic in U and F_- defines a function holomorphic in $\mathbb{C}_\infty \setminus K$ which vanishes at ∞. It is a consequence of Liouville's theorem that this decomposition is unique. That is,

$$H(U \setminus K) \cong H(U) \oplus H_0(\mathbb{C}_\infty \setminus K), \tag{2.1}$$

where $H(U)$ is the space of all holomorphic in the set U and $H_0(\mathbb{C} \setminus K)$ is the space of functions holomorphic in $\mathbb{C} \setminus K$ which vanish at ∞. These spaces are Fréchet spaces when equipped with the topology of uniform convergence on compact sets. One immediately notices that decomposition (2.1) holds not only algebraically, but also topologically. This implies that

$$\lim \operatorname{ind} H(U \setminus K) \cong \lim \operatorname{ind} H(U) \oplus \lim \operatorname{ind} H_0(\mathbb{C}_\infty \setminus K)$$

in the category of locally convex spaces. We refer the reader to [6, Section 3] for the details. Recall that

$$\lim \operatorname{ind} H(U) \cong \mathcal{A}(\mathbb{R}).$$

Furthermore, the space

$$\lim \operatorname{ind} H_0(\mathbb{C}_\infty \setminus K),$$

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sometimes denoted $H_0(\mathbb{C}\setminus\mathbb{R})$, is algebraically and topologically isomorphic with the dual space $\mathcal{A}(\mathbb{R})'_b$, equipped with the topology of uniform convergence on bounded sets of $\mathcal{A}(\mathbb{R})$,

$$\mathcal{A}(\mathbb{R})'_b \cong \operatorname{lim\,ind} H_0(\mathbb{C}_\infty\setminus K). \tag{2.2}$$

The duality between $\mathcal{A}(\mathbb{R})$ and $\operatorname{lim\,ind} H_0(\mathbb{C}_\infty\setminus K)$ is given by

$$(f,F)\mapsto\langle f,F\rangle := \int_\gamma f(z)F(z)dz.$$

Here, f is real analytic, so holomorphic in some open set $U\supset\mathbb{R}$ (which may be assumed simply connected), and F is holomorphic in $\mathbb{C}_\infty\setminus K$ for some compact set $K\subset\mathbb{R}$ (which may be assumed connected), and vanishes at ∞. The C^∞ smooth Jordan curve γ is contained in U and satisfies $K\subset I(\gamma)$. The definition is correct. It depends neither on the representatives of f and F chosen nor the curve γ. The fact the isomorphism (2.2) holds true is a consequence of the fundamental Köthe-Grothendieck-da Silva duality. We refer the reader to [19, pp 372–378] for the details. We conclude that

$$\mathcal{X}(\mathbb{R})\cong\mathcal{A}(\mathbb{R})\oplus\mathcal{A}(\mathbb{R})'_b$$

both algebraically and topologically.

Let $F\in\mathcal{A}(\mathbb{R})$ be represented by a function (denoted by the same symbol) $F\in H(U\setminus K)$. Let also F_+,F_- be the corresponding decomposition (2.1). Let p be a continuous seminorm on the space $\mathcal{A}(\mathbb{R})$. Then, trivially,

$$p(F_+) = p(T_{F_+}(1)).$$

In other words,

$$p(F_+) = \sup_{f\in B_1} p(T_{F_+}(f)),$$

where $B_1=\{1\}\subset\mathcal{A}(\mathbb{R})$ is a bounded set. Furthermore,

$$T_{F_+}(1)(z) = \frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{F_+(\zeta)}{\zeta-z}d\zeta = \frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{F(\zeta)}{\zeta-z}d\zeta,$$

since F_- vanishes at ∞. That is,

$$p(F_+) = \sup_{f\in B_1} p(T_F(f)).$$

Consider now the function F_-. Recall that

$$\operatorname{lim\,ind} H_0(\mathbb{C}_\infty\setminus K)\cong\mathcal{A}(\mathbb{R})'_b.$$

As a result, for every continuous seminorm r on $\operatorname{lim\,ind} H_0(\mathbb{C}_\infty\setminus K)$, there is a bounded set $B_2\subset\mathcal{A}(\mathbb{R})$ such that

$$r(F_-)\le C\sup_{f\in B_2}|\langle f,F_-\rangle|.$$

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By Cauchy's theorem, we have

$$\sup_{f\in B_2}|\langle f,F_-\rangle| = \sup_{f\in B_2}\Big|\int_\gamma f(z)F_-(z)dz\Big|$$
$$= \sup_{f\in B_2}\Big|\int_\gamma f(z)F(z)dz\Big| = \sup_{f\in B_2}\Big|\int_\gamma f(z)z\cdot\frac{F(z)}{z-0}dz\Big|,$$

since we may choose the curve γ to enclose the set K and (for instance) the point 0. As a result,

$$\sup_{f\in B_2}|\langle f,F_-\rangle| = \sup_{f\in B_2'}|(T_F f)(0)|,$$

where $B_2' = \{z\cdot f : f\in B\}$. Notice that the set B_2' is bounded in $\mathcal{A}(\mathbb{R})$. Indeed, the operator of multiplication $M_z : \mathcal{A}(\mathbb{R}) \ni f \mapsto zf \in \mathcal{A}(\mathbb{R})$ is continuous, since it factorizes through the operator of multiplication by z, denoted again by M_z, on every space $H(U)$. The operator M_z is obviously continuous on $H(U)$. We have $B_2' = M_z(B_2)$.

The evaluation at 0 is continuous on the space $\mathcal{A}(\mathbb{R})$. Hence, there is a continuous seminorm q on $\mathcal{A}(\mathbb{R})$ such that

$$\sup_{f\in B_2'}|(T_F f)(0)| \le \sup_{f\in B_2'} q(T_F f).$$

We conclude that for every continuous seminorm r on $\mathcal{X}(\mathbb{R})$, there are continuous seminorms p and q on $\mathcal{A}(\mathbb{R})$ such that

$$r(F) \le C\Big(\sup_{f\in B_1} p(T_F f) + \sup_{f\in B_2'} q(T_F f)\Big)$$
$$\le C\sup_{f\in B_3}(\max\{p,q\})(T_F f),$$

where $B_3 := \{1\}\cup B_2'$ and $\max\{p,q\}$ is a continuous seminorm on $\mathcal{A}(\mathbb{R})$. We showed that the map

$$\mathcal{T}(\mathbb{R}) \ni T_F \mapsto F \in \mathcal{X}(\mathbb{R})$$

is continuous as a map from the space of all Toeplitz operators equipped with the topology t_{ub} of uniform convergence on bounded sets to the symbol space.

We prove now that the assignment

$$H(U\setminus K) \ni F \mapsto T_F \in \mathcal{L}(\mathcal{A}(\mathbb{R}))$$

is continuous for every $U \supset \mathbb{R}$ open and $K \subset \mathbb{R}$ compact. In view of [23, Proposition 24.7], this implies that the assignment

$$\mathcal{X}(\mathbb{R}) \ni F \mapsto T_F \in (\mathcal{T}(\mathbb{R}), t_{ub})$$

is continuous.

Let now $B \subset \mathcal{A}(\mathbb{R})$ be bounded. It is a very important fact that there exists an open set $V \supset \mathbb{R}$ such that $B \subset H(V)$ and B is bounded in $H(V)$ – see [5, Corollary 1.22] and also [23, Proposition 25.19]. Inductive limits which possess this property are called regular.

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The embedding

$$H(V \cap U) \hookrightarrow \mathcal{A}(\mathbb{R})$$

is continuous by definition of the inductive topology of the space $\mathcal{A}(\mathbb{R})$. Therefore, for every continuous seminorm p on the space $\mathcal{A}(\mathbb{R})$, there is a compact set $L \subset V \cap U$ such that

$$p(f) \le C \sup_{z \in L} |f(z)|$$

for every $f \in H(V \cap U)$. Choose now a C^∞ smooth Jordan curve γ contained $V \cap U$ such that $K \cup L \subset I(\gamma)$. Then

$$\begin{aligned}\sup_{f \in B} p(T_F f) &\le C \sup_{f \in B} \sup_{z \in L} \left| \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{F(\zeta) \cdot f(\zeta)}{\zeta - z} d\zeta \right| \\ &\le C \sup_{\zeta \in \gamma} |f(\zeta)| \sup_{\zeta \in \gamma} |F(\zeta)| \le C \sup_\gamma |F|.\end{aligned}$$

Observe that the estimate is uniform, since the curve γ is the same for every function $f \in B$. This is a consequence of the fact that B is bounded.

We proved that

$$(\mathcal{T}(\mathbb{R}), t_{ub}) \cong \mathcal{X}(\mathbb{R})$$

as locally convex spaces. ∎

3 The case of the space of all holomorphic functions on a finitely connected domain in $\mathbb{C}$.

Proof of Theorem 1.3 We first consider the case of the space $X = H(D)$, when $D = I(\gamma_0) \cap E(\gamma_1) \cap \cdots \cap E(\gamma_n)$ and the space $X = H_0(D)$, when $D = E(\gamma_1) \cap \cdots \cap E(\gamma_n)$. The curves $\gamma_0, \gamma_1, \ldots, \gamma_n$ are assumed to satisfy the assumptions of Definition 1.1.

Recall that in these cases,

$$\mathfrak{S}(D) = \bigoplus_{i=0}^{n} \lim \operatorname{ind} H(U_i \cap D)$$

is the symbol space. In the case of the space $X = H_0(D)$ with D unbounded, the term containing $H(U_0 \cap D)$ is absent. The same remark concerns the arguments given below.

Again by Cauchy's theorem and Liouville's theorem, we have

$$H(U_0 \cap D) \cong H(I(\gamma_0)) \oplus H_0(E(\gamma_0) \cup U_0),$$

where U_0 is an open neighborhood of the curve γ_0. Similarly,

$$H(U_i \cap D) \cong H_0(E(\gamma_i)) \oplus H(I(\gamma_i) \cup U_i), \quad i = 1, \ldots, n,$$

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where this time U_i is an open neighborhood of the curve γ_i. As a result,

$$\begin{aligned}\text{lim ind}\, H(U_0 \cap D) &\cong H(I(\gamma_0)) \oplus H_0(I(\gamma_0)^c)\\ &= H(I(\gamma_0)) \oplus H_0(\overline{E(\gamma_0)}),\end{aligned}$$

and

$$\begin{aligned}\text{lim ind}\, H(U_i \cap D) &\cong H_0(E(\gamma_i)) \oplus H(E(\gamma_i)^c)\\ &= H_0(E(\gamma_i)) \oplus H(\overline{I(\gamma_i)})\end{aligned}$$

for $i = 1, \dots, n$. We remark that unfortunately there is a misprint in [16, (18)] – the symbol $H(I(\gamma_i)^c)$ appears instead of $H(E(\gamma_i)^c) = H(\overline{I(\gamma_i)})$. Recall that if $K \subset \mathbb{C}_\infty$ is compact, then $H(K)$ stands for the space of all germs of holomorphic functions on K. In particular, $H(I(\gamma_0)^c) = H(\overline{E(\gamma_0)})$ and $H(E(\gamma_i)^c) = H(\overline{H(I(\gamma_i))})$ are spaces of germs. These spaces carry the natural inductive topologies. We refer the reader to [2] for more information on this subject. Furthermore, it follows from Cauchy's integral formula and Liouville's theorem

$$H(D) \cong H(I(\gamma_0)) \oplus H_0(E(\gamma_1)) \oplus \cdots \oplus H_0(E(\gamma_n))$$

and, by the Köthe-Grothendieck-da Silva duality,

$$H(D)_b' \cong H_0(I(\gamma_0)^c) \oplus H(E(\gamma_1)^c) \oplus \cdots \oplus H(E(\gamma_n)^c).$$

We conclude [16, Proposition 4.1] that, as locally convex spaces,

$$\mathfrak{S}(D) \cong H(D) \oplus H(D)_b'. \tag{3.1}$$

At this moment, we can essentially repeat the arguments which justified Theorem 1.2. Indeed, let $F \in \mathfrak{S}(D)$ be a symbol and let $F_+ \in H(D)$ the holomorphic function part corresponding to the decomposition (3.1). Furthermore, assume that p is a continuous seminorm on the space $H(D)$. Then

$$p(F_+) = p(T_{F_+}(1))$$

and

$$T_{F_+}(1)(z) = \frac{1}{2\pi \mathrm{i}} \sum_{i=0}^{n} \int_{(\gamma_i)_\varepsilon} \frac{F_+(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi \mathrm{i}} \sum_{i=0}^{n} \int_{(\gamma_i)_\varepsilon} \frac{F(\zeta)}{\zeta - z} d\zeta. \tag{3.2}$$

Here, $F = F_+ \oplus F_-$ with

$$\begin{aligned}F_- &= F_{-0} \oplus \cdots \oplus F_{-n}\\ &\in H_0(E(\gamma_0) \cup U_0)) \oplus H(I(\gamma_1) \cup U_1)) \oplus \cdots \oplus H(I(\gamma_n) \cup U_n).\end{aligned}$$

Indeed, for instance,

$$\int_{(\gamma_0)_\varepsilon} \frac{F_{-0}(\zeta)}{\zeta - z} d\zeta = 0,$$

since $z \in I((\gamma_0)_\varepsilon)$ and F_{-0} vanishes in ∞.

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That is,

$$p(F_+) = \sup_{f \in B_1} p(T_F f),$$

where $B_1 = \{1\}$.

Let $f \in H(D)$. Then, by Cauchy's theorem,

$$\sum_{i=0}^{n} \int_{(\gamma_i)_\varepsilon} f(z) F_+(z) dz = 0$$

and, as a result,

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \sum_{i=0}^{n} \int_{(\gamma_i)_\varepsilon} f(z) F_{-i}(z) dz &= \frac{1}{2\pi \mathrm{i}} \sum_{i=0}^{n} \int_{(\gamma_i)_\varepsilon} f(z)(F_{-i}(z) + F_+(z)) dz \\ &= \frac{1}{2\pi \mathrm{i}} \sum_{i=0}^{n} \int_{(\gamma_i)_\varepsilon} f(z) F_i(z) dz. \end{aligned}$$

Let now $B_2 \subset H(D)$ be a bounded set. Then, for some point $a \in D$,

$$\begin{aligned} &\sup_{f \in B_2} |\langle f, F_- \rangle| = \sup_{f \in B_2} \left| \frac{1}{2\pi \mathrm{i}} \sum_{i=0}^{n} \int_{(\gamma_i)_\varepsilon} f(z) F_{-i}(z) dz \right| \\ &\quad = \sup_{f \in B_2} \left| \frac{1}{2\pi \mathrm{i}} \sum_{i=0}^{n} \int_{(\gamma_i)_\varepsilon} f(z) F_i(z) dz \right| = \sup_{f \in B_2} \left| \frac{1}{2\pi \mathrm{i}} \sum_{i=0}^{n} \int_{(\gamma_i)_\varepsilon} f(z)(z-a) \frac{F_i(z)}{z-a} dz \right|. \end{aligned}$$

The one point set $\{a\}$ is compact. Also, the set $\{f(z)(z-a) : f \in B_2\}$ is bounded in $H(D)$. It follows from (3.1) that for every continuous seminorm on $\mathfrak{S}(D)$, say r, there are continuous seminorms p and q on $H(D)$ and $H(D)'_b$, respectively, such that

$$r(F) \leq C(p(F_+) + q(F_-)).$$

Hence, for every continuous seminorm r on $\mathfrak{S}(D)$, we have

$$\begin{aligned} r(F) \leq C(p(F_+) + q(F_-)) &\leq C(\sup_{f \in B_1} p(T_F f) + \sup_{f \in B_2'} \sup_{z \in \{a\}} |(T_F f)(z)|) \\ &\leq C \sup_{f \in B_3} \sup_{z \in K \cup \{a\}} |(T_F f)(z)|, \end{aligned}$$

where $B_3 = \{1\} \cup B_2'$ and we assumed that $p(f) = \sup_{z \in K} |f(z)|$ for some $K \subset D$ compact.

The fact that the assignment

$$\mathfrak{S}(D) \ni F \mapsto T_F \in (\mathfrak{T}(D), t_{ub})$$

is continuous is elementary. One simply shows that the map

$$\bigoplus_{i=0}^{n} H(U_i \cap D) \ni F = \bigoplus_{i=0}^{n} F_i \mapsto T_F \in (\mathfrak{T}(D), t_{ub})$$

is continuous for every neighborhood U_i of γ_i. In view of the definition of the inductive topology, this shows that $F \mapsto T_F$ is continuous as a map from $\mathfrak{S}(D)$ to $(\mathfrak{T}(D), u_{ub})$.

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Let $B \subset H(D)$ be a bounded set and $K \subset D$ compact. Then, for a sufficiently small $\varepsilon > 0$,

$$\sup_{f\in B}\sup_{z\in K}|(T_F f)(z)| = \sup_{f\in B}\sup_{z\in K}\Big|\frac{1}{2\pi\mathrm{i}}\sum_{i=0}^{n}\int_{(\gamma_i)_\varepsilon}\frac{F_i(\zeta)\cdot f(\zeta)}{\zeta - z}d\zeta\Big|$$
$$\leq C\sup\Big\{|f(z)|: z\in(\gamma_0)_\varepsilon\cup\cdots\cup(\gamma_n)_\varepsilon, f\in B\Big\}\max_{i=0,\dots,n}\sup\{|F_i(z)|: z\in(\gamma_i)_\varepsilon\}$$
$$\leq C\max_{i=0,\dots,n}\sup\{|F_i(z)|: z\in(\gamma_i)_\varepsilon\}.$$

Let us now consider the case $X = H(D)$, when $D = E(\gamma_1)\cap\cdots\cap E(\gamma_n)$. First of all, by Cauchy's integral formula and Liouville's theorem,

$$H(D)\cong H_0(E(\gamma_1))\oplus\cdots\oplus H_0(E(\gamma_n))\oplus H(\mathbb{C}).$$

By the Köthe-Grothendieck-da Silva duality,

$$H(D)_b'\cong H(E(\gamma_1)^c)\oplus\cdots\oplus H(E(\gamma_n)^c)\oplus H_0(\infty),$$

where the symbol $H_0(\infty)$ is the space of germs of holomorphic functions on ∞ which vanish at ∞. That is,

$$H_0(\infty) = \lim\operatorname{ind} H_0(U\setminus\{\infty\})$$

and U run through open in the Riemann sphere neighborhoods of ∞. We again conclude that also in this case,

$$\mathfrak{S}(D)\cong H(D)\oplus H(D)_b'.$$

This is the key observation. Essentially at this moment, the above arguments can be repeated verbatim. Some care has to be taken with the $\mathfrak{S}_\infty$ term. We provide some comments only. Observe that if $F_{-\infty}\in H_0(\infty)$, then

$$\frac{1}{2\pi\mathrm{i}}\int_{|\zeta|=R}\frac{F_{-\infty}(\zeta)}{\zeta - z}d\zeta\equiv 0.$$

Also, if $F_+\in H(D)$, then by Cauchy's theorem,

$$\frac{1}{2\pi\mathrm{i}}\sum_{i=1}^{n}\int_{(\gamma_i)_\varepsilon}f(\zeta)F_+(\zeta)d\zeta + \frac{1}{2\pi\mathrm{i}}\int_{|\zeta|=R}f(\zeta)F_+(\zeta)d\zeta\equiv 0$$

for every $f\in H(D)$.

These two observations allow us to repeat the whole argument. This completes the proof of Theorem 1.3. ■

Proof of Theorem 1.4 Assume that

$$D = I(\gamma_0)\cap E(\gamma_1)\cap\cdots\cap E(\gamma_n),$$

where $\gamma_0,\gamma_1,\dots,\gamma_n$ are C^∞ smooth Jordan curves which satisfy the assumptions from Definition 1.1. In order to prove the theorem, it suffices to show that there exists a sequence C_n of elements in the commutator ideal $\mathfrak{C}(D)$ which converges to the identity operator in the topology of uniform convergence on bounded sets. Indeed, assume that $C_n\in\mathfrak{C}(D)$ and $C_n\to I$ in this topology. Let $T\in\operatorname{Alg}\mathfrak{T}(D)$ be an element

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of the Toeplitz algebra. Then, for every bounded set $B \subset H(D)$ and every compact set $K \subset D$,

$$\sup_{f\in B}\sup_{z\in K}|(Tf)(z)-(C_nTf)(z)| = \sup_{z\in K}\sup_{g\in T(B)}|g(z)-(C_ng)(z)| \to 0,$$

since the image of the set B under the operator T is bounded.

We shall construct a compact exhaustion of the set D and the operators C_n together. Let Φ_0 be the Riemann map of the simply connected set $I(\gamma_0)$ and Φ_i, $i = 1,\dots,n$ the Riemann maps of $E(\gamma_i)$. That is,

$$\Phi_0\colon I(\gamma_0) \to \mathbb{D},$$

and

$$\Phi_i\colon E(\gamma_i) \to \mathbb{D}$$

with $\Phi_i(\infty) = 0$ for $i = 1,\dots,n$. There is $\tau_0 \in [0,1)$ such that the C^∞ smooth Jordan curve $\gamma_t^0 := \Phi_0^{-1}(|w| = t)$ is contained in D for $\tau_0 < t < 1$. Similarly, for every $i = 1,\dots,n$, there is τ_i such that $\gamma_t^i := \Phi_i^{-1}(|w| = t)$ is contained in D for $\tau_i < t < 1$. Put

$$K_t := \overline{I(\gamma_t^0)} \cap \overline{E(\gamma_t^1)} \cap \cdots \cap \overline{E(\gamma_t^n)}$$

for $t_0 := \max_i \tau_i < t < 1$. The sets (K_t) form a compact exhaustion of the domain D. Hence, the seminorms

$$p_{K_t}(f) := \sup_{z\in K_t}|f(z)|, \quad f \in H(D)$$

for $t_0 < t < 1$ form a fundamental system of seminorms in $H(D)$. Also, the boundary of K_t is

$$\gamma_t^0 \cup \gamma_t^1 \cup \cdots \cup \gamma_t^n.$$

Furthermore, for every $f \in H(D)$,

$$f(z) = \sum_{i=0}^{n}\frac{1}{2\pi \mathrm{i}}\int_{\gamma_t^i}\frac{f(\zeta)}{\zeta - z}d\zeta,$$

provided $z \in \operatorname{int} K_t$. Consider the projections

$$H(D) \ni f \mapsto \frac{1}{2\pi \mathrm{i}}\int_{\gamma_i^t}\frac{f(\zeta)}{\zeta - z}d\zeta,$$

$i = 0,1,\dots,n$, $z \in D$ and t chosen in such a way that $z \in I(\gamma_0^t) \cap E(\gamma_1^t) \cap \cdots \cap E(\gamma_n^t)$. Denote them by P_i. Then

$$P_0\colon H(D) \to H(I(\gamma_0)),$$

and

$$P_i\colon H(D) \to H_0(E(\gamma_i))$$

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for $i = 1, \ldots, n$. Observe that the operator P_i is the Toeplitz operators with the symbol

$$0 \oplus \cdots \oplus 0 \oplus \overset{i}{1} \oplus 0 \oplus \cdots \oplus 0.$$

Also, by Cauchy's integral formula,

$$P_0 + P_1 + \cdots + P_n = I.$$

When $f \in H(D)$, we shall write f_i, $i = 0, 1, \ldots, n$ to denote $P_i f$.

Let A_m^i be the Toeplitz operator the symbol of which is

$$0 \oplus \cdots \oplus 0 \oplus \overset{i}{(\Phi_i)^m} \oplus 0 \oplus \cdots \oplus 0,$$

and let B_m^i be the Toeplitz operator with the symbol

$$0 \oplus \cdots \oplus 0 \oplus \overset{i}{\frac{1}{(\Phi_i)^m}} \oplus 0 \oplus \cdots \oplus 0.$$

Observe that

$$A_m^0, B_m^0 : H(D) \to H(I(\gamma_0)), \tag{3.3}$$

and

$$A_m^i, B_m^i : H(D) \to H_0(E(\gamma_i)), \tag{3.4}$$

for $i = 1, \ldots, n$.

Let

$$A_m := \sum_{i=0}^{n} A_m^i \circ P_i$$

and

$$B_m := \sum_{i=0}^{n} B_m^i \circ P_i.$$

Then

$$A_m(f)(z) = \sum_{i=0}^{n} \frac{1}{2\pi \mathrm{i}} \int_{\gamma_i^t} \frac{\Phi_i^m(\zeta) \cdot f_i(\zeta)}{\zeta - z} d\zeta = \sum_{i=0}^{n} \Phi_i^m(z) \cdot f_i(z)$$

by Cauchy's integral formula and

$$(B_m f)(z) = \sum_{i=0}^{n} \frac{1}{2\pi \mathrm{i}} \int_{\gamma_i^t} \frac{\Phi_i^{-m}(\zeta) \cdot f_i(\zeta)}{\zeta - z} d\zeta.$$

Notice that

$$P_j\Big(\sum_{i=0}^{n} \Phi_i^m \cdot f_i\Big) = \Phi_j^m \cdot f_j$$

for $j = 0, 1, \ldots, n$.

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Consequently,

$$(B_m \circ A_m)(f)(z) = \sum_{i=0}^{n} \frac{1}{2\pi \mathrm{i}} \int_{\gamma_i^t} \frac{\Phi_i^{-m}(\zeta) \cdot \Phi_i^m(\zeta) \cdot f_i(\zeta)}{\zeta - z} d\zeta$$
$$= \sum_{i=0}^{n} \frac{1}{2\pi \mathrm{i}} \int_{\gamma_i^t} \frac{f_i(\zeta)}{\zeta - z} d\zeta = f_0(z) + f_1(z) + \cdots + f_n(z) = f(z).$$

That is, $B_m \circ A_m = I$ for every $m \in \mathbb{N}$ and, as a result,

$$I - [B_m, A_m] = A_m \circ B_m.$$

Furthermore, it follows from (3.3) and (3.4) that

$$(A_m \circ B_m f)(z) = \sum_{i=0}^{n} \Phi_i^m(z) \cdot \frac{1}{2\pi \mathrm{i}} \int_{\gamma_i^t} \frac{\Phi_i^{-m}(\zeta) \cdot f_i(\zeta)}{\zeta - z} d\zeta.$$

We show that for every bounded set $B \subset H(D)$ and every compact set K_t, $t_0 < t < 1$, it holds that

$$\sup_{f \in B} \sup_{z \in K_t} |(A_m \circ B_m f)(z)| \to 0,$$

as $m \to \infty$. For a fixed t choose t_1 with $t_0 < t < t_1 < 1$. Then

$$\sup_{f \in B} \sup_{z \in K_t} \Big| \sum_{i=0}^{n} \Phi_i^m(z) \frac{1}{2\pi \mathrm{i}} \int_{\gamma_i^{t_1}} \frac{\Phi_i^{-m}(\zeta) \cdot f_i(\zeta)}{\zeta - z} d\zeta \Big| \le c(n+1) \Big(\frac{t}{t_1} \Big)^m \to 0$$

as $m \to \infty$.

It remains to notice that $[B_m, A_m]$ belongs to the commutator ideal $\mathfrak{C}(D)$. Observe that

$$[B_m, A_m] = \sum_{i=0}^{n} [B_m^i, A_m^i] \circ P_i.$$

That is, $[B_m, A_m]$ is of the form

$$\sum_{i=0}^{n} [T_{F_i}, T_{G_i}] \circ T_{H_i}$$

for the symbols F_i, G_i, and H_i defined above.

The proof in the case $X = H_0(D)$, $D = E(\gamma_1) \cap \cdots \cap E(\gamma_n)$ is the same. Some comment is needed when $X = H(D)$ for such a domain D. Then the projection

$$P_\infty : H(D) \to H(\mathbb{C})$$

is defined by the formula

$$(P_\infty f)(z) := \frac{1}{2\pi \mathrm{i}} \int_{|\zeta| = R} \frac{f(\zeta)}{\zeta - z} d\zeta$$

for $R > 0$ appropriately large. This is a Toeplitz operator with the symbol $0 \oplus 1$. A compact exhaustion of the domain D is now

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$$\overline{E(\gamma_1^t)} \cap \cdots \cap \overline{E(\gamma_n^t)} \cap \{|\zeta| \le r\},$$

where the C^∞ smooth Jordan curves γ_i^t were defined above.

The Toeplitz operators A_m^∞ and B_m^∞ are defined in the following way:

$$(A_m^\infty f)(z) := z^m f(z)$$
$$(B_m^\infty f)(z) := \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=R} \frac{1}{\zeta^m} \frac{f(\zeta)}{\zeta - z} d\zeta$$

and

$$A_m := \sum_{i=1}^{n} A_m^i \circ P_i + A_m^\infty \circ P_\infty$$
$$B_m := \sum_{i=1}^{n} B_m^i \circ P_i + B_m^\infty \circ P_\infty,$$

with A_m^i, B_m^i as before. Most of the arguments can now be repeated. Again, a comment is needed in the ∞ case. Observe that for every $B \subset H(D)$ bounded,

$$\sup_{f\in B} \sup_{z\in K} |(A_m^\infty \circ B_m^\infty f)(z)|$$
$$= \sup_{f\in B} \sup_{z\in K} \Big|\frac{1}{2\pi \mathrm{i}} z^m \cdot \int_{|\zeta|=R} \frac{1}{\zeta^m} \frac{f(\zeta)}{\zeta - z} d\zeta\Big| \le C\Big(\frac{r}{R}\Big)^m \to 0,$$

as $m \to \infty$. Here,

$$K = \overline{E(\gamma_1^t)} \cap \cdots \cap \overline{E(\gamma_n^t)} \cap \{|\zeta| \le r\}$$

and $r < R$. This suffices to complete the argument. ∎

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